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THE MEASURE-VALUED DYSON SERIES AND ITS STABILITY THEOREM

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ABSTRACT. In this article, we establish the existence theorem for measure-valued Dyson series and show that it satisfies the Volterra-type integral equation. Furthermore, we prove the stability theorems for measure-valued Dyson series.

1. Introduction

The Feynman-Kac formula plays the key role in the evolution of the theories of quantum mechanics and from the stability theorems for it, we can obtain the valuable information on the behavior of solution near given point.

For a sake of study for the Feynman-Kac formula, Cameron and Storvick introduced some definitions and some theories, related to the operator-valued Feynman integral on the Wiener space in [4]. Since then, the theory of this integral was investigated deeply by many mathematicians. In particular, Johnson and Lapidus proved the existence theorem for the generalized Dyson series and its stability theorems in [10, 11, 14, 15, 16].

Recently, the authors presented the definition of a complex-valued analogue of Wiener measure ω_φ on $C[0,t]$, the space of all continuous functions on a closed interval $[0,t]$, associated with a complex-valued measure φ on $\mathbb{R}$ in [22]. Indeed, if φ is the Dirac measure δ_0 at the origin in $\mathbb{R}$ then ω_φ is the concrete Wiener measure. In that paper, the

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authors derived the measure-valued measure V_φ on $C[0,t]$ and establish the measure-valued Feynman-Kac formula, which satisfies a kind of Volterra integral equation.

This article consists of five sections. In section 2, we introduce some notations, some definitions and some basic facts which are needed to understand the contents of the subsequent sections. In section 3, we establish the existence theorem of the generalized Dyson series associated with a measure-valued measure V_φ and show that it satisfies an integral equation under the some conditions for given potential functions. In section 4, we find the relation between the Bartle integral with respect to V_φ and the conditional ω_φ-integral. In the last section, we prove the stability theorems for the generalized Dyson series, treated in section 3.

2. Preliminaries

In this section, we introduce some notations, definitions and facts which are needed to understand the subsequent sections. Insofar as possible, we adopt the definitions and notation of [21].

Let $\mathbb{N}$ be the natural number system and let $\mathbb{R}$ be the real number system. For a natural number n, let $\mathbb{R}^n$ be the n-times product space of $\mathbb{R}$. Let $\mathcal{B}(\mathbb{R})$ be the set of all Borel measurable subsets of $\mathbb{R}$ and let m_L be the Lebesgue measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = i$ and $\alpha_4 = -i$.

For a positive real number t, let $C[0,t]$ be the space of all real-valued continuous functions on a closed bounded interval $[0,t]$ with the supremum norm $\|\cdot\|_\infty$.

Let $\mathcal{M}(\mathbb{R})$ be the space of all finite complex-valued countably additive measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For p in $\mathbb{R}$, let δ_p be the Dirac measure concentrated at p with total mass one. For μ in $\mathcal{M}(\mathbb{R})$ and for E in $\mathcal{B}(\mathbb{R})$, the total variation $|\mu|(E)$ on E is defined by

$$|\mu|(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|, \tag{2.1}$$

where the supremum is taken over all finite sequences $\langle E_i \rangle$ of disjoint sets in $\mathcal{B}(\mathbb{R})$. Then $|\mu|$ is in $\mathcal{M}(\mathbb{R})$ and, by the Jordan decomposition theorem [9, (19.13) Theorem, p.307], there are unique nonnegative measures μ_j $(j = 1, 2, 3, 4)$ in $\mathcal{M}(\mathbb{R})$ such that

$$\mu = \sum_{j=1}^{4} \alpha_j \mu_j. \tag{2.2}$$

By [5, Theorem 4.1.7, p.128] $(\mathcal{M}(\mathbb{R}), |\cdot|(\mathbb{R}))$ is a complex Banach space. Let $\mathcal{RM}(\mathbb{R})$ be the space of all finite complex-valued measures μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are absolutely continuous with respect to m_L.

Let $\mathbb{B}$ be a complex Banach space and let $\mathbb{B}^*$ be the dual space of $\mathbb{B}$. For a $\mathbb{B}$-valued countably additive measure ν on $(X, \mathcal{B})$ and for E in $\mathcal{B}$, the semivariation $\| \nu \| (E)$ of ν on E is given by

$$\| \nu \| (E) = \sup\{|x^*\nu|(E) \mid x^* \text{ is in } \mathbb{B}^* \text{ and } \| x^* \|_{\mathbb{B}^*} \leq 1\}, \tag{2.3}$$

where $|x^*\nu|(E)$ is the total variation on E of the complex-valued measure $x^*\nu$.

Let $\mathbb{B}$ be a complex Banach space and let $(X, \mathcal{B}, \mu)$ be a complex measure space. A function $f : X \to \mathbb{B}$ is called μ-measurable if there exists a sequence $\langle f_n \rangle$ of $\mathbb{B}$-valued simple functions with

$$\lim_{n\to\infty} \| f_n - f \|_{\mathbb{B}} = 0 \quad |\mu|\text{-a.e.} \tag{2.4}$$

A function f is called μ-weakly measurable if x^*f is μ-measurable for each x^* in $\mathbb{B}^*$, the dual space of $\mathbb{B}$. We say that f is μ-Bochner integrable if there exists a sequence $\langle f_n \rangle$ of $\mathbb{B}$-valued simple functions such that $\langle f_n \rangle$ converges to f in the norm sense in $\mathbb{B}$ for $|\mu|$-a.e. and $\lim_{n\to\infty} \int_X \| f(t) - f_n(t) \|_{\mathbb{B}} \, d|\mu|(t) = 0$.

From [21], we have the following theorem.

THEOREM 2.1. *Let $(X, \mathcal{B}, \mu)$ be a complex measure space and let $f : X \to \mathcal{M}(\mathbb{R})$ be a μ-Bochner integrable function. Then for E in $\mathcal{B}(\mathbb{R})$, $[f(t)](E)$ is a complex-valued μ-integrable function of t and*

$$\left[(Bo) - \int_X f(t) \, d\mu(t)\right](E) = \int_X [f(t)](E) \, d\mu(t). \tag{2.5}$$

Let $\mathbb{B}$ be a complex Banach space and let $(Y, \mathcal{C}, \nu)$ be a $\mathbb{B}$-valued measure space. Let g be a complex-valued $\| \nu \|$-measurable function on Y, that is, there exists a sequence $\langle g_n \rangle$ of complex-valued simple functions with $\lim_{n\to\infty} |g_n - g| = 0$ $\| \nu \|$-a.e. We say that g is ν-Bartle integrable if there exists a sequence $\langle g_n \rangle$ of simple functions such that $\langle g_n \rangle$ converges to g $\| \nu \|$-a.e. and the sequence $\langle \int_Y g_n(s) d\nu(s) \rangle$ is Cauchy in the norm sense. In this case, $(Ba) - \int_Y g(s) d\nu(s)$ is defined by

$$(Ba) - \int_Y g(s) \, d\nu(s) = \lim_{n\to\infty} \int_Y g_n(s) \, d\nu(s), \tag{2.6}$$

where the limit means the limit in the norm sense.

Let φ be in $\mathcal{M}(\mathbb{R})$ and η be a complex-valued Borel measure on $[0,t]$. A complex-valued Borel measurable function θ on $[0,t]\times\mathbb{R}$ is said to belong to $L_{\varphi;\infty,1;\eta}$ (or $L^t_{\varphi;\infty,1;\eta}$) if

$$\| \theta \|_{\varphi;\infty,1;\eta} = \int_{[0,t]} \| \theta(s,\cdot) \|_{\varphi;\infty} \; d|\eta|(s) \tag{2.7}$$

is finite, where $\| \theta(0,\cdot) \|_{\varphi;\infty}$ is $\inf\{\lambda > 0 \mid |\varphi|(\{\xi \text{ in } \mathbb{R} \mid |\theta(0,\xi)| > \lambda\}) = 0\}$ and $\| \theta(s,\cdot) \|_{\varphi;\infty}$ is $\inf\{\lambda > 0 \mid m_L(\{\xi \text{ in } \mathbb{R} \mid |\theta(s,\xi)| > \lambda\}) = 0\}$ for $0 < s \leq t$. If θ is bounded Borel measurable, then θ is in $L_{\varphi;\infty,1;\eta}$.

For θ in $L^\infty(\mathbb{R}, m_L)$, we consider an operator M_θ from $\mathcal{RM}(\mathbb{R})$ into itself such that

$$[M_\theta(\mu)](E) = \int_E \frac{d\mu}{dm_L}(\xi)\theta(\xi)\; dm_L(\xi) \tag{2.8}$$

for E in $\mathcal{B}(\mathbb{R})$ and for μ in $\mathcal{RM}(\mathbb{R})$. Then

$$\frac{dM_\theta(\mu)}{dm_L}(\xi) = \frac{d\mu}{dm_L}(\xi)\theta(\xi); \tag{2.9}$$

so M_θ is well-defined. Since $|M_\theta(\mu)|(\mathbb{R}) \leq \int_{\mathbb{R}} |\frac{d\mu}{dm_L}(\xi)|\; |\theta(\xi)|\; dm_L(\xi) \leq \| \theta \|_\infty \; |\mu|(\mathbb{R})$, M_θ is a bounded linear operator.

For $s > 0$, we let

$$P_s(E) = \int_E \frac{1}{\sqrt{2\pi s}} \exp\left\{-\frac{u^2}{2s}\right\} \; dm_L(u) \tag{2.10}$$

for E in $\mathcal{B}(\mathbb{R})$.

For $s > 0$, we consider an operator S_s from $\mathcal{RM}(\mathbb{R})$ into itself such that

$$\begin{aligned} [S_s(\mu)](E) &= (\mu * P_s)(E) \\ &= \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \left[\int_E \exp\left\{-\frac{(u-v)^2}{2s}\right\} \; dm_L(u)\right] \; d\mu(v). \end{aligned} \tag{2.11}$$

Then $\frac{dS_s(\mu)}{dm_L}(\xi) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \exp\{-\frac{(\xi-v)^2}{2s}\} \; d\mu(v)$. It is not hard to show that S_s is a bounded linear operator and the operator norm $\| S_s \|$ of S_s is less than or equal to one.

Let s_1 and s_2 be two positive real numbers. Then by the Chapman-Kolmogorov equation in [10, Proposition 3.2.3, p.37] and the classical Fubini theorem, we have

$$S_{s_1} \circ S_{s_2} = S_{s_1+s_2} \; . \tag{2.12}$$

For $s > 0$, for φ in $\mathcal{M}(\mathbb{R})$, for a Borel measurable $|\varphi|$-essentially bounded function θ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and for E in $\mathcal{B}(\mathbb{R})$, we let

(2.13)

$$[T(s,\varphi,\theta)](E) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \left[\int_E \theta(v) \exp\left\{ -\frac{(u-v)^2}{2s} \right\} dm_L(u) \right] d\varphi(v) .$$

Then $T(s,\varphi,\theta)$ is in $\mathcal{RM}(\mathbb{R})$ and

$$\frac{dT(s,\varphi,\theta)}{dm_L}(u) = \frac{1}{\sqrt{2\pi s}} \int_{\mathbb{R}} \theta(v) \exp\left\{ -\frac{(u-v)^2}{2s} \right\} d\varphi(v) . \tag{2.14}$$

Here we will introduce a complex-valued analogue of the Wiener measure ω_φ on $C[0,t]$.

Let t be a positive real number and n a nonnegative integer. For $\vec{t} = (t_0, t_1, \ldots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be a function with

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n)) . \tag{2.15}$$

For B_j $(j = 0, 1, 2, \ldots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval. Let $\mathcal{I}$ be the set of all intervals. For a nonnegative finite Borel measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$\begin{aligned} &m_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)) \\ &= \int_{B_0} \left[\int_{\prod_{j=1}^n B_j} W(n+1; \vec{t}; u_0, u_1, \ldots, u_n) \right. \\ &\quad \left. d\prod_{j=1}^n m_L(u_1, \ldots, u_n) \right] d\varphi(u_0), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} &W(n+1; \vec{t}; u_0, u_1, \ldots, u_n) \\ &= \left(\prod_{j=1}^n \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \right) \exp\left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\}. \end{aligned}$$

By [19, Theorem 5.1, p.144] and [19, Theorem 2.1, p.212], $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive measure ω_φ on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I) = m_\varphi(I)$ for all I in $\mathcal{I}$.

For φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^{4} \alpha_j \varphi_j$, let $\omega_\varphi = \sum_{j=1}^{4} \alpha_j \omega_{\varphi_j}$. We say that ω_φ is the complex-valued analogue of the Wiener measure on $(C[0,t], \mathcal{B}(C[0,t]))$, associated with φ.

By the change of variables formula, we can easily prove the following theorem.

THEOREM 2.2. (The Wiener integration formula) *If $f : \mathbb{R}^{n+1} \to \mathbb{C}$ is a Borel measurable function, then the following equality holds:*

$$\begin{aligned} &\int_{C[0,t]} f(x(t_0), x(t_1), \ldots, x(t_n))\, d\omega_\varphi(x) \\ (2.17) \qquad &\overset{*}{=} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \ldots, u_n) W(n+1; \vec{t}; u_0, u_1, \ldots, u_n) \\ &d(\prod_{j=1}^{n} m_L \times \varphi)((u_1, u_2, \ldots, u_n), u_0), \end{aligned}$$

where $\overset{*}{=}$ means that if one side exists, then both sides exist and the two values are equal.

Let φ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let n be a non-negative integer. Let X be a $\mathbb{R}^{n+1}$-valued measurable function on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$. We write P_X for a measure on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$ determined by X, that is,

$$P_X(E) = \omega_\varphi(X^{-1}(E)) \tag{2.18}$$

for E in $\mathcal{B}(\mathbb{R}^{n+1})$. If $X(x) = x(t)$, then

$$P_X(E) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \int_E \exp\{-\frac{(\xi - u_0)^2}{2}\}\, dm_L(\xi) d\varphi(u_0)\ . \tag{2.19}$$

Let Z be an integrable function on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$. The conditional ω_φ-integral of Z given X, written $E^{\omega_\varphi}(Z|X)$, is defined to be any real-valued Borel measurable and P_X-integrable function ψ on $\mathbb{R}^{n+1}$ such that

$$\int_{X^{-1}(H)} Z(x) d\omega_\varphi(x) = \int_H \psi(\xi) dP_X(\xi) \tag{2.20}$$

for H in $\mathcal{B}(\mathbb{R})$. By the Radon-Nikodym theorem, we know that such a function ψ always exists.

In this paper, we will treat the case $X : C[0,t] \to \mathbb{R}$ given by $X(x) = x(t)$. We can easily check the following two facts by the definition of $E^{\omega_\varphi}(Z|X)$. If Z_1 and Z_2 are bounded measurable then for a, b in $\mathbb{R}$,

$$E^{\omega_\varphi}(aZ_1 + bZ_2|X) = aE^{\omega_\varphi}(Z_1|X) + bE^{\omega_\varphi}(Z_2|X)$$

and if Z_1 and Z_2 are real-valued bounded measurable with $Z_1 \le Z_2$ ω_φ-a.e. and φ is a probability measure then $E^{\omega_\varphi}(Z_1|X) \le E^{\omega_\varphi}(Z_2|X)$ m_L-a.e.

From Yeh's paper in [25], we have the following lemma.

LEMMA 2.3. *Let Y be an integrable function on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$. If $\int_{C[0,t]} e^{iuX(x)}Y(x)\, d\omega_\varphi(x)$ is a Lebesgue integrable function of u in $\mathbb{R}$, $E^{\omega_\varphi}(Y|X)$ is given by*

$$\begin{aligned}&E^{\omega_\varphi}(Y|X)(\xi)\frac{dP_X}{dm_L}(\xi)\\ =&\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iu\xi}\left(\int_{C[0,t]} e^{iuX(x)}Y(x)\, d\omega_\varphi(x)\right) dm_L(u).\end{aligned}$$

By the elementary calculus, we obtain the following lemmas.

LEMMA 2.4. *Let $(\Omega, \mathcal{B}, P)$ be a probability measure space, and let X, f and g be three real-valued measurable functions on Ω such that $f(\omega) \ge g(\omega) \ge 0$ for all ω in Ω. If $\int_\Omega e^{-iuX(\omega)}f(\omega)\, dP(\omega)$ is m_L-integrable of u, then $\int_\Omega e^{-iuX(\omega)}g(\omega)\, dP(\omega)$ is also m_L-integrable.*

LEMMA 2.5. *For E in $\mathcal{B}(C[0,t])$, $\int_{C[0,t]} e^{iuX(x)}\chi_E(x)\, d\omega_\varphi(x)$ is a Lebesgue integrable function of u.*

From the above lemmas, we can easily check the following facts.

(1) For any simple function s on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$, $\int_{C[0,t]} e^{iuX(x)}s(x)\, d\omega_\varphi(x)$ is a bounded continuous and Lebesgue integrable function of u.

(2) For any real-valued bounded measurable f on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$, $\int_{C[0,t]} e^{iuX(x)}f(x)\, d\omega_\varphi(x)$ is a bounded and continuous and Lebesgue integrable function of u.

(3) Let f be real-valued bounded measurable on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$ and let $\langle s_n\rangle$ be an increasing sequence of simple functions such that $\langle s_n\rangle$ converges to f uniformly. Then $\langle \int_{C[0,t]} e^{iuX(x)}s_n(x)\, d\omega_\varphi(x)\rangle$ converges uniformly and converges to $\int_{C[0,t]} e^{iuX(x)}\, f(x)\, d\omega_\varphi(x)$ in L_1-norm sense.

(4) Since the Fourier transform from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$ is continuous,

$$\begin{aligned}&\left\langle \int_{\mathbb{R}} e^{-iu\xi}\int_{C[0,t]} e^{iuX(x)}s_n(x)d\omega_\varphi(x)dm_L(u)\right\rangle\\ &\to \int_{\mathbb{R}} e^{-iu\xi}\int_{C[0,t]} e^{iuX(x)}f(x)d\omega_\varphi(x)dm_L(u)\end{aligned}$$

as $n \to +\infty$ in L_∞-norm sense. Therefore $\langle E^{\omega_\varphi}(s_n|X)(\xi)\rangle$ converges to $E^{\omega_\varphi}(f|X)(\xi)$ m_L-a.e.

(5) Since $E^{\omega_\varphi}(s_n|X)$ and $E^{\omega_\varphi}(f|X)$ are all in $L_1(\mathbb{R}, m_L)$ for all n in $\mathbb{N}$, $\langle E^{\omega_\varphi}(s_n|X)\rangle$ converges to $E^{\omega_\varphi}(f|X)$ in L_1-norm sense. Using the results in above, we obtain the following theorem.

THEOREM 2.6. *Let f be real-valued bounded measurable on $(C[0,t], \mathcal{B}(C[0,t]))$, and let $\langle s_n\rangle$ be an increasing sequence of simple functions on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $\langle s_n\rangle$ converges to f ω_φ-a.e. x. Then*

$$\int_E E^{\omega_\varphi}(f|X)(\xi)\ dP_X(\xi) = \lim_{n\to\infty}\int_E E^{\omega_\varphi}(s_n|X)(\xi)\ dP_X(\xi).$$

THEOREM 2.7. *If a sequence $\langle\varphi_n\rangle$ of non-negative finite measures, converges to φ in the sense of total variation norm then a sequence $\langle\omega_{\varphi_n}\rangle$ converges to ω_φ in the total variation norm.*

Proof. From [21, Theorem 4.3], we have

$$|\omega_{\varphi_n} - \omega_\varphi|(C[0,t]) = |\omega_{\varphi_n - \varphi}|(C[0,t]) = |\varphi_n - \varphi|(\mathbb{R}),$$

so we have desired. □

From [2, p.20], we can find a sequence $\langle P_n\rangle$ of measures on $C[0,t]$ such that $\langle P_n\rangle$ doesn't converges to P weakly even though every finite dimensional measures of P_n converges to some finite dimensional measure of P weakly. Here, we want to find the conditions such that $\langle\omega_{\varphi_n}\rangle$ converges to ω_φ weakly whenever $\langle\varphi_n\rangle$ converges to φ weakly.

LEMMA 2.8. *Let $X : [0,t]\times C[0,t] \to \mathbb{R}$ be a function with $X(s,x) = x(s)$. Then*

(1) *X is a continuous stochastic process,*
(2) *if $0 = t_0 < t_1 < \cdots < t_n \le t$, then $X(t_j,\cdot) - X(t_{j-1},\cdot)$ $(j = 1,2,\ldots,n)$ are independent and*
(3) *for $0 \le t_1 \le t_2 \le t$, $X(t_2,\cdot) - X(t_1,\cdot)$ is normal distributed with the mean zero and the variance $t_2 - t_1$.*

Proof. The statement (1) is clear from the definition of continuous random variable. By [21, Example 3.3], the statement (2) holds. We will prove the statement (3).

Firstly, we treat the case $t_1 = 0$. Let y be in $\mathbb{R}$. Let $E = \{(u_0, u_2)|u_2 - u_0 < y\}$. Then $\{x|x(t_2) - x(t_1) < y\} = \{x|(x(0), x(t_2)) \in E\}$. So, by

[21, Theorem 2.2] and by the substitution of $v = u_2 - u_0$,

$$\begin{aligned} &\omega_\varphi(\{x|x(t_2) - x(t_1) < y\}) \\ (2.21) \qquad &= \int_{\mathbb{R}} \int_{-\infty}^{u_0+y} \frac{1}{\sqrt{2\pi t_2}} \exp\left\{ - \frac{(u_2 - u_0)^2}{2t_2} \right\} dm_L(u_2) d\varphi(u_0) \\ &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi t_2}} \exp\left\{ - \frac{v^2}{2t_2} \right\} dm_L(v). \end{aligned}$$

So we proved the case $t_1 = 0$. Now we suppose $0 < t_1$. Let y be in $\mathbb{R}$. Let $E = \{(u_1, u_2)|u_2 - u_1 < y\}$. Then by the substitution of $v = u_2 - u_1$,

$$\begin{aligned} &\omega_\varphi(\{x|x(t_2) - x(t_1) < y\}) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sqrt{(2\pi)^2 (t_2 - t_1) t_1}} \exp\left\{ - \frac{(u_1 - u_0)^2}{2t_1} \right\} \\ (2.22) \qquad &\quad \int_{-\infty}^{u_1+y} \exp\left\{ - \frac{(u_2 - u_1)^2}{2(t_2 - t_1)} \right\} dm_L(u_2) dm_L(u_1) d\varphi(u_0) \\ &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left\{ - \frac{v^2}{2(t_2 - t_1)} \right\} dm_L(v), \end{aligned}$$

so we proved the case $t_1 > 0$ as desired. □

LEMMA 2.9. *Under the assumptions in Lemma 2.8, for $0 < t_1 \le t$ and for $\epsilon > 0$,*

$$\omega_\varphi(\{x|\sup\{x(s) - x(0)|0 \le s \le t_1\} \ge \epsilon\}) \le \frac{1}{\epsilon} \sqrt{\frac{2t_1}{\pi}} \exp\left\{ - \frac{\epsilon^2}{2t_1} \right\}.$$

Proof. Let $S(x) = \sup\{x(s) - x(0)|0 \le s \le t_1\}$. Then there exists a dense sequence $\langle s_n \rangle$ in $[0, t_1]$ such that $S(x) = \sup_n \{x(s_n) - x(0)\}$. Let $S_n(x) = \max\{x(s_k) - x(0)|k = 1, 2, \ldots, n\}$. Then $S_n(x) \to S(x)$ ω_φ-a.e. x. Let us relabel $s_1, s_2, \ldots, s_n$ as $\tau_{n,1}, \tau_{n,2}, \cdots, \tau_{n,n}$ such that $\tau_{n,0} = 0 < \tau_{n,1} < \tau_{n,2} < \cdots < \tau_{n,n} \le t_1$. Let $X_{n,j}(x) = x(\tau_{n,j}) - x(\tau_{n,j-1})$, $j = 1, 2, \ldots, n$ and $S_{n,j}(x) = X_{n,1}(x) + \cdots + X_{n,j}(x)$. Then $S_{n,k}(x) = x(\tau_{n,k}) - x(0)$. Since $S_{n,k} - S_{n,n} = x(\tau_{n,k}) - x(\tau_{n,n})$, $S_{n,k} - S_{n,n}$ is normal distributed with the mean zero and the variance $\tau_{n,n} - \tau_{n,k}$, the median of $S_{n,k} - S_{n,n}$ is zero. By Levy inequality [24, p.137], for $\epsilon > 0$, letting $E_\epsilon = \{(u_0, u_1)|u_1 - u_0 \ge \epsilon\}$ and let $u_1 - u_0 = v$,

$$(2.23) \qquad \omega_\varphi(\{x|S_n(x) > \epsilon\}) \le 2\omega_\varphi(\{x|S_{n,n} > \epsilon\})$$

$$
\begin{aligned}
&= 2\omega_\varphi(\{x|(x(0),x(\tau_{n,n}))\in E_\epsilon\}) \\
&= \frac{2}{\sqrt{2\pi\tau_{n,n}}}\int_{\mathbb{R}}\int_{\epsilon+u_0}^{\infty}\exp\left\{-\frac{(u_1-u_0)^2}{2\tau_{n,n}}\right\}dm_L(u_1)d\varphi(u_0) \\
&= \frac{2}{\sqrt{2\pi\tau_{n,n}}}\int_{\mathbb{R}}\int_{\epsilon}^{\infty}\exp\left\{-\frac{v^2}{2\tau_{n,n}}\right\}dm_L(v)d\varphi(u_0) \\
&\le \frac{2}{\epsilon\sqrt{2\pi\tau_{n,n}}}\int_{\epsilon}^{\infty}v\exp\left\{-\frac{v^2}{2\tau_{n,n}}\right\}dm_L(v) \\
&= \sqrt{\frac{2\tau_{n,n}}{\pi}}\frac{1}{\epsilon}\exp\left\{-\frac{\epsilon^2}{2\tau_{n,n}}\right\}.
\end{aligned}
$$

Not only does $S_n(x)\to S(x)$ a.e., but

$$\{x|S_n(x)\ge\epsilon\}\to\{x|S(x)\ge\epsilon\},\quad \text{a.e.},$$

$$\omega_\varphi(\{x|S_n(x)\ge\epsilon\})\to\omega_\varphi(\{x|S(x)\ge\epsilon\}),$$

so,

$$P(S(x)\ge\epsilon)<\frac{1}{\epsilon}\sqrt{\frac{2t_1}{\pi}}\exp\left\{-\frac{\epsilon^2}{2t_1}\right\}.$$

□

From the essentially same method as in the proof of Lemma 3 in [24, p.256], we can prove the following lemma.

LEMMA 2.10. *For $\epsilon>0$ and $\lambda>0$,*

$$\omega_\varphi\left(\left\{x\middle|\sup_{0\le t\le\epsilon}|x(t)-x(\epsilon)|\le\lambda\right\}\right)=\omega_\varphi\left(\left\{x\middle|\sup_{0<t<\frac{\epsilon}{2}}|x(t)-x(0)|\le\lambda\right\}\right)^2.$$

COROLLARY 2.11.

$$\omega_\varphi\left(\left\{x\middle|\sup_{0\le s\le t_1}|x(s)-x(t_1)|>\lambda\right\}\right)\le\frac{1}{\lambda}\sqrt{\frac{t_1}{\pi}}e^{-\frac{\lambda^2}{t_1}}\left(2-\frac{1}{\lambda}\sqrt{\frac{t_1}{\pi}}e^{-\frac{\lambda^2}{t_1}}\right).$$

COROLLARY 2.12. *For each positive ϵ and η, there exists a δ with $0<\delta<1$ such that for s_1,s_2 in $[0,t]$*

$$\omega_\varphi\left(\left\{x\middle|\sup_{|s_1-s_2|<\delta}|x(s_1)-x(s_2)|\ge\epsilon\right\}\right)\le\eta.$$

From [2, pp. 54–55], we find the following lemma.

LEMMA 2.13. *The sequence $\langle P_n \rangle$ of probability measures on $C[0,t]$ is tight, that is, for positive ϵ there exists a compact set K such that $P_n(K) > 1-\epsilon$ for all natural number n, if and only if*

(i) *for each positive η, there exists an α such that*

$$P_n(\{x||x(0)| > \alpha\}) \le \eta$$

for all n and

(ii) *for each positive $\epsilon > 0$ and η, there exists a δ with $0 < \delta < 1$ and a natural number n_0 such that for $n \ge n_0$,*

$$P_n(\{x| \sup_{|s_1-s_2|<\delta} |x(s_1)-x(s_2)| \ge \epsilon\}) \le \eta.$$

LEMMA 2.14. *Let P_n, P be probability measures on $(C[0,t], \mathcal{B}(C[0,t]))$. If the finite dimensional distributions of P_n converge weakly to those of P, and if $\langle P_n \rangle$ is tight, then $\langle P_n \rangle$ converges to P weakly.*

By the above results, we have the following theorems.

THEOREM 2.15. *Suppose $\langle \varphi_n \rangle$ is tight. Then $\langle \omega_{\varphi_n} \rangle$ is also tight.*

LEMMA 2.16. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be bounded continuous. Let $\vec{t} = (t_0, t_1, \ldots, t_n)$ be a vector in $\mathbb{R}^{n+1}$ with $t_0 = 0 < t_1 < \cdots, t_n \le t$ and let $J_{\vec{t}}: C[0,t] \to \mathbb{R}^n$ be a function with $J_{\vec{t}}(x) = (x(t_0), x(t_1), \ldots, x(t_n))$. Suppose $\langle \varphi_n \rangle$ converges to φ weakly. Then*

$$\begin{aligned}
&\lim_{m\to\infty} \int_{\mathbb{R}^{n+1}} f(u_0, u_1, \ldots, u_n) d\omega_{\varphi_m} J_{\vec{t}}^{-1}(u_0, \ldots, u_n) \\
&= \lim_{m\to\infty} \int_{C[0,t]} f(J_{\vec{t}}(x)) d\omega_{\varphi_m}(x) \\
&= \lim_{m\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0, u_1, \ldots, u_n) \frac{1}{\prod_{j=1}^n \sqrt{2\pi(t_j - t_{j-1})}} \\
&\quad \exp\{-\sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\} d\prod_{j=1}^n m_L(u_1, \ldots, u_n) d\varphi_m(u_0).
\end{aligned} \tag{2.24}$$

THEOREM 2.17. *If $\langle \varphi_n \rangle$ is tight and $\langle \varphi_n \rangle$ converges to φ weakly, $\langle \omega_{\varphi_n} \rangle$ converges to ω_φ weakly.*

For φ in $\mathcal{M}(\mathbb{R})$ and for B in $\mathcal{B}(C[0,t])$, we let $[V_\varphi(B)](E) = \omega_\varphi(B \cap X^{-1}(E))$. Then V_φ is a measure-valued measure on $(C[0,t], \mathcal{B}(C[0,t]))$ in the total variation norm sense.

From [21], we have the following theorem.

THEOREM 2.18. *Let φ be in $\mathcal{M}(\mathbb{R})$ and let $\vec{t} = (t_0, t_1, \dots, t_n)$ be a vector in $\mathbb{R}^{n+1}$ with $0 = t_0 < \cdots < t_n = t$. Let $f : \mathbb{R}^{n+1} \to \mathbb{C}$ be a Borel measurable function such that $f(u_0, u_1, \dots, u_n)W(n+1; \vec{t}; u_0, \dots, u_n)$ is $|\varphi| \times \prod_{j=1}^{n} m_L$-integrable. Let $F : C[0,t] \to \mathbb{C}$ be a function with $F(x) = (f \circ J_{\vec{t}})(x) = f(x(t_0), x(t_1), \dots, x(t_n))$. Then F is V_φ-Bartle integrable on $C[0,t]$ and for E in $\mathcal{B}(\mathbb{R})$,*

$$\begin{aligned}
&\left[(Ba) - \int_{C[0,t]} F(x)\, dV_\varphi(x)\right](E) \\
(2.25) \qquad &= \int_E \left\{ \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} f(u_0, u_1, \dots, u_n) W(n+1; \vec{t}; u_0, \dots, u_n) \right.\right. \\
&\left.\left. \cdot\, d\varphi(u_0) \right) d\Big(\prod_{j=1}^{n-1} m_L\Big)(u_1, \dots, u_{n-1}) \right\} dm_L(u_n).
\end{aligned}$$

3. The generalized Dyson series and Volterras's integral equation

The purpose of this section is to establish the existence theorem of the generalized Dyson series associated with a measure-valued measure V_φ and is to show that it satisfies an integral equation, similar to Volterra's integral equation, under the some conditions for given functions.

Recently, the authors derived the Feynman-Kac formula associated with a measure valued measure V_φ in [21] and proved that this formula satisfies the Volterra-type integral equation. Here, we will prove the existence theorem of the generalized Dyson series associated with V_φ. However, the solution of Volterra's equation is unique in usual case, we will show that our Dyson series for given many analytic functions satisfy the Volterra-type integral equation.

By Theorem 2 in [23], we can easily check the following lemma.

LEMMA 3.1. *Let $\langle F_n \rangle$ be a sequence of bounded measurable functions on $(C[0,t], \mathcal{B}(C[0,t]))$. If $\sum_{n=1}^{\infty} F_n$ converges to F unconditionally in the uniform convergence topology. Then*

$$(3.1) \qquad \sum_{n=1}^{\infty} (Ba) - \int_{C[0,t]} F_n(x)\, dV_\varphi(x)$$

converges to

$$(3.2) \qquad (Ba)-\int_{C[0,t]} F(x)\, dV_\varphi(x)$$

in the total variation norm sense.

Proof. Letting $T(F) = (Ba) - \int_{C[0,t]} F(x)\, dV_\varphi(x)$ for bounded measurable function F on $(C[0,t], \mathcal{B}(C[0,t]))$, since V_φ is bounded variation by Remark 4.1 (4) of [21], T is absolutely summing operator, so we have our result. □

Let $\eta = \mu + \nu$ be a complex-valued Borel measure such that μ is the continuous part of η and $\nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$, where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = t$ and c_p $(p = 0, 1, \ldots, n)$ are complex numbers.

Using the parallel method as in the proof of Lemma 5.3, Theorem 5.4 and the proof of Theorem 5.5 in [21], we have following theorem.

THEOREM 3.2. *Let g be an analytic function with the radius of convergence $\| \theta \|_{\varphi;\infty,1;\eta}$, say $g(z) = \sum_{m=0}^{\infty} a_m z^m$. Then $(Ba) - \int_{C[0,t]} g(\theta(s, x(s)))\, d\eta(s)$ is Bartle integrable and for E in $\mathcal{B}(\mathbb{R})$,*

$$\left[(Ba)-\int_{C[0,t]} g\left(\int_{[0,t]} \theta(s,x(s))\, d\eta(s)\right) dV_\varphi(x)\right](E)$$

$$(3.3) \qquad = \sum_{m=0}^{\infty} a_m m! \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_n=q_{n+1}} \int_{\triangle_{q_{n+1};j_1,\ldots,j_n}} [(L_n \circ L_{n-1} \circ \cdots \circ L_1)(T(s_{1,1}, \varphi, \theta(0,\cdot)^{q_0}))](E)\, d\left(\prod_{i=1}^{n}\prod_{j=1}^{j_i} \mu\right)(s_{1,1}, \ldots, s_{n,j_n}) .$$

Here, for $k = 2, 3, \ldots, n$,

$$L_k = M_{\theta(\tau_k)^{q_k}} \circ S_{\tau_k - s_{k,j_k}} \circ M_{\theta(s_{k,j_k})} \circ S_{s_{k,j_k}-s_{k,j_k-1}} \circ \cdots \circ M_{\theta(s_{k,1})} \circ S_{s_{k,1}-s_{k,0}}$$

and

$$L_1 = M_{\theta(\tau_1)^{q_1}} \circ S_{\tau_1 - s_{1,j_1}} \circ M_{\theta(s_{1,j_1})} \circ S_{s_{1,j_1}-s_{1,j_1-1}} \circ \cdots \circ M_{\theta(s_{1,1})}.$$

Moreover,

$$\text{(3.4)}\qquad \begin{aligned}&\left|(Ba)-\int_{C[0,t]} g\left(\int_{[0,t]}\theta(s,x(s))\,d\eta(s)\right)\,dV_\varphi(x)\right|(\mathbb{R})\\ &\leq 4|\varphi|(\mathbb{R})[\sum_{m=0}^{\infty}|a_m|\parallel\theta\parallel^m_{\varphi;\infty,1;\eta}]\,.\end{aligned}$$

Proof. From Theorem 2.18, we have $(Ba)-\int_{C[0,t]} K\,dV_\varphi(x) = KS_t(\varphi)$ for any constant K. Let m be a natural number. By Theorem 2.3, Lemma 5.2, Lemma 5.3, Theorem 5.4 and Theorem 5.5 in [21], $\left(\int_{[0,t]}\theta(s,x(s))\,d\eta(s)\right)^m$ is Bartle integrable with respect to V_φ and

$$\text{(3.5)}\qquad \begin{aligned}&\left|(Ba)-\int_{C[0,t]}\left(\int_{[0,t]}\theta(s,x(s))\,d\eta(s)\right)^m\,dV_\varphi(x)\right|(\mathbb{R})\\ &\leq 4|\varphi|(\mathbb{R})[\parallel\theta\parallel^m_{\varphi;\infty,1;\eta}]\,.\end{aligned}$$

Since the series g has the radius of convergence greater than $\parallel\theta\parallel_{\varphi;\infty,1;\eta}$, the series

$$\text{(3.6)}\qquad \sum_{m=0}^{\infty}|a_m|\parallel\theta\parallel^m_{\varphi;\infty,1;\eta}$$

converges absolutely. Since every absolutely convergent series is unconditionally convergent series, by Lemma 3.1, we have equality (3.3). The proof of rest part is trivial. □

In [21], we established the measure-valued Feynman-Kac formula, a kind of extension of the classical Feynman-Kac formula and showed that it satisfies a Volterra-type integral equation. Here, setting $g_k(u) = \sum_{m=k}^{\infty}\frac{1}{m!}u^m$ for any natural number k, $\int_{C[0,t]} g_k\left(\int_{[0,t]}\theta(s,x(s))\,d\eta(s)\right)$ $dV_\varphi(x)$ satisfies a Volterra-type integral equation. In the other word, the solution of Volterra's integral equation is unique in the usual case but in our case, there are solutions of a Volterra-type integral equation.

THEOREM 3.3. *Let $0=\tau_0<\tau_1<\tau_2<\cdots<\tau_n<t<\tilde{t}$ and let η be a Borel measure on $[0,\tilde{t}]$ such that $\eta=\mu+\nu$, where μ is the continuous part of η and $\nu=\sum_{p=0}^{n}c_p\delta_{\tau_p}$. Furthermore, let θ be in $L^{\tilde{t}}_{\varphi;\infty,1;\eta}$ and let*

$$\text{(3.7)}\qquad u(t') = (Ba)-\int_{C[0,t']} g_k\left\{\int_{[0,t']}\theta(s,x(s))\,d\eta(s)\right\}\,dV_\varphi(x)$$

for $t < t' \leq \widetilde{t}$. For $t < t' \leq \widetilde{t}$, $u(t')$ satisfies a Volterra-type integral equation, that is,

$$u(t') = S_{t'-t}(u(t)) + (Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)}) u(s) \, d\mu(s). \tag{3.8}$$

Proof. Using the parallel method as in the proof of Theorem 6.1 in [21], so, we give a sketch of proof of our theorem. By Corollary 5.8 [21], for $t < s \leq \widetilde{t}$,

$$\begin{aligned} u(s) &= \sum_{m=k}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_{n+1}=q_{n+1}} (Bo) \\ & \quad - \int_{\triangle^{(s)}_{q_{n+1};j_1,\ldots,j_{n+1}}} [S_{s-s_{n+1,j_{n+1}}} \circ M_{\theta(s_{n+1,j_{n+1}})} \\ & \quad \circ \cdots \circ S_{s_{n+1,1}-s_{n+1,0}} \circ L_n \circ \cdots \circ L_1](T(s_{1,1}, \varphi, \theta(0,\cdot)^{q_0})) \\ & \quad d\Big(\prod_{i=1}^{n+1} \prod_{j=1}^{j_i} \mu\Big)(s_{1,1}, \ldots, s_{n+1,j_{n+1}}), \end{aligned} \tag{3.9}$$

where $\triangle^{(s)}_{q_{n+1};j_1,\ldots,j_{n+1}} = \{(s_{1,1}, \ldots, s_{n+1,j_{n+1}}) \mid 0 = s_{0,0} = \tau_0 < s_{1,1} < \cdots < s_{1,j_1} < \tau_1 < s_{2,1} < \cdots < \tau_n < s_{n+1,1} < \cdots < s_{n+1,j_{n+1}} < s \}$ and for $k = 1, 2, \ldots, n$, L_k is given in Theorem 5.5 [21]. For $t < s \leq \widetilde{t}$, let

$$\begin{aligned} & Y(s; q_0, \ldots, q_{n+1}; j_1, \ldots, j_{n+1}) \\ = \; & (Bo) - \int_{\triangle^{(s)}_{q_{n+1};j_1,\ldots,j_{n+1}}} [S_{s-s_{n+1,j_{n+1}}} \circ M_{\theta(s_{n+1,j_{n+1}})} \circ \cdots \\ & \circ S_{s_{n+1,1}-s_{n+1,0}} \circ L_n \circ \cdots \circ L_1] \\ & (T(s_{1,1}, \varphi, \theta(0,\cdot)^{q_0})) \, d\Big(\prod_{i=1}^{n+1} \prod_{j=1}^{j_i} \mu\Big)(s_{1,1}, \ldots, s_{n+1,j_{n+1}}). \end{aligned} \tag{3.10}$$

For $t < t' \leq \widetilde{t}$, let

$$u_1(t') = \sum_{m=k}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} Y(t'; q_0, \ldots, q_n, 0; 0, 0, \ldots, 0),$$

$$\begin{aligned} u_2(t') &= \sum_{m=k}^{\infty} \sum_{\substack{q_0+\cdots+q_{n+1}=m \\ q_{n+1}\geq 1}} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_n=q_{n+1}} \\ &\qquad Y(t';q_0,\ldots,q_n,q_{n+1};j_1,\ldots,j_n,0), \\ u_3(t') &= \sum_{m=k+1}^{\infty} \sum_{\substack{q_0+\cdots+q_{n+1}=m \\ q_{n+1}\geq 1}} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{\substack{j_1+\cdots+j_{n+1}=q_{n+1} \\ j_{n+1}\geq 1}} \\ &\qquad Y(t';q_0,\ldots,q_{n+1};j_1,\ldots,j_{n+1}). \end{aligned}$$

Then

$$\begin{aligned} &u_1(t') + u_2(t') \\ (3.11) \qquad &= S_{t'-t}\Bigg(\sum_{m=k}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \\ &\qquad \times \sum_{j_1+\cdots+j_n=q_{n+1}} Y(t';q_0,\ldots,q_n,q_{n+1};j_1,\ldots,j_n)\Bigg) \\ &= S_{t'-t}(u(t)). \end{aligned}$$

And

$$\begin{aligned} (3.12) \qquad &(Bo)-\int_{(t,t']} (S_{t'-s}\circ M_{\theta(s)})(u(s))\, d\mu(s) \\ &\overset{(1)}{=} \sum_{m=k}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_{n+1}=q_{n+1}} (Bo) \\ &\quad - \int_{(t,t']} (S_{t'-s}\circ M_{\theta(s)})(Y(s;q_0,\ldots,q_{n+1};j_1,\ldots,j_{n+1}))\, d\mu(s) \\ &\overset{(2)}{=} \sum_{m=k}^{\infty} \sum_{q_0+\cdots+q_{n+1}=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} \sum_{j_1+\cdots+j_{n+1}=q_{n+1}} \\ &\quad Y(t';q_0,\ldots,q_{n+1}+1;j_1,\ldots,j_{n+1}+1) \\ &\overset{(3)}{=} \sum_{m'=k+1}^{\infty} \sum_{q_0'+q_1'+\cdots+q_{n+1}'=m'} \frac{\prod_{p=0}^{n} c_p^{q_p'}}{\prod_{p=0}^{n} q_p'!} \sum_{j_1'+\cdots+j_{n+1}'=q_{n+1}'} \\ &\quad Y(t';q_0',\ldots,q_{n+1}';j_1',\ldots,j_{n+1}') \\ &= u_3(t')\ . \end{aligned}$$

Step (1) follows from (3.3). By an elementary calculation, we have Step (2). If $q_{n+1} \geq k+1$ then the condition "$m = k$" has no meaning, so by making the substitution ; $j_i = j_i'$ $(1 \leq i \leq n)$, $j_{n+1} + 1 = j_{n+1}'$, $q_i = q_i'$ $(1 \leq i \leq n)$, $q_{n+1} + 1 = q_{n+1}'$ and $m + 1 = m'$, we obtain Step (3).

Hence, for $t < t' \leq \widetilde{t}$,

$$\begin{aligned} & u(t') \\ & = (u_1(t') + u_2(t')) + u_3(t') \\ & = S_{t'-t}(u(t)) + (Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s))\ d\mu(s)\ , \end{aligned} \tag{3.13}$$

as desired. □

COROLLARY 3.4. *Under the assumptions in Theorem 3.3 and we assume that* $\eta = \mu$, *an arbitrary continuous measure on* $[0, t]$, *for* $0 < t' \leq \widetilde{t}$, $u(t')$ *satisfies a Volterra integral equation, that is,*

$$u(t') = (Bo) - \int_{(0,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s))\ d\mu(s)\ . \tag{3.14}$$

COROLLARY 3.5. *Under the assumptions in Theorem 3.3 and we assume that* $\eta = \nu = \sum_{p=0}^{n} c_p \delta_{\tau_p}$, *a discrete measure on* $[0, t]$, *for* $0 < t' \leq \widetilde{t}$,

$$\begin{aligned} u(t') \ = \ & \sum_{m=k}^{\infty} \sum_{q_0+\cdots+q_n=m} \frac{\prod_{p=0}^{n} c_p^{q_p}}{\prod_{p=0}^{n} q_p!} [S_{t'-t} \circ M_{\theta(\tau_n)^{q_n}} \circ S_{\tau_n-\tau_{n-1}} \circ \cdots \\ & \circ S_{\tau_2-\tau_1} \circ M_{\theta(\tau_1)^{q_1}}](T(\tau_1, \varphi, \theta(0,\cdot)^{q_0})), \end{aligned} \tag{3.15}$$

$$u(t') \ = \ S_{t'-t}(u(t)) \tag{3.16}$$

and

$$\begin{aligned} & (Bo) - \int_{(t,t']} (S_{t'-s} \circ M_{\theta(s)})(u(s))\ d\mu(s) \\ & = 0, \ \textit{the zero operator.} \end{aligned} \tag{3.17}$$

4. A new formula for the Bartle integral with respect to a vector measure V_φ

In this section, we will show that the Bartle integral with respect to V_φ can be written as an iterated integral with respect to a complex-valued measure. From this, we recognize the relation between the Bartle integral and the conditional ω_φ-integral (see [22]).

THEOREM 4.1. *Let φ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let f be bounded measurable on $(C[0,t], \mathcal{B}(C[0,t]))$. Then*

$$\begin{aligned}
&\big[(Ba) - \int_{C[0,t]} f(x)\ dV_\varphi(x)\big](E) \\
&= \int_E E^{\omega_\varphi}(f|X)(\xi)\ dP_X(\xi) \\
&= \int_E E^{\omega_\varphi}(f|X)(\xi)\frac{dP_X}{dm_L}(\xi)\ dm_L(\xi) \\
&= \frac{1}{2\pi}\int_E\int_{\mathbb{R}} e^{-i\xi u}\int_{C[0,t]} e^{iux(t)} f(x)\ d\omega_\varphi(x)\ dm_L(u)\ dm_L(\xi)
\end{aligned} \tag{4.1}$$

for E in $\mathcal{B}(\mathbb{R})$.

Proof. Let f be bounded measurable on $(C[0,t], \mathcal{B}(C[0,t]))$. Then there is an increasing sequence $\langle s_n\rangle$ of simple functions on $(C[0,t], \mathcal{B}(C[0,t]))$ such that $\langle s_n\rangle$ converges to f. By the basic properties of Bartle integral and Lebesgue integral, we have

$$\begin{aligned}
&\big[(Ba) - \int_{C[0,t]} s_n(x)\ dV_\varphi(x)\big](E) \\
&= \int_E E^{\omega_\varphi}(s_n|X)(\xi)\ dm_L(\xi) \\
&= \frac{1}{2\pi}\int_E\int_{\mathbb{R}} e^{-i\xi u}\int_{C[0,t]} e^{iux(t)} s_n(x)\ d\omega_\varphi(x)\ dm_L(u)\ dm_L(\xi)
\end{aligned} \tag{4.2}$$

for E in $\mathcal{B}(\mathbb{R})$. Hence for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
&\big[(Ba) - \int_{C[0,t]} f(x)\ dV_\varphi(x)\big](E) \\
&\overset{(1)}{=} \big[\lim_{n\to\infty}\int_{C[0,t]} s_n(x)\ dV_\varphi(x)\big](E) \\
&\overset{(2)}{=} \lim_{n\to\infty}\big[\int_{C[0,t]} s_n(x)\ dV_\varphi(x)\big](E)
\end{aligned} \tag{4.3}$$

$$\overset{(3)}{=}\lim_{n\to\infty}\frac{1}{2\pi}\int_E\int_{\mathbb{R}}e^{-i\xi u}\int_{C[0,t]}e^{iux(t)}s_n(x)\,d\omega_\varphi(x)\,dm_L(u)\,dm_L(\xi)$$

$$\overset{(4)}{=}\frac{1}{2\pi}\int_E\lim_{n\to\infty}\Big(\int_{\mathbb{R}}e^{-i\xi u}\int_{C[0,t]}e^{iux(t)}s_n(x)\,d\omega_\varphi(x)\,dm_L(u)\Big)\,dm_L(\xi)$$

$$\overset{(5)}{=}\frac{1}{2\pi}\int_E\int_{\mathbb{R}}e^{-i\xi u}\Big(\lim_{n\to\infty}\int_{C[0,t]}e^{-iux(t)}s_n(x)\,d\omega_\varphi(x)\Big)\,dm_L(u)\,dm_L(\xi)$$

$$\overset{(6)}{=}\frac{1}{2\pi}\int_E\int_{\mathbb{R}}e^{-i\xi u}\int_{C[0,t]}\lim_{n\to\infty}e^{iux(t)}s_n(x)\,d\omega_\varphi(x)\,dm_L(u)\,dm_L(\xi)$$

$$\overset{(7)}{=}\frac{1}{2\pi}\int_E\int_{\mathbb{R}}e^{-i\xi u}\int_{C[0,t]}e^{iux(t)}f(x)\,d\omega_\varphi(x)\,dm_L(u)\,dm_L(\xi).$$

Step (1) results from the dominated convergence theorem for Bartle integral. Step (2) follows from the Vitali-Hahn-Saks theorem. From the above equality (4.2), we obtain Step (3). By Theorem 2.6, we have Step (4). Step (5) is true because the Fourier transform from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$, the space of all continuous function on $\mathbb{R}$ which vanish zero as approach to $\pm\infty$, is a bounded operator. Step (6) holds by the dominated convergence theorem for Lebesgue integral and Step (7) is trivial. By Lemma 2.3, we obtain the equalities (4.1). □

For a non-negative finite real valued measure in $\mathcal{M}(\mathbb{R})$, let φ^N be a normalized measure of φ, that is, $\varphi^N(E) = \frac{\varphi(E)}{|\varphi|(\mathbb{R})}$ for E in $\mathcal{B}(\mathbb{R})$ if φ is a non-zero measure and φ^N is a zero measure if φ is a zero measure. For φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^4 \alpha_j\varphi_j$, $\omega_\varphi = \sum_{j=1}^4 \alpha_j\omega_{\varphi_j}$ and for $j = 1,2,3,4$, $\omega_{\varphi_j} = |\varphi_j|(\mathbb{R})\varphi_j^N$. Hence, for φ in $\mathcal{M}(\mathbb{R})$ with the Jordan decomposition $\varphi = \sum_{j=1}^4 \alpha_j\varphi_j$, for B in $\mathcal{B}(C[0,t])$ and for E in $\mathcal{B}(\mathbb{R})$,

$$\begin{aligned}\big[V_\varphi(B)\big](E) &= \sum_{j=1}^4 \alpha_j\omega_{\varphi_j}(B\cap J_t^{-1}(E))\\ &= \sum_{j=1}^4 \alpha_j|\varphi_j|(\mathbb{R})\omega_{\varphi_j^N}(B\cap J_t^{-1}(E))\\ &= \Big[\sum_{j=1}^4 \alpha_j V_{\varphi_j}(B)\Big](E).\end{aligned}\tag{4.4}$$

THEOREM 4.2. *Let φ in $\mathcal{M}(\mathbb{R})$ and for a bounded measurable function f on $(C[0,t],\mathcal{B}(C[0,t]))$. Then*

$$
\begin{aligned}
&\big[(Ba) - \int_{C[0,t]} f(x)\ dV_\varphi(x)\big](E) \\
&= \int_E E^{\omega_\varphi}(f|X)(\xi)\ dP_X(\xi) \\
&= \int_E E^{\omega_\varphi}(f|X)(\xi) \frac{dP_X}{dm_L}(\xi)\ dm_L(\xi) \\
&= \frac{1}{2\pi} \int_E \int_{\mathbb{R}} e^{-i\xi u} \int_{C[0,t]} e^{iux(t)} f(x)\ d\omega_\varphi(x)\ dm_L(u)\ dm_L(\xi)
\end{aligned} \tag{4.5}
$$

for E in $\mathcal{B}(\mathbb{R})$.

Proof. From Theorem 4.1, we have

$$
\begin{aligned}
&\big[(Ba) - \int_{C[0,t]} f(x)\ dV_\varphi(x)\big](E) \\
&= \big[\sum_{j=1}^{4} \alpha_j |\varphi_j|(\mathbb{R})(Ba) - \int_{C[0,t]} f(x)\ dV_{\varphi_j^N}(x)\big](E) \\
&= \sum_{j=1}^{4} \alpha_j |\varphi_j|(\mathbb{R}) \\
&\quad \times \frac{1}{2\pi} \int_E \int_{\mathbb{R}} e^{-i\xi u} \int_{C[0,t]} e^{iux(t)} f(x) d\omega_{\varphi_j^N}(x) dm_L(u) dm_L(\xi) \\
&= \frac{1}{2\pi} \int_E \int_{\mathbb{R}} e^{-i\xi u} \int_{C[0,t]} e^{iux(t)} f(x)\ d\omega_\varphi(x)\ dm_L(u)\ dm_L(\xi),
\end{aligned} \tag{4.6}
$$

as desired. The rest part of the proof is trivial from Lemma 2.3. □

REMARK 4.3. Putting $\varphi = \delta_0$, $\omega_\varphi = \omega$, the classical Wiener measure and

$$
\begin{aligned}
&\big[(Ba) - \int_{C[0,t]} f(x)\ dV_\varphi(x)\big](E) \\
&= \frac{1}{2\pi} \int_E \int_{\mathbb{R}} e^{-i\xi u} \int_{C[0,t]} e^{iux(t)} f(x)\ d\omega_\varphi(x)\ dm_L(u)\ dm_L(\xi) \\
&= \int_E E^{\omega_\varphi}(f|X)(\xi)\ dm_L(\xi) \\
&= \int_{X^{-1}(E)} f(x)\ d\omega(x).
\end{aligned} \tag{4.7}
$$

Here f is a bounded measurable function and E is in $\mathcal{B}(\mathbb{R})$.

5. The stability theorems

In this section, we will concern with the stability theorems for measure valued Dyson series, given in Section 3.

5.1. The stability theorem for a convergent sequence $\langle\varphi_n\rangle$ in the total variation norm sense

In this part, we will concern with the stability theorem for total variation norm convergent sequence $\langle\varphi_n\rangle$.

THEOREM 5.1. *Let $\langle\varphi_n\rangle$ be a sequence in $\mathcal{M}(\mathbb{R})$ which converges to φ in the total variation norm sense. If F is bounded measurable on $C[0,t]$ then $\langle\int_{C[0,t]} F(x)\,dV_{\varphi_n}(x)\rangle$ converges to $\int_{C[0,t]} F(x)\,dV_\varphi(x)$ in the total variation norm sense.*

Proof. Let F be bounded by K. Then

$$(5.1)\qquad \begin{aligned} &\left|\int_{C[0,t]} F(x)\,dV_{\varphi_n}(x) - \int_{C[0,t]} F(x)\,dV_\varphi(x)\right| \\ \leq\ & K \parallel V_{\varphi_n} - V_\varphi \parallel \\ =\ & K \parallel V_{\varphi_n - \varphi} \parallel \\ \leq\ & 4K|\varphi_n - \varphi| \to 0. \end{aligned}$$

as $n \to \infty$, so, the proof is finished. □

THEOREM 5.2. *Let $\langle\varphi_n\rangle$ be a sequence in $\mathcal{M}(\mathbb{R})$ which converges to φ in the total variation norm sense. Let θ be in $\cap_{n=1}^\infty L_{\varphi_n;\infty,1;\eta}$ such that $\langle\parallel \theta(0,\cdot) \parallel_{\varphi_n;\infty}\rangle$ is bounded. Then θ belongs to $L_{\varphi;\infty,1;\eta}$.*

Proof. We let $A_\lambda = \{\xi \text{ in } \mathbb{R} \,|\, |\theta(0,\xi)| > \lambda\}$ for each positive real number λ. For each natural number n, let $B_{n,\lambda} = \{\lambda > 0 \,|\, |\varphi_n|(A_\lambda) = 0\}$ and let $B_\lambda = \{\lambda > 0 \,|\, |\varphi|(A_\lambda) = 0\}$. We suppose that λ_0 is a positive real number and $\langle n_u\rangle$ is a subsequence of $\langle n\rangle$ such that $|\varphi_{n_u}|(A_{\lambda_0}) = 0$ for all u in $\mathbb{N}$. Then since $\langle|\varphi_n|(E)\rangle$ converges to $|\varphi|(E)$ for all E in $\mathcal{B}(\mathbb{R})$, $|\varphi|(A_\lambda) = 0$ which implies that $\cup_{n=1}^\infty \cap_{k=n}^\infty B_k$ is a subset of B. Now, we assume that $\inf(\cup_{n=1}^\infty \cap_{k=n}^\infty B_k) > \lim_{n\to\infty} \inf \parallel \theta(0,\cdot) \parallel_{\varphi_n;\infty}$. We can pick v such that $\lim_{n\to\infty} \inf \parallel \theta(0,\cdot) \parallel_{\varphi_n;\infty} < v < \inf(\cup_{n=1}^\infty \cap_{k=n}^\infty B_k)$. Then there is a convergent subsequence $\langle\parallel \theta(0,\cdot) \parallel_{\varphi_{n_u};\infty}\rangle$ of $\langle\parallel \theta(0,\cdot) \parallel_{\varphi_n;\infty}\rangle$ such that $\lim_{u\to\infty} \parallel \theta(0,\cdot) \parallel_{\varphi_{n_u};\infty} = \lim_{n\to\infty} \inf \parallel \theta(0,\cdot) \parallel_{\varphi_n;\infty}$ and $\parallel \theta(0,\cdot) \parallel_{\varphi_{n_u}} < v$ for all u in $\mathbb{N}$. So, we obtain $\inf_{\lambda>0} B_{n_u,\lambda} < v$ for all u in $\mathbb{N}$. Since $B_{n,\lambda}$ is an interval, there is a positive real number $\lambda_1 < v$ such that λ_1 is belong to

$\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k$, which is contradicted by $v < \inf(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k)$. Hence we have $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k \leq \lim_{n\to\infty} \inf \| \theta(0,\cdot) \|_{\varphi_n;\infty}$. Therefore,

$$
\begin{aligned}
&\| \theta \|_{\varphi;\infty,1;\eta} \\
&= \int_{[0,t]} \| \theta(s,\cdot) \|_{\varphi;\infty} \; d|\eta|(s) \\
&= \| \theta(0,\cdot) \|_{\varphi;\infty} \; |\eta|(\{0\}) \int_{(0,t]} \| \theta(s,\cdot) \|_{\varphi;\infty} \; d|\eta|(s) \\
&\leq \big(\inf_{\lambda>0} B_\lambda\big) \; |\eta|(\{0\}) \int_{(0,t]} \| \theta(s,\cdot) \|_{m_L;\infty} \; d|\eta|(s) \\
&\leq \big(\lim_{n\to\infty} \inf \| \theta(0,\cdot) \|_{\varphi_n;\infty} \big) \; |\eta|(\{0\}) \\
&\times \int_{(0,t]} \| \theta(s,\cdot) \|_{m_L;\infty} \; d|\eta|(s) < +\infty,
\end{aligned}
\tag{5.2}
$$

as desired. □

THEOREM 5.3. (The stability theorem for convergence of $\langle \varphi_n \rangle$ in the total variation norm sense) *Under the assumptions of Theorem 5.2, let g be analytic with a radius of convergence less than $K = 2\sup(\{\| \theta \|_{\varphi_n;\infty,1;\eta} \, | n \in \mathbb{N}\} \cup \{\| \theta \|_{\varphi;\infty,1;\eta}\})$. Then the sequence $\langle (Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s))dV_{\varphi_n}(x) \rangle$ of measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, converges to a measure $(Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s))dV_\varphi(x)$ in the total variation norm sense.*

Proof. From Theorem 5.1 in [21, p.4938],

$$\Big| g\Big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\Big) \Big| \leq \sum_{n=0}^{\infty} |a_n| \big(\| \theta \|_{\varphi;\infty,1;\eta} \big)^n$$

for $|\omega_\varphi|$-a.e. x and for any natural number k,

$$\Big| g\Big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\Big) \Big| \leq \sum_{n=0}^{\infty} |a_n| \big(\| \theta \|_{\varphi_k;\infty,1;\eta} \big)^n$$

for $|\omega_{\varphi_k}|$-a.e. x. Since $\| V_\varphi \| (B) \leq |\omega_\varphi|(B)$ for B in $\mathcal{B}(C[0,t])$, by Theorem 5.1, the Bartle integral $(Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s))dV_\varphi(x)$ and $(Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s, x(s))d\eta(s))dV_{\varphi_k}(x)$, $k \in \mathbb{N}$ exist. Clearly, $\| \theta(s,\cdot) \|_{\varphi_k-\varphi;\infty} \leq \| \theta(s,\cdot) \|_{\varphi_k;\infty} + \| \theta(s,\cdot) \|_{\varphi;\infty}$, $\| \theta \|_{\varphi_k-\varphi;\infty,1;\eta} \leq$

$\| \theta \|_{\varphi_k;\infty,1;\eta} + \| \theta \|_{\varphi;\infty,1;\eta} \le K$. So from Theorem 5.1,

$$\begin{aligned}
&\Big|(Ba) - \int_{C[0,t]} g\Big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\Big)dV_{\varphi_k}(x) \\
&\qquad - (Ba) - \int_{C[0,t]} g\Big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\Big)dV_{\varphi}(x)\Big|(\mathbb{R}) \\
&= \Big|(Ba) - \int_{C[0,t]} g\Big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\Big)dV_{\varphi_k-\varphi}(x)\Big|(\mathbb{R}) \qquad (5.3)\\
&\le 4\,|\varphi_k - \varphi|(\mathbb{R}) \sum_{n=0}^{\infty} |a_n|(\| \theta \|_{\varphi_k-\varphi;\infty,1;\eta})^n \\
&\le 4\,|\varphi_k - \varphi|(\mathbb{R}) \sum_{n=0}^{\infty} |a_n| K^n \to 0,
\end{aligned}$$

as $k \to +\infty$, as desired. □

From Theorem 5.3, directly we have the following corollary.

COROLLARY 5.4. *Suppose $\varphi = \varphi^c + \varphi^d$ where φ^c is the continuous part of φ and φ^d is the discrete part of φ. For each natural number n, we let $\varphi_n = \varphi^c + \sum_{p=1}^{n} c_p \delta_{\tau_p}$. Let θ be in $L_{\varphi;\infty,1;\eta}$ and let g be analytic having a radius of convergence less than $2 \| \theta \|_{\varphi;\infty,1;\eta}$, say $g(u) = \sum_{m=0}^{\infty} a_m u^m$. Then the sequence $\langle (Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s)) dV_{\varphi_n}(x)\rangle$ of measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, converges to $(Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s, x(s))d\eta(s))dV_{\varphi}(x)$ in the total variation norm sense. Moreover, for each natural number n,*

$$\begin{aligned}
&\Big|(Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s))dV_{\varphi_n}(x) \\
&\qquad - (Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s))dV_{\varphi}(x)\Big|(\mathbb{R}) \qquad (5.4)\\
&\le 4\Big(\sum_{p=n+1}^{\infty} |c_p|(\| \theta \|_{\varphi;\infty,1;\eta})^p\Big)\Big(\sum_{n=0}^{\infty} |a_n|(\| \theta \|_{\varphi;\infty,1;\eta})^n\Big).
\end{aligned}$$

5.2. The stability theorem for convergence of $\langle \varphi_n \rangle$ weakly

In this part, we will establish the stability theorem for weakly convergent sequence $\langle \varphi_n \rangle$.

THEOREM 5.5. *Let F be a bounded continuous on $C[0,t]$. We suppose $\langle \varphi_n \rangle$ is tight and is a weakly convergent sequence $\langle \varphi_n \rangle$ to φ. For* *a closed set E in $\mathbb{R}$ such that for the closed continuity set E, $\langle\big[(Ba)-\int_{C[0,t]}F(x)\ dV_{\varphi_n}(x)\big](E)\rangle$ converges to $\big[(Ba)-\int_{C[0,t]}F(x)\ dV_{\varphi}(x)\big](E)$.*

Proof. Let $b(B)$ the set of all boundary points of B for B in $\mathcal{B}(\mathbb{R})$. Let E be a closed continuity set. Then $X^{-1}(E)$ is closed and $X^{-1}(E-b(E))$ is open in $C[0,t]$. Since boundary points of $X^{-1}(E)$ contained in $X^{-1}(E-b(B))$, $X^{-1}(E)$ is a closed continuity set. From Theorem 2.17, we have $\langle\omega_{\varphi_n}\rangle$ converges to ω_φ weakly. Since $E-b(B)$ is m_L-null, $\omega_\varphi(X^{-1}(E-b(B)))=0$. Hence, by Helly's theorem [1, p.81],

$$\lim_{n\to\infty}\int_{X^{-1}(E)}F(x)d\omega_{\varphi_n}(x)=\int_{X^{-1}(E)}F(x)d\omega_{\varphi}(x).$$

By Theorem 4.1, $\langle\big[(Ba)-\int_{C[0,t]}F(x)\ dV_{\varphi_n}(x)\big](E)\rangle$ converges to $\big[(Ba)-\int_{C[0,t]}F(x)\ dV_{\varphi}(x)\big](E)$. □

The following corollary follows from Theorem 5.5.

COROLLARY 5.6. (The stability theorem for weakly convergent sequence $\langle\varphi_n\rangle$) *We suppose $\langle\varphi_n\rangle$ is tight and is a weakly convergent sequence $\langle\varphi_n\rangle$ to φ. Let θ be bounded continuous on $[0,t]\times C[0,t]$ and let g be an analytic function with the norm the radius of convergence $\|\ \theta\ \|_{\varphi;\infty,1;\eta}$*, say $g(z)=\sum_{m=0}^{\infty}a_m z^m$. *Then for each closed continuity set E of φ, $\langle\Big[(Ba)-\int_{C[0,t]}g\big(\int_{[0,t]}\theta(s,x(s))d\eta(s)\big)dV_{\varphi_n}(x)\Big](E)\rangle$ converges to $\Big[(Ba)-\int_{C[0,t]}g(\int_{[0,t]}\theta(s,x(s))d\eta(s))dV_{\varphi}(x)\Big](E)$.*

Using Theorem 4.2 and Theorem 5.5, we have the following corollary.

COROLLARY 5.7. (The bounded convergence theorem) *We suppose $\langle\varphi_n\rangle$ is tight and is a weakly convergent sequence $\langle\varphi_n\rangle$ to φ. For a closed continuity subset E of $\mathbb{R}$ for φ, and for a bounded continuous function f on $C[0,t]$,*

$$\begin{aligned}&\lim_{n\to\infty}\int_E\int_{\mathbb{R}}e^{-i\xi u}\int_{C[0,t]}e^{iux(t)}f(x)d\omega_{\varphi_n}(x)dm_L(u)dm_L(\xi)\\ &=\int_E\int_{\mathbb{R}}e^{-i\xi u}\int_{C[0,t]}e^{iux(t)}f(x)d\omega_{\varphi}(x)dm_L(u)dm_L(\xi).\end{aligned}$$

5.3. The stability theorem for convergent sequence $\langle \eta_n \rangle$ in the total variation norm sense

In this part, we will establish the stability theorem for convergent sequence $\langle \eta_n \rangle$ of measures on $([0,t], \mathcal{B}([0,t]))$ in the total variation norm sense.

THEOREM 5.8. *Let $\langle \eta_n \rangle$ be a sequence of measures on $([0,t], \mathcal{B}([0,t]))$ in the total variation norm sense. Let θ be a bounded Borel measurable function on $[0,t]$, bounded by K and let g be analytic having a disk of convergence containing $\{z : |z| \le 2K \sup\{|\eta|(\mathbb{R}) : n \in \mathbb{N}\}\}$. Suppose that a sequence $\langle \eta_n \rangle$ of measures on $([0,t], \mathcal{B}([0,t]))$, converges to η in the total variation norm sense. Then the sequence $\langle (Ba)\text{-}\int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta_n(s))dV_\varphi(x)\rangle$ converges to*

$$(Ba) - \int_{C[0,t]} g(\int_{[0,t]} \theta(s,x(s))d\eta(s))dV_\varphi(x)$$

in the total variation norm sense.

Moreover, for a natural number n,

$$\begin{aligned}
&\Big|(Ba) - \int_{C[0,t]} g\Big(\int_{[0,t]} \theta(s,x(s))d\eta_n(s)\Big)dV_\varphi(x) \\
(5.5)\qquad &\quad - (Ba) - \int_{C[0,t]} g\Big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\Big)dV_\varphi(x)\Big|(\mathbb{R}) \\
&\le 4\, C_n|\varphi|(\mathbb{R}).
\end{aligned}$$

Here $C_n = \sup\{|g(z_1) - g(z_2)| : |z_1| \le K|\eta_n - \eta|([0,t])$ and $|z_2| \le K|\eta_n - \eta|([0,t])\}$

Proof. For $\| V_\varphi \|$-a.e. x and for a natural number n,

$$\begin{aligned}
&\Big|\int_{[0,t]} \theta(s,x(s))d\eta_n(s) - \int_{[0,t]} \theta(s,x(s))d\eta(s))\Big| \\
(5.6)\qquad &= \Big|\int_{[0,t]} \theta(s,x(s))d(\eta_n - \eta)(s)\Big| \\
&\le K\, |\eta_n - \eta|([0,t]),
\end{aligned}$$

so

$$\Big|g(\int_{[0,t]} \theta(s,x(s))d\eta_n(s)) - g(\int_{[0,t]} \theta(s,x(s))d\eta(s))\Big| \le C_n.$$

Therefore,

$$
\begin{aligned}
&\Big|(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta_n(s)\big)dV_\varphi(x) \\
(5.7)\qquad &- (Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\big)dV_\varphi(x)\Big|(\mathbb{R}) \\
&\le C_n \parallel V_\varphi \parallel (\mathbb{R}) \le 4\ C_n|\varphi|(\mathbb{R}).
\end{aligned}
$$

Since g is uniformly continuous in the closed disk $|z| \le K$ and $\lim_{n\to\infty} |\eta_n - \eta|([0,t]) = 0$, $\lim_{n\to\infty} C_n = 0$, as desired. □

COROLLARY 5.9. (The stability theorem for convergent sequence $\langle\eta_n\rangle$ in the total variation norm sense) *Under the assumption of Theorem 5.8, let $\eta = \mu + \nu$ be a complex Borel measure on $[0,t]$ such that μ is the continuous part of η and $\nu = \sum_{p=0}^{\infty} c_p\delta_{\tau_p}$. For a natural number n, let $\sigma_n : \{0,1,2,\dots,n\} \to \{0,1,2,\dots,n\}$ be a bijection with $0 = \tau_{\sigma(0)} < \tau_{\sigma(1)} < \cdots < \tau_{\sigma(n)} \le t$, let $\eta_n = \mu + \sum_{p=1}^{n} c_p\delta_{\tau_p}$ where $\langle c_p\rangle$ is summable. Then $\langle\eta_n\rangle$ converges to η in the total variation norm sense and $(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\big)\ dV_\varphi(x)$ and $(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta_n(s)\big)\ dV_\varphi(x)\quad (n \in \mathbb{N})$ exist.*

Moreover, $\langle(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta_n(s)\big)dV_\varphi(x)\rangle$ converges to $(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\big)dV_\varphi(x)$ in the total variation norm sense and

$$
\begin{aligned}
&\Big|(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta_n(s)\big)dV_\varphi(x) \\
(5.8)\qquad &- (Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s,x(s))d\eta(s)\big)dV_\varphi(x)\Big|(\mathbb{R}) \\
&\le 4\ C_n|\varphi|(\mathbb{R}).
\end{aligned}
$$

The sequence, given by right side of (3.3) from the changing p, τ_p and η to $\sigma_n(p)$, $\tau_{\sigma_n(p)}$ and η_n, respectively, converges to $(Ba) - \int_{C[0,t]} g\big(\int_{[0,t]} \theta(s, x(s))d\eta(s)\big)dV_\varphi(x)$, in the total variation norm sense.

5.4. The stability theorem for weakly convergent sequence $\langle\eta_n\rangle$

In this part, we prove the stability theorem for a weakly convergence sequence $\langle\eta_n\rangle$ of measures on $([0,t], \mathcal{B}([0,t]))$.

THEOREM 5.10. (The stability theorem for weakly convergent sequence $\langle\eta_n\rangle$) *Let $\langle\eta_n\rangle$ be a bounded sequence such that $\langle\eta_n\rangle$ converges* *to η weakly. Suppose $\langle|\eta_n|([0,t])\rangle$ is a bounded sequence. Let θ be a bounded continuous function on $[0,t]$, bounded by K on $[0,t]\times\mathbb{R}$. Let g be an analytic function. Then for $n\in\mathbb{N}$, the Bartle integral*

$$(5.9)\qquad (Ba)-\int_{C[0,t]} g\Big(\int_{[0,t]}\theta(s,x(s))d\eta(s)\Big)dV_{\varphi}(x)$$

and

$$(5.10)\qquad (Ba)-\int_{C[0,t]} g\Big(\int_{[0,t]}\theta(s,x(s))d\eta_n(s)\Big)dV_{\varphi}(x),$$

exist and the sequence $\langle(Ba)\text{-}\int_{C[0,t]} g(\int_{[0,t]}\theta(s,x(s))d\eta_n(s))dV_{\varphi}(x)\rangle$ of measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ converges to $(Ba)\text{-}\int_{C[0,t]} g(\int_{[0,t]}\theta(s,x(s))d\eta(s))$ $dV_{\varphi}(x)$ in the total variation norm sense.

Proof. We suppose $\sup\{|\eta_n|([0,t])|n\in\mathbb{N}\}\le M$. Then $|\int_{[0,t]}\theta(s,x(s))$ $d\eta_n(s)|\le K\ |\eta_n|([0,t])\le\ KM$. Setting $g(z)=\sum_{m=0}^{\infty}a_m z^m$, $|g(\int_{[0,t]}\theta(s,$ $x(s))d\eta_n(s))|\le\sum_{m=0}^{\infty}|a_m|\ (KM)^m$. Since $\int_{C[0,t]} c\, dV_{\varphi}(x)=c\, V_{\varphi}(C[0,t])=$ $cS_t(\varphi)$ for a constant c, the Bartle integrals (5.9) and (5.10), in our theorem, exist. By the dominated convergence theorem for Bartle integral, we obtain that a sequence $\langle(Ba)\text{-}\int_{C[0,t]} g(\int_{[0,t]}\theta(s,x(s))d\eta_n(s)\ )dV_{\varphi}(x)\rangle$ of measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, converges to $(Ba)\text{-}\int_{C[0,t]} g(\int_{[0,t]}\theta(s,x(s))d\eta(s))$ $dV_{\varphi}(x)$ in the total variation norm sense. □

5.5. The stability theorem for potential functions

In this part, we treat the stability theorem for potential functions.

THEOREM 5.10. (The stability theorem for potential function) *Let $\bar{\theta}$ be in $L_{\varphi;\infty,1;\eta}$ and let $\langle\theta_n\rangle$ be a sequence of Borel measurable functions on $[0,t]\times\mathbb{R}$ such that $\|\ \theta_n(\cdot,\cdot)\ \|_{\varphi;\infty}\le\|\ \bar{\theta}\ \|_{\varphi;\infty}$ on $[0,t]\times\mathbb{R}$ and $\langle\theta_n\rangle$ converge to θ. Let g be an analytic function having a radius of convergence less then $\|\ \bar{\theta}\ \|_{\varphi;\infty,1;\eta}$. Then θ and θ_n, $n\in\mathbb{N}$, belong to $L_{\varphi;\infty,1;\eta}$ and a sequence $\langle\int_{C[0,t]} g\ (\int_{[0,t]}\theta_n(s,x(s))\ d\eta(s))\ dV_{\varphi}\ (x)\rangle$ converges to $\int_{C[0,t]}\ g(\int_{[0,t]}\theta(s,x(s))\ d\eta(s))\ dV_{\varphi}(x)$ in the total variation norm sense.*

Proof. Clearly, θ and θ_n, $n\in\mathbb{N}$, belong to $L_{\varphi;\infty,1;\eta}$. By the dominated convergence theorem, $\lim_{n\to\infty}\int_{[0,t]}\theta_n(s,x(s))d\eta(s)=\int_{[0,t]}\theta(s,x(s))d\eta(s)$.

Since $g(z)$ is continuous in $|z| <\| \bar{\theta} \|_{\varphi;\infty,1;\eta}$,

$$(5.11)\qquad \begin{aligned} &\lim_{n\to\infty} g\Big(\int_{[0,t]} \theta_n(s,x(s))d\eta(s)\Big) \\ &= g\Big(\lim_{n\to\infty}\int_{[0,t]} \theta_n(s,x(s))d\eta(s)\Big). \end{aligned}$$

By the dominated convergence theorem for Bartle integral,

$$(5.12)\qquad \begin{aligned} &\lim_{n\to\infty}\int_{C[0,t]} g\bigg(\int_{[0,t]} \theta_n(s,x(s))d\eta(s)\bigg)dV_\varphi(x) \\ &= \int_{C[0,t]} \lim_{n\to\infty} g\bigg(\int_{[0,t]} \theta_n(s,x(s))d\eta(s)\bigg)dV_\varphi(x). \end{aligned}$$

as desired. □

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