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Long Memory and Covariance Stationarity of Asymmetric Power FIGARCH Model[1)]

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Abstract

In this paper, we study an asymmetric power fractionally integrated GARCH model and find a region on which the process is stationary ergodic and has long memory property.

Keywords : Long memory, Covariance stationary, APFIGARCH model, ARCH(∞) model

1. Introduction

The generalized autoregressive conditional heteroscedastic (GARCH) model was proposed by Engle(1982) and Bollerslev(1986) to represent the dynamic evolution of conditional variances and has been mainly applied to represent time series of high frequency financial returns. For classical GARCH model, the returns series X_t is given by $X_t=\psi_t\varepsilon_t$ where ε_t is independent white noise process with mean zero and unit variance and ψ_t is the volatility, specified as a linear function of the past squared returns. Recently, it is observed empirically that autocorrelations of observations in various fields tend to decay very slowly and remain fairly large for long lags. As a consequence, many researchers have proposed extensions of generalized GARCH models which can produce such long memory behaviour (see, for example, Ding and Granger(1996), Baillie et al(1996), Bollerslev and Mikkelsen(1996), Hosking(1996), Robinson and Zaffaroni(1997), Robinson and Henry(1999), Giraitis et al(2000b), Giraitis et al(2005) etc.). Koulikov (2003) introduce martingale-difference

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autoregressive conditionally heteroskedastic (MD-ARCH(∞)) model which has long-memory property in the sense that the process has non-summable autocovariance functions and covariance stationarity. On the other hand, another generalization of the GARCH model is related with asymmetric response of ψ_t^2 to positive and negative returns.

In this paper, to represent simultaneously asymmetric volatility and long memory, following asymmetric power fractionally integrated GARCH(p,d,q)(abbreviated as APFIGARCH(p,d,q)) model is proposed:

$$X_t=\psi_t\varepsilon_t,\quad (1-\phi(L))(1-L)^d(|X_t|-\gamma X_t)^\delta=\alpha_0+(1-\beta(L))v_t, \tag{1.1}$$

where L denotes the lag operator, $d\in(0,1/2)$, $\phi(L)=\sum_{i=1}^{p}\phi_i L^i$, $\beta(L)=\sum_{i=1}^{q}\beta_i L_i$ and v_t is a martingale difference sequence.

The objective of this paper is to consider a model (1.1) and to find a sufficient condition under which the process is covariance stationary and has long memory property. For APFIGARCH(1,d,1) process, we find and illustrate a coefficient region on which the process is stationary and has long memory property.

2. Covariance Stationary and Long Memory

In this section, we introduce an asymmetric power fractionally integrated GARCH model and examine its covariance stationarity and long memory property.

Giraitis et al(2000a) and Kazakevičius and Leipus (2002) show that a wide class of GARCH model can be expressed in the framework of ARCH(∞) process. ARCH(∞) process is given by

$$X_t=\psi_t\varepsilon_t,\quad \psi_t=a+\sum_{j=1}^{\infty}\pi_{j-1}X_{t-j}, \tag{2.2}$$

where $a>0$, $\pi_j:j\geq 0\subseteq R_0^+$ and $\varepsilon_t:t\in Z$ is a sequence of i.i.d non-negative random variables.

Stationarity condition of ARCH(∞) sequences given by Giraitis et al(2000a) imply absolute summability of the coefficients $\pi_i:j\geq 0$ and ultimately, short memory nature of the process. For second-order stationary time series $X_t:t\in Z$ with mean $EX_t=\mu$ and lag-k autocovariance $\gamma_k=E(X_t-\mu)(X_{t+k}-\mu)$, we say that X_t has short memory or long memory according to whether

$\sum_{k=-\infty}^{\infty}|\gamma_k|$ is convergent or divergent. Absolute summability of $\pi_j;j\geq 0$ is necessary to ensure convergence of the infinite series in the definition of ψ_t in (2.2).

Asymmetric power GARCH model is given by

$$X_t=\psi_t\varepsilon_t,\quad \psi_t^{\delta}=\alpha_0+\sum_{i=1}^{p}\alpha_i(|X_{t-i}|-\gamma X_{t-i})^{\delta}+\sum_{j=1}^{q}\beta_j\psi_{t-j}^{\delta}, \tag{2.3}$$

where $|\gamma|<1$, $\delta>0$, $\alpha_0>0$, $\alpha_i,\beta_j\geq 0\ (i=1,\cdots,p,\ j=1,\cdots,q)$ and ε_t is i.i.d with mean 0 and variance 1. $\sum\alpha_i E(|\varepsilon_t|-\gamma\varepsilon_t)^{\delta}+\sum\beta_j<1$ implies the strictly stationarity and geometric ergodicity of the process and in that case the process has short memory.

Equations (2.3) can be rewritten in the form of

$$(1-\beta(L)-m\alpha(L))(|X_t|-\gamma X_t)^{\delta}=m\alpha_0+(1-\beta(L))v_t, \tag{2.4}$$

where $\alpha(L)=\sum_{i=1}^{p}\alpha_i L^i$, $\beta(L)=\sum_{j=1}^{q}\beta_j L^j$, $m=E(|\varepsilon_t|-\gamma\varepsilon_t)^{\delta}$, $v_t=(|X_t|-\gamma X_t)^{\delta}-m\psi_t^{\delta}$.

Asymmetric power FIGARCH model which is assumed to have both long memory and asymmetry is defined by

$$(1-\phi(L))(1-L)^{\delta}(|X_t|-\gamma X_t)^{\delta}=m\alpha_0+(1-\beta(L))v_t, \tag{2.5}$$

with $d\in(0,1/2)$. Alternative representations of the APFIGARCH model (2.5) are as follows:

$$m\psi_t^{\delta}=\alpha^*+(1-\frac{1-\phi(L)}{1-\beta(L)}(1-L)^{\delta})(|X_t|-\gamma X_t)^{\delta}, \tag{2.6}$$

or

$$m\psi_t^{\delta}=a+(\frac{1-\beta(L)}{1-\phi(L)}(1-L)^{-d}-1)v_t. \tag{2.7}$$

Here equation (2.6) is a type of ARCH(∞).

From now on, we focus our attention on the following alternative to (2.7)

$$X_t=\psi_t\varepsilon_t,\quad \psi_t^*=a+\sum_{j=1}^{\infty}\Theta_{j-1}(X_{t-j}^*-\psi_{t-j}^*), \tag{2.8}$$

where $X^*_t=(|X_t|-\gamma X_t)^\delta$, $\varepsilon^*_t=\frac{1}{m}(|\varepsilon_t|-\gamma\varepsilon_t)^\delta$, $\psi^*_t=m\psi^\delta_t$ and

$$\frac{1-\beta L}{1-\phi L}(1-L)^{-d}-1=\sum_{j=1}^{\infty}\Theta_{j-1}L^j.$$

We make the assumptions:

A1. $\varepsilon_t: t\in Z$ is defined on the common probability space $(\Omega,\mathcal{F},P)$, and consists of i.i.d. copies of a random variable ε_t with $E(\varepsilon^*_0)=1<\infty$.

A2. $a>0$ and $\Theta_j: j\geq 0\subseteq R^+_0$.

Then $X^*_t-\psi^*_t: t\in Z$ is a sequence of zero-centered innovations, where $X^*_t-\psi^*_t=\psi^*_t(\varepsilon^*_t-1)$ and ψ^*_t and ε^*_t are independent for each $t\in Z$. It follows that $E[X^*_t-\psi^*_t]=0$ and $E[X^*_t-\psi^*|\mathcal{F}t-1]$ for each $t\in Z$, $\mathcal{F}t$ being the process filtration, and hence $X^*_t-\psi^*_t: t\in Z$ is a sequence of martingale differences innovations. Because of this structure of innovations, the infinite series in (2.8) converges without assuming the absolute summability of $\Theta_j: j\geq 0$. Model (2.8) is a type of MD-ARCH(∞) model which extends the covariance stationary GARCH sequence to the case of non-summable autocovariance. Since the sequence of innovations $X^*_t-\psi^*_t: t\in Z$ is formulated in terms of its past history, it is useful to use Volterra series representation of (2.8) as Giraitis et al(2000a) and Kazakevičius and Leipus (2002) have shown. Model (2.8) can be written in the form of

$$X^*_t=\psi^*_t\varepsilon^*_t,\quad \psi^*_t=a\sum_{k=0}^{\infty}M(k,t), \tag{2.9}$$

where for each $t\in Z$, sequence $M(k,t): k\geq 0$ is defined as:

$$M(0,t):=1,$$

$$M(k,t):=\sum_{j_1\cdots j_k}^{\infty}\Theta_{j_1-1}\cdots\Theta_{j_k-1}(\varepsilon^*_{t-j_1}-1)\cdots(\varepsilon^*_{t-j_1-\cdots-j_k}-1)(k\geq 1). \tag{2.10}$$

Since ψ^*_t in (2.8) involves the infinite series of weighted zero-centered innovation $X^*_t-\psi^*_t$, nonnegativity of the process is not immediate from the definition. Nonnegativity of ψ^*_t in (2.9) with probability 1 and following Theorem 2.1 are due to Koulikov(2003).

Theorem 2.1 Under A1-A2 and

$$\sum_{j=1}^{\infty}[\log j]^2\Theta_j^2<\infty, \tag{2.11}$$

$$E(\varepsilon_0^*-1)^2\sum_{j=0}^{\infty}\Theta_j^2<1, \tag{2.12}$$

the sequence $\{(X_t^*,\psi_t^*):t\in Z\}$ defined in (2.8), equivalently (2.9)-(2.10), converges a.e. on $(\Omega,\Im,P)$, and is stationary ergodic.

Theorem 2.2 Assume A1-A2 and (2.12). Then the sequence $\{(X_t^*,\psi_t^*):t\in Z\}$ defined in (2.8) is covariance stationary, where for each $t\in Z$ and $k\geq 0$:

$$EX_t^*=a,\quad E\psi_t^*=a,$$

$$E[(\psi_{t+k}^*-a)(\psi_t^*-a)]=\frac{a^2E(\varepsilon_0^*-1)^2}{1-E(\varepsilon_0^*-1)^2\sum_{j=0}^{\infty}\Theta_j^2}\sum_{j=0}^{\infty}\Theta_j\Theta_{j+k},$$

$$E[(X_{t+k}^*-a)(X_t^*-a)]=E[(\psi_{t+k}^*-a)(\psi_t^*-a)]+\frac{a^2E(\varepsilon_0^*-1)^2}{1-E(\varepsilon_0^*-1)^2\sum_{j=0}^{\infty}\Theta_j^*}\Theta_k^*$$

and $\{\Theta_k^*:k\geq 0\}$ is defined as $\Theta_0^*:=\Theta_{k-1}$ for $k\geq 1$.

Proof. The proof of Theorem 2.2 is essentially the same as that of Theorem 2 in Koulikov (2003) and hence is omitted.

3. APFIGARCH(1,d,1) Process

In most practical applications, relatively simple models such as APFIGARCH(0,d,0), APFIGARCH(0,d,1), APFIGARCH(1,d,0), APFIGARCH(1,d,1) provide a good representation of the real data.

In this section, we consider in detail the APFIGARCH(1,d,1) process with $\phi(L)=\phi L$, $\beta(L)=\beta L$ in equation (2.7):

$$X_t^*=\psi_t^*\varepsilon_t^* \tag{3.13}$$

$$\psi_t^* = a + \left(\frac{1-\beta L}{1-\phi L}(1-L)^{-d} - 1\right)(X_t^* - \psi_t^*) \tag{3.14}$$

$$= a + \sum_{j=1}^{\infty} \Theta_{j-1} v_{t-j}, \tag{3.15}$$

where $E(|\varepsilon_t| - \gamma\varepsilon_t)^\delta = m$, $X_t^* = (|X_t| - \gamma X_t)^\delta$, $\psi_t^* = m\psi_t^\delta$, $\varepsilon_t^* = \frac{1}{m}(|\varepsilon_t| - \gamma\varepsilon_t)^\delta$, $v_t = X_t^* - \psi_t^*$.

Note that

$$(1-L)^d = F(-d, 1; 1; L) = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(-d)\Gamma(j+1)} L^j = \sum_{j=0}^{\infty} \binom{d}{j}(-1)^j L^j$$

where F denotes the hypergeometric function, i.e.

$$F(a,b;c;x) = a + \frac{a\cdot b}{1\cdot c}x + \frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c=1)(c+2)}x^3 + \cdots$$

and $\Gamma(\cdot)$ the gamma function.

Define that

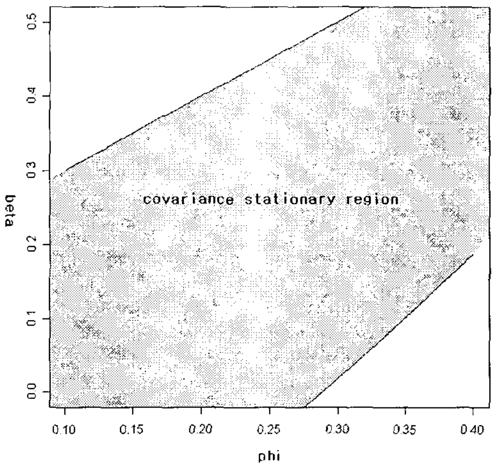
R0: $E(X_t^*) = a > 0$, $d \in (0, 1/2)$

R1: $\Phi < \frac{1-d}{2}$, $\beta < d$, $E(\varepsilon_t^* - 1)^2 \sum_{j=0}^{\infty} \Theta_j^2 < 1$, where $\sum_{j=0}^{\infty} \Theta_j^2 = (1+\beta^2)\sum_{j=0}^{\infty} \Theta_j'^2 - 2\beta \sum_{j=0}^{\infty} \Theta_j' \Theta_{j-1}' + \beta^2$ and $\Theta_j' = \frac{\Gamma(1+j+d)}{\Gamma(j+2)\Gamma(d)} F(1, -j-1; -d-j; \phi)$.

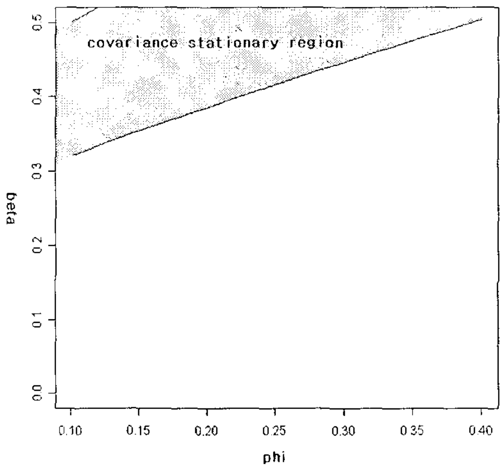
On the region which satisfies the conditions R0 and R1, (X_t^*, ψ_t^*) given by (3.13)-(3.15) is stationary ergodic and has long memory.

Since

$\sum_{j=0}^{\infty} \Theta_j^2 \le \frac{(\alpha-\beta)^2 + (\alpha-\beta)}{(1-\alpha-\alpha^2)}(\alpha^2 + 2\alpha d + B - 1 + \alpha(B - 1 - d^2)) + (\alpha - \beta + 1)(B-1) - (\alpha - \beta)d^2 + (\alpha-\beta)^2 + 2(\alpha-\beta)d$, where $B = \frac{\Gamma(1-2d)}{\Gamma(1-d)^2}$, if we assume that $E(\varepsilon_0^* - 1)^2 = 2$, region for long memory and covariance stationarity can be illustrated as follows:



<Figure1> APFIGARCH(1,d,1) with d=0.2



<Figure2> APFIGARCH(1,d,1) with d=0.4

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