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THE GROUPS OF SELF PAIR HOMOTOPY EQUIVALENCES

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Abstract. In this paper, we extend the concept of the group $\mathcal{E}(X)$ of self homotopy equivalences of a space X to that of an object in the category of pairs. Mainly, we study the group $\mathcal{E}(X, A)$ of pair homotopy equivalences from a CW-pair (X, A) to itself which is the special case of the extended concept. For a CW-pair (X, A), we find an exact sequence $1 \to G \to \mathcal{E}(X, A) \to \mathcal{E}(A)$ where G is a subgroup of $\mathcal{E}(X, A)$. Especially, for CW homotopy associative and inversive H-spaces X and Y, we obtain a split short exact sequence $1 \to \mathcal{E}(X) \to \mathcal{E}(X \times Y, Y) \to \mathcal{E}(Y) \to 1$ provided the two sets $[X \wedge Y, X \times Y]$ and $[X, Y]$ are trivial.

1. Introduction and preliminaries

Let X be a connected CW-complex with the base point $*$ and $\mathcal{E}(X)$ the set of homotopy classes of self homotopy equivalences of X. Then the set $\mathcal{E}(X)$ is a group with group operation given by composition of homotopy classes. This group has been studied by several authors. For instances, M. Arkowitz [1], K. Maruyama [4], S. Oka [5], J. Rutter [6], N. Sawashita [8], A. Sieradski [10] and K. Tsukiyama [12], et al..

It is a well-known fact that $\mathcal{E}(S^n) = \mathbb{Z}_2$ and $\mathcal{E}(K(\pi, n)) = \mathrm{Aut}(\pi)$ where $K(\pi, n)$ is an Eilenberg-Mclane space and $\mathrm{Aut}(\pi)$ is the group of automorphisms on π.

In this paper, we extend the concept of the group of self homotopy equivalences of a space to that of a map as an object in the category of pairs.

In the category of pairs, the "objects" are maps $(X_1, *) \to (X_2, *)$ and "morphism" from $\alpha : X_1 \to X_2$ to $\beta : Y_1 \to Y_2$ is a pair of maps

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(f_1, f_2) such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & X_2 \\ f_1\downarrow & & \downarrow f_2 \\ Y_1 & \xrightarrow[\beta]{} & Y_2 \end{array}$$

is commutative, i.e., $\beta f_1 = f_2\alpha$. A homotopy of (f_1, f_2) is just a pair of homotopies (f_{1t}, f_{2t}) such that $\beta f_{1t} = f_{2t}\alpha$. This category reduces to the category of ordinary pairs of spaces (with base point) if we restrict ourselves to maps α which are inclusions. If (f_1, f_2) is homotopic to (g_1, g_2) by the homotopy (f_{1t}, f_{2t}), we denote by

$$(f_{1t}, f_{2t}) : (f_1, f_2) \simeq (g_1, g_2).$$

We denote by $[f_1, f_2]$ the homotopy class of the morphism $(f_1, f_2) : \alpha \to \beta$ and by $\Pi(\alpha, \beta)$ the set of all homotopy classes from α to β. (f_1, f_2) is called a *homotopy equivalent morphism*, or simply a *homotopy equivalence* if there is a morphism (g_1, g_2) such that $(g_1, g_2) \circ (f_1, f_2) \simeq (id_{X_1}, id_{X_2})$ and $(f_1, f_2) \circ (g_1, g_2) \simeq (id_{Y_1}, id_{Y_2})$. Such morphism (g_1, g_2) is called a *homotopy inverse* of (f_1, f_2). Furthermore, (f_1, f_2) is called a *self homotopy equivalent morphism*, or simply a *self homotopy equivalence* if $\alpha = \beta$ and a *self pair homotopy equivalent morphism* or simply, a *self pair homotopy equivalence* if $\alpha = \beta = i : A \to X$ is the inclusion.

For an object α, we define the subset $\mathcal{E}(\alpha)$ of $\Pi(\alpha, \alpha)$ by

$$\mathcal{E}(\alpha) = \{[f_1, f_2] \in \Pi(\alpha, \alpha) \mid (f_1, f_2) \text{ is a homotopy equivalence}\}.$$

Especially, for a CW-pair (X, A), if $\alpha = i : A \to X$ is the inclusion, we denote $\mathcal{E}(i)$ by $\mathcal{E}(X, A)$. We define the subset $\mathcal{E}(X, A; id_A)$ by

$$\mathcal{E}(X, A; id_A) = \{[id_A, f] \in \mathcal{E}(X, A) \mid id_A \text{ is the identity on A}\}.$$

In Section 2, we show that all these sets are groups, homotopy invariants in the category of pairs and generalizations of several concepts of the group of self homotopy equivalences. As one of the main results, we show that there exists an exact sequence

$$1 \to \mathcal{E}(X, A; id_A) \to \mathcal{E}(X, A) \to \mathcal{E}(A).$$

In Section 3, we obtain the conditions of the CW-pair (X, A) that the above sequence is a short exact sequence or a split short exact sequence, i.e., the following sequence is exact or split exact:

$$1 \to \mathcal{E}(X, A; id_A) \to \mathcal{E}(X, A) \to \mathcal{E}(A) \to 1.$$

In Section 4, we use the sequence in Section 3 and the method of Sieradski [10] to obtain the following results.

THEOREM 4.6. *Let X and Y be connected CW homotopy associative and inversive H-spaces such that the two sets $[X \wedge Y, X \times Y]$ and $[X, Y]$ are trivial. Then there exists a split short exact sequence*

$$1 \to \mathcal{E}(X) \to \mathcal{E}(X \times Y, Y) \to \mathcal{E}(Y) \to 1.$$

Especially, we have $\mathcal{E}(X \times Y, Y) \cong \mathcal{E}(X) \oplus \mathcal{E}(Y)$ if $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are abelian groups.

COROLLARY 4.7. *For two cyclic groups H and G, let $K(G, n)$ and $K(H, m)$ be Eilenberg-Maclane spaces with $n > m \geq 1$. Then we have*

$$\mathcal{E}(K(G, n) \times K(H, m), K(H, m)) \cong \mathrm{Aut}(G) \oplus \mathrm{Aut}(H),$$

where $\mathrm{Aut}(G)$ *is the group of automorphisms on* G.

Moreover,

$$\begin{aligned} \mathcal{E}(S^1 \times K(G, n), K(G, n)) &\cong \mathcal{E}(K(G, n) \times S^1, S^1) \\ &\cong \mathcal{E}(K(G, n) \times S^1) \cong \mathbb{Z}_2 \oplus \mathrm{Aut}(G). \end{aligned}$$

Throughout this paper, all spaces are based connected CW-complexes, all maps and all homotopies are based and all topological pairs are CW-pairs.

2. The self homotopy equivalences in the category of pairs

In the first place, we show that the set $\mathcal{E}(\alpha)$ has a group structure.

THEOREM 2.1. *Let $\alpha : X_1 \to X_2$ be an object in the category of pairs. Then the set $\mathcal{E}(\alpha)$ has a group structure induced by the composition of morphisms.*

Proof. Let $[f_1, f_2]$ and $[g_1, g_2]$ be elements of $\mathcal{E}(\alpha)$. Then

$$[f_1, f_2] \circ [g_1, g_2] = [f_1 g_1, f_2 g_2] \in \mathcal{E}(\alpha),$$

since $(f_1 g_1, f_2 g_2)$ is a self homotopy equivalent morphism on α. For each $[f_1, f_2] \in \mathcal{E}(\alpha)$, let (h_1, h_2) be a homotopy inverse morphism of (f_1, f_2). Then $[h_1, h_2]$ is the inverse element of $[f_1, f_2]$. Moreover, $[id_{X_1}, id_{X_2}]$ is the identity element of $\mathcal{E}(\alpha)$. □

Next we show that the group $\mathcal{E}(\alpha)$ is a homotopy invariant.

THEOREM 2.2 *If α and β have same homotopy type, then $\mathcal{E}(\alpha)$ and $\mathcal{E}(\beta)$ are isomorphic.*

Proof. Suppose that $\alpha : X_1 \to X_2$ and $\beta : Y_1 \to Y_2$ have the same homotopy type by a homotopy equivalent morphism $(e_1, e_2) : \alpha \to \beta$

with the homotopy inverse morphism $(e_1', e_2') : \beta \to \alpha$. Define $\Psi : \mathcal{E}(\alpha) \to \mathcal{E}(\beta)$ by

$$\Psi[f_1, f_2] = [(e_1, e_2) \circ (f_1, f_2) \circ (e_1', e_2')].$$

Then Ψ is a homomorphism, since for each pair $[f_1, f_2]$, $[g_1, g_2] \in \mathcal{E}(\alpha)$, we have

$$\begin{aligned}\Psi([f_1, f_2] \circ [g_1, g_2]) &= [e_1 f_1 g_1 e_1', e_2 f_2 g_2 e_2'] \\ &= [e_1 f_1 e_1' e_1 g_1 e_1', e_2 f_2 e_2' e_2 g_2 e_2'] \\ &= [e_1 f_1 e_1', e_2 f_2 e_2'] \circ [e_1 g_1 e_1', e_2 g_2 e_2'] \\ &= \Psi[f_1, f_2] \circ \Psi[g_1, g_2].\end{aligned}$$

If we define a homomorphism $\Phi : \mathcal{E}(\beta) \to \mathcal{E}(\alpha)$ by $\Phi[h_1, h_2] = [(e_1', e_2') \circ (h_1, h_2) \circ (e_1, e_2)]$, then Φ is an inverse homomorphism of Ψ. Thus Ψ is an isomorphism. □

The group $\mathcal{E}(\alpha)$ is called *the group of the self homotopy equivalent morphisms* in the category of pairs. This group is a generalization of the group of self homotopy equivalences for a space. This fact is explained in the following remark.

REMARK. Let X be a CW-complex and $\alpha : * \to X$ the constant map. Then we have $\mathcal{E}(\alpha) = \mathcal{E}(X)$. For any self homotopy equivalence $f : X \to X$ and any homotopy $h_t : X \to X$ of f, $(*, f)$ is a self homotopy equivalent morphism and $(*, h_t)$ is a homotopy of $(*, f)$ from α to itself. Thus we can identify $[f] \in \mathcal{E}(X)$ with $[*, f] \in \mathcal{E}(\alpha)$. Similarly, if $\alpha : X \to *$ is a constant map, then we have $\mathcal{E}(\alpha) = \mathcal{E}(X)$. Moreover, for the identity map $id_X : X \to X$, we have $\mathcal{E}(id_X) = \mathcal{E}(X)$. Since for any self homotopy equivalence $f : X \to X$ and for any homotopy h_t of f, (f, f) is a self homotopy equivalent morphism from id_X to itself and (h_t, h_t) is a homotopy of (f, f), we can identify $[f] \in \mathcal{E}(X)$ to $[f, f] \in \mathcal{E}(id_X)$.

For a CW-pair (X, A), if $\alpha = i : A \to X$ is the inclusion, we denote $\mathcal{E}(i)$ by $\mathcal{E}(X, A)$. If (f_1, f_2) is a morphism from the inclusion i to itself, then $f_1|_A = f_2$. Thus we can consider the morphism (f_1, f_2) as the pair map $f_1 : (X, A) \to (X, A)$. So the group $\mathcal{E}(X, A)$ is just the group of pair homotopy equivalences, i.e.,

$$\mathcal{E}(X, A) = \{[f] | f : (X, A) \to (X, A) \text{ is a pair homotopy equivalence}\}.$$

We define a subset $\mathcal{E}(X, A; id_A)$ by

$$\mathcal{E}(X, A; id_A) = \{[id_A, f] \in \mathcal{E}(X, A) \mid id_A \text{ is the identity on } A\}.$$

This subset is actually a subgroup of $\mathcal{E}(X,A)$.

PROPOSITION 2.3. *$\mathcal{E}(X,A;id_A)$ is a subgroup of $\mathcal{E}(X,A)$.*

Proof. It is sufficient to show that for each element in $\mathcal{E}(X,A;id_A)$, it's homotopy inverse element in $\mathcal{E}(X,A)$ belongs to $\mathcal{E}(X,A;id_A)$ again. Let $[id_A,f] \in \mathcal{E}(X,A;id_A)$ and $[h,g]$ be the inverse element of $[id_A,f]$ in $\mathcal{E}(X,A)$. Since

$$(id_A,f)\circ(h,g) = (h, f\circ g) \sim (id_A, id_X),$$

there exists a homotopy $(H_A,H) : i\times id_I \to i$ between $(h, f\circ g)$ and (id_A,id_X). That is, the diagram

$$\begin{array}{ccc} X\times I & \xrightarrow{H} & X \\ i\times id_I \uparrow & & \uparrow i \\ A\times I & \xrightarrow[H_A]{} & A \end{array}$$

is commutative, $H(x,0) = (f\circ g)(x)$, $H(x,1) = x$ for all $x\in X$, $H_A(a,0) = h(a)$, $H_A(a,1) = a$ for all $a\in A$ and $H(*,t) = *$, where $*$ is the base point of X, $i : A\to X$ is the inclusion and $id_I : [0,1]\to[0,1]$ is the identity.

Define a map $g\sqcup H_A : X\times 0\sqcup A\times I \to X$ by

$$(g\sqcup H_A)|_{X\times 0} = g \text{ and } (g\sqcup H_A)|_{A\times I} = iH_A.$$

Since $g(a) = h(a) = H_A(a,0)$ for all $a\in A$, $g\sqcup H_A$ is well-defined and has an extension $F : X\times I\to X$. Thus we have

$$\begin{aligned} &F(x,0) = g(x),\\ &F(a,0) = H_A(a,0) = h(a),\\ &F(a,1) = H_A(a,1) = a,\\ &F(*,t) = H_A(*,t) = *. \end{aligned}$$

Let $g' = F(\cdot,1) : X\to X$. Then (h,g) is homotopic to (id_A,g') by the homotopy (H_A,F). Thus we have

$$[h,g] = [id_A,g'] \in \mathcal{E}(X,A;id_A).$$

□

Now we fit three groups $\mathcal{E}(X,A;id_A)$, $\mathcal{E}(X,A)$ and $\mathcal{E}(A)$ together into an exact sequence.

THEOREM 2.4. *For a CW-pair (X,A), there exists an exact sequence*

$$(1) \qquad 1\to\mathcal{E}(X,A;id_A)\to\mathcal{E}(X,A)\to\mathcal{E}(A).$$

Proof. Let $\Phi : \mathcal{E}(X, A; id_A) \to \mathcal{E}(X, A)$ be the inclusion. Then it is trivial that Φ is a monomorphism. Define $\Psi : \mathcal{E}(X, A) \to \mathcal{E}(A)$ by

$$\Psi[f_1, f_2] = [f_1]$$

for $[f_1, f_2] \in \mathcal{E}(X, A)$. Then Ψ is well-defined. Let $[f_1, f_2] = [g_1, g_2] \in \mathcal{E}(X, A)$. Then there exists a homotopy $(F|_A, F) : i \times id_I \to i$ between (f_1, f_2) and (g_1, g_2), where $i : A \to X$ is the inclusion and id_I is the identity on the unit interval $[0, 1]$. Since $F|_A : f_1 \simeq g_1$, we have

$$\Psi[f_1, f_2] = [f_1] = [g_1] = \Psi[g_1, g_2].$$

Furthermore, Ψ is a homomorphism, since the group operations of $\mathcal{E}(X, A)$ and $\mathcal{E}(A)$ are induced by the composition of maps.

Now we show the exactness at $\mathcal{E}(X, A)$. The image of Φ is contained in the kernel of Ψ, since

$$\Psi\Phi[id_A, f] = \Psi[id_A, f] = [id_A] \in \mathcal{E}(A).$$

Thus it remains for us to show that the kernel of Ψ is contained in the image of Φ. That is, each element $[f_1, f_2] \in \mathcal{E}(X, A)$ such that $[f_1] = [id_A] \in \mathcal{E}(A)$ belongs to $\mathcal{E}(X, A; id_A)$. Let $[f_1, f_2]$ be such an element. Since $f_1 \simeq id_A$ relative to $*$ in A, there exists a homotopy $H : A \times I \to A$ such that $H(a, 0) = f_1(a)$, $H(a, 1) = a$ and $H(*, t) = *$. Then the map $f_2 \sqcup iH : X \times 0 \sqcup A \times I \to X$ defined by $(f_2 \sqcup iH)|_{X \times 0} = f_2$ and $(f_2 \sqcup iH)|_{A \times I} = iH$ has an extension $F : X \times I \to X$. Let $\overline{f} = F(\cdot, 1)$. Then, for each $a \in A$, we have

$$\overline{f}(a) = F(a, 1) = H(a, 1) = a.$$

So $(id_A, \overline{f})$ is a morphism from i to itself, where $i : A \to X$ is the inclusion . But (f_1, f_2) is homotopic to $(id_A, \overline{f})$ by the homotopy (H, F) in the category of pairs. Therefore, $[f_1, f_2] = [id_A, \overline{f}] \in \mathcal{E}(X, A; id_A)$. □

3. Homotopy equivalence extendable pairs and a short exact sequence

In this section, we will find certain sufficient conditions for the sequence (1) to be a short exact sequence or a split short exact sequence.

DEFINITION 3.1. A CW-pair (X, A) is called *a homotopy equivalence extendable pair* if, for every homotopy equivalence $f : A \to A$, there exists a homotopy equivalence $\overline{f} : X \to X$ such that $(f, \overline{f}) : i \to i$ is a self equivalent morphism in the category of pair, where $i : A \to X$ is the inclusion. In this case, $\overline{f}$ is called a *homotopy equivalence extension* of f.

In the following proposition, we introduce a homotopical property of homotopy equivalence extensions.

PROPOSITION 3.2. *Let (X, A) be a homotopy equivalence extendable pair, and f and g self homotopy equivalences on A. If f and g are homotopic relative to $*$, then there are homotopy equivalence extensions $\overline{f}$ and $\overline{g}$ of f and g respectively such that $(f, \overline{f})$ and $(g, \overline{g})$ are homotopic in the category of pairs.*

Proof. Let $H : A \times I \to A$ be a homotopy between f and g. Then we have $H(a, 0) = f(a)$, $H(a, 1) = g(a)$ and $H(*, t) = *$. Since (X, A) is a homotopy equivalence extendable pair, there exists a homotopy equivalence extension $\overline{f} : X \to X$ of f. Define $\overline{f} \sqcup iH : X \times 0 \sqcup A \times I \to X$ by $(\overline{f} \sqcup iH)|_{X\times 0} = \overline{f}$ and $(\overline{f} \sqcup iH)|_{A\times I} = iH$, where $i : A \to X$ is the inclusion. Then it is well-defined, since $\overline{f}(a) = f(a) = H(a, 0)$, for each $a \in A$. Since the inclusion $i : A \to X$ is a cofibration, the map $(\overline{f} \sqcup iH)$ has an extension $\overline{H} : X \times I \to X$. Define $\overline{g} : X \to X$ by $\overline{g}(x) = \overline{H}(x, 1)$. Then $\overline{g}(a) = \overline{H}(a, 1) = H(a, 1) = g(a)$. So $(g, \overline{g})$ is a morphism. Since $\overline{g}$ is homotopic to $\overline{f}$ by the homotopy $\overline{H}$, $\overline{g}$ is a self homotopy equivalence. Furthermore, we have $(\overline{H}, H) : (\overline{f}, f) \simeq (g, \overline{g})$, since $\overline{H} \circ (i \times id_I) = i \circ H$, where $i : A \to X$ is the inclusion. Therefore, $\overline{g}$ is a homotopy equivalence extension of g. □

EXAMPLE 3.3. The pair $(X, *)$ is a homotopy equivalence extendable pair. More generally, if A is a strong deformation retract of X, the pair (X, A) is a homotopy equivalence extendable pair. Let $r : X \to A$ be a retraction and f a self homotopy equivalence on A. Then $i \circ f \circ r$ is a homotopy equivalence extension of f.

In the following theorem, we show that the homotopy equivalence extendability is a sufficient condition for the sequence (1) to be a short exact sequence.

THEOREM 3.4. *If (X, A) is a homotopy equivalence extendable pair, then we have the following short exact sequence:*

$$1 \to \mathcal{E}(X, A; id_A) \xrightarrow{\Phi} \mathcal{E}(X, A) \xrightarrow{\Psi} \mathcal{E}(A) \to 1, \tag{2}$$

where Φ is the inclusion and Ψ is a homomorphism defined by $\Psi[f, \overline{f}] = [f]$.

Proof. It is sufficient to show that Ψ is onto. Let $[f] \in \mathcal{E}(A)$. By the hypothesis and Proposition 3.2, there exists an element $[f, \overline{f}] \in \mathcal{E}(X, A)$. Moreover, $\Psi[f, \overline{f}] = [f]$ by the definitions of Ψ. □

In general, the sequence (2) is not split. But the sequence (2) is split for some CW-pairs.

THEOREM 3.5. *Let X and Y be CW-complexes. Then we have the following split exact sequence:*

$$1 \to \mathcal{E}(X \times Y, Y; id_Y) \xrightarrow{\Phi} \mathcal{E}(X \times Y, Y) \underset{\underset{J}{\longleftarrow}}{\xrightarrow{\Psi}} \mathcal{E}(Y) \to 1. \tag{3}$$

Proof. First, we show that $(X \times Y, Y)$ is a homotopy equivalence extendable pair. Let $f : Y \to Y$ be a homotopy equivalence. Then the map $id_X \times f : X \times Y \to X \times Y$ is a homotopy equivalence and $(f, id_X \times f) : i \to i$ is a morphism in the category of pairs, where $i : Y \to X \times Y$ is the inclusion given by $i(y) = (*, y)$. Furthermore, if g is a homotopy inverse, then $(g, id_X \times g)$ is a homotopy inverse of $(f, id_X \times f)$. So $id_X \times f$ is a homotopy equivalence extension of f.

Define $J : \mathcal{E}(Y) \to \mathcal{E}(X \times Y, Y)$ by $J[f] = [f, id_X \times f]$. Then J is well-defined. In fact, if $H : Y \times I \to Y$ is a homotopy between f and g, then the pair $(H, id_X \times H) : i \times id_I \to i$ is a homotopy between $(f, id_X \times f)$ and $(g, id_X \times g)$ in the category of pairs.

Moreover, J is a homomorphism, since

$$\begin{aligned} J([f] \cdot [g]) &= J[f \circ g] = [f \circ g, id_X \times (f \circ g)] \\ &= [f \circ g, (id_X \times f) \circ (id_X \times g)] \\ &= [f, id_X \times f] \cdot [g, id_X \times g] \\ &= J[f] \cdot J[g]. \end{aligned}$$

By the definitions of Ψ and J, $\Psi \circ J = id_{\mathcal{E}(Y)}$. So the sequence (3) is split. □

Theorem 3.5 shows that there exists a one-to-one correspondence between the direct product $\mathcal{E}(X \times Y, Y; id_Y) \times \mathcal{E}(Y)$ and the group $\mathcal{E}(X \times Y, Y)$. In fact, if we define a map $\Theta : \mathcal{E}(X \times Y, Y; id_Y) \times \mathcal{E}(Y) \to \mathcal{E}(X \times Y, Y)$ by $\Theta(\alpha, \beta) = \Phi(\alpha) \cdot J(\beta)$ for $(\alpha, \beta) \in \mathcal{E}(X \times Y, Y; id_Y) \times \mathcal{E}(Y)$, then the map Θ is bijective by basic properties of algebra. Furthermore, if the groups $\mathcal{E}(X \times Y, Y; id_Y)$, $\mathcal{E}(Y)$ and $\mathcal{E}(X \times Y, Y)$ are abelian, the map Θ is an isomorphism. But the commutativity of $\mathcal{E}(X \times Y, Y; id_Y) \times \mathcal{E}(Y)$ implies that of $\mathcal{E}(X \times Y, Y)$. Thus we have the following corollary.

COROLLARY 3.6. *Let X and Y be CW-complexes such that $\mathcal{E}(X \times Y, Y; id_Y)$ and $\mathcal{E}(Y)$ are abelian groups. Then we have*

$$\Theta : \mathcal{E}(X \times Y, Y; id_Y) \oplus \mathcal{E}(Y) \equiv \mathcal{E}(X \times Y, Y). \tag{4}$$

We can use the formula (4) to calculate one of three groups in relation to the others in the formula. From this point of view, we need to concrete the groups in the formula. Here, we find a relation between the groups $\mathcal{E}(X \times Y, Y; id_Y)$ and $\mathcal{E}(X)$. First, we have the following proposition in general.

PROPOSITION 3.7. *Let X and Y be CW-complexes. Then $\mathcal{E}(X)$ is isomorphic to a subgroup of $\mathcal{E}(X \times Y, Y; id_Y)$.*

Proof. It is sufficient to show that there is a monomorphism from $\mathcal{E}(X)$ to $\mathcal{E}(X \times Y, Y; id_Y)$. Define $\Gamma : \mathcal{E}(X) \to \mathcal{E}(X \times Y, Y; id_Y)$ by $\Gamma[f] = [id_Y, f \times id_Y]$.

First, we show that Γ is well-defined. It is easy to show $(id_Y, f \times id_Y) : i_2 \to i_2$ is a morphism in the category of pair, where $i_2 : Y \to X \times Y$ is the inclusion given by $i_2(y) = (*, y)$ for each $y \in Y$. Let $H : f \simeq g$ rel $*$. Then

$$H(x,0) = f(x), \;\; H(x,1) = g(x) \;\text{ and }\; H(*,t) = *.$$

Define $\overline{H} : X \times Y \times I \to X \times Y$ by $\overline{H}(x,y,t) = (H(x,t), y)$. Then

$$\overline{H} \circ (i_2 \times id_I)(y,t) = \overline{H}(*,y,t) = (H(*,t),y) = (*,y) = i_2(y)$$

and

$$\begin{aligned}
\overline{H}(x,y,0) &= (H(x,0),y) = (f(x),y) = (f \times id_Y)(x,y),\\
\overline{H}(x,y,1) &= (H(x,1),y) = (g(x),y) = (g \times id_Y)(x,y),\\
\overline{H}(*,*,t) &= (H(*,t),*) = (*,*).
\end{aligned}$$

Therefore, $(id_Y, \overline{H})$ is a homotopy between $(id_Y, f \times id_Y)$ and $(id_Y, g \times id_Y)$ in the category of pairs. Furthermore, $(id_Y, f \times id_Y)$ is a self homotopy equivalence on i_2 in the category of pairs if f is a self homotopy equivalence on X. So Γ is well defined. Moreover, Γ is a homomorphism. In fact,

$$\begin{aligned}
\Gamma([f] \cdot [g]) &= \Gamma[f \circ g] = [id_Y, (f \circ g) \times id_Y]\\
&= [id_Y, f \times id_Y] \cdot [id_Y, g \times id_Y]\\
&= \Gamma[f] \cdot \Gamma[g].
\end{aligned}$$

Finally, we show that Γ is monic. Let $\Gamma[f] = \Gamma[g]$ for $[f],\ [g] \in \mathcal{E}(X)$. By the definition of the Γ, this means

$$(id_Y, f \times id_Y) \simeq (id_Y, g \times id_Y)$$

in the category of pair. So there exists a homotopy of pair $(F, \overline{F})$: $i_2 \times id_I \to i_2$ such that

$$\overline{F} : (f \times id_Y) \simeq (g \times id_Y) \text{ rel } (*, *)$$

and

$$F : id_Y \simeq id_Y,$$

where $i_2 : Y \to X \times Y$ is the inclusion and id_I is the identity on $[0, 1]$. Define $G : X \times I \to X$ by $G(x, t) = p_1\overline{F}(x, *, t)$ for $(x, t) \in X \times I$, where $p_1 : X \times Y \to X$ is the projection. Then

$$\begin{aligned} G(x, 0) &= p_1(\overline{F}(x, *, 0)) = p_1(f(x), *) = f(x), \\ G(x, 1) &= p_1(\overline{F}(x, *, 1)) = p_1(g(x), *) = g(x), \\ G(*, t) &= p_1(\overline{F}(*, *, t)) = p_1(*, *) = *. \end{aligned}$$

Thus $[f] = [g]$ in $\mathcal{E}(X)$. □

Proposition 3.7 implies that the group $\mathcal{E}(X)$ gives always a lower bound of the group $\mathcal{E}(X \times Y, Y; id_Y)$. Now we are interested in the monomorphism Γ defined in the Proposition 3.7. Is the map Γ an isomorphism under the conditions in Proposition 3.7? Otherwise, when is it onto, i.e., an isomorphism? What are the conditions of X and Y under which the homomorphism Γ is bijective?

In the next section, we will discuss about them and introduce one of the sufficient conditions under which the map Γ is onto.

4. H-group structures and self pair homotopy equivalences

In this section, in order to study the condition that the groups $\mathcal{E}(X)$ and $\mathcal{E}(X \times Y, Y; id_Y)$ are isomorphic, we will use a part of method which was used earlier by Sieradski [10]. So we will introduce a part of Sieradski's method which is necessary to develop this section until further notice.

Throughout this section, spaces X and Y are connected CW-complexes which admit homotopy inversive, homotopy associative multiplication $m_X : X \times X \to X$ and $m_Y : Y \times Y \to Y$ and we will not distinguish in our notation between a base point preserving map and its homotopy class.

First, we note that each set $[A, B]$, with B a homotopy inversive, homotopy associative H-space, of homotopy classes of base point preserving maps receives a group structure whose operation will be denoted additively and will be referred to as "addition". Thus for any space S, the set $[S, X \times Y]$ is an additive group, since the product $X \times Y$ inherits from X and Y a "coordinate-wise multiplication" which is homotopy inversive and homotopy associative. Consider the wedge sum $X \vee Y$ and the inclusion $i : X \vee Y \to X \times Y$. Then we obtain from a mapping cone sequence for the inclusion i a short exact sequence of additive groups

(5) $0 \to [X \wedge Y, X \times Y] \xrightarrow{q^\#} [X \times Y, X \times Y] \xrightarrow{i^\#} [X \vee Y, X \times Y] \to 0,$

where $q : X \times Y \to X \wedge Y$ is the inclusion of $X \times Y$ onto base of the mapping cone $X \wedge Y$ of i. It proves convenient to identify $[X \vee Y, X \times Y]$ with the set of 2×2 matrices

$$(h_{AB}) = \begin{bmatrix} h_{XX} & h_{XY} \\ h_{YX} & h_{YY} \end{bmatrix}$$

with entries h_{AB} from the homotopy sets $[A, B]$ for $A, B = X, Y$, via the correspondence of $h : X \vee Y \to X \times Y$ with the matrix $(p_B \circ h \circ i_A)$, where $i_A : A \to X \vee Y$ $(A = X, Y)$ are the two inclusions of the summands into the sum and $p_B : X \times Y \to B$ $(B = X, Y)$ are the two projections of the product onto the factors. We observe that the composition of homotopy classes determines an associative operation in $[X \times Y, X \times Y]$ which is written multiplicatively. This operation has a unit $id_{X \times Y}$, is generally noncommutative, and distributes over additive from one side: $(g + h) \circ f = g \circ f + h \circ f$. Thus the self-homotopy equivalence on $X \times Y$ is just an element in $[X \times Y, X \times Y]$ which has an invertible element with respect to the multiplication. The identification of the set $[X \vee Y, X \times Y]$ with the set of 2×2 matrices (h_{AB}) with entries h_{AB} from the sets $[A, B]$ (for $A, B = X, Y$) makes it possible to introduce a matrix multiplication in $[X \vee Y, X \times Y]$:

$$\begin{bmatrix} h_{XX} & h_{XY} \\ h_{YX} & h_{YY} \end{bmatrix} \begin{bmatrix} k_{XX} & k_{XY} \\ k_{YX} & k_{YY} \end{bmatrix}$$
$$= \begin{bmatrix} k_{XX} \circ h_{XX} + k_{YX} \circ h_{XY} & k_{XY} \circ h_{XX} + k_{YY} \circ h_{XY} \\ k_{XX} \circ h_{YX} + k_{YX} \circ h_{YY} & k_{XY} \circ h_{YX} + k_{YY} \circ h_{YX} \end{bmatrix},$$

where the multiplication $k_{XB} \circ h_{AX}$ is the composition of the map h_{AX} in $[A, X]$ and the map k_{XB} in $[X, B]$ and the indicated addition $k_{XB} \circ h_{AX} + k_{YB} \circ h_{AY}$ takes place in $[A, B]$. The matrix multiplication need not be associative, but does admit a unit

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where 0 is the homotopy class of the constant map $*$ and 1 is the homotopy class of the identity map id_A. So we can refer to invertible matrices

$(h_{AB}) : X \vee Y \to X \times Y$. The induced map

$$i^{\#} : [X \times Y, X \times Y] \to [X \vee Y, X \times Y]$$

is said to be a *multiplicative homomorphism* if it is a homomorphism from the composition multiplication to the matrix multiplication. Here, we quote two theorems in [10].

THEOREM 4.1([10], p.793). *Let $i^{\#} : [X \times Y, X \times Y] \to [X \vee Y, X \times Y]$ be a multiplicative homomorphism. Then $h : X \times Y \to X \times Y$ is a homotopy equivalence if and only if $h \circ i = (h_{AB}) : X \vee Y \to X \times Y$ is an invertible matrix.*

THEOREM 4.2([10], p.796). *The map $i^{\#} : [X \times Y, X \times Y] \to [X \vee Y, X \times Y]$ is a multiplicative homomorphism if and only if kernel $i^{\#}$ is a right ideal in $[X \times Y, X \times Y]$.*

Let us return to the investigation about $\mathcal{E}(X \times Y, Y; id_Y)$. We know that for each class $h \in [X \times Y, X \times Y]$, $i^{\#}(h)$ can be identified with the matrix

$$\begin{bmatrix} h_{XX} & h_{XY} \\ h_{YX} & h_{YY} \end{bmatrix},$$

where $h_{AB} = p_B \circ h \circ i \circ i_A$ for the inclusion $i_A : A \to X \vee Y$ $(A, B = X, Y)$.

PROPOSITION 4.3. *Let $[id_Y, h]$ be an element of $\mathcal{E}(X \times Y, Y; id_Y)$ and $[X, Y] = 0$. Then for some $h_{XX} \in \mathcal{E}(X)$, $i^{\#}(h)$ can be identified with the matrix*

$$\begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix}$$

provided that $i^{\#}$ is a multiplicative homomorphism.

Proof. If h is the class in $[X \times Y, X \times Y]$ such that $(id_Y, h) : i \circ i_Y \to i \circ i_Y$ is a morphism in the category of pairs, then $i^{\#}(h)$ can be identified with

$$\begin{bmatrix} h_{XX} & h_{XY} \\ 0 & 1 \end{bmatrix}.$$

In fact, $h_{YX} = p_X \circ h \circ i \circ i_Y = p_X \circ i_Y = *$ and $h_{YY} = p_Y \circ h \circ i \circ i_Y = p_X \circ i_Y = id_Y$. Moreover, if $[X, Y] = 0$, $i^{\#}(h)$ can be identified with

$$\begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix},$$

since $h_{XY} \in [X, Y]$.

Let $[id_Y, h] \in \mathcal{E}(X \times Y, Y; id_Y)$. Then $(id_Y, h) : i \circ i_Y \to i \circ i_Y$ is a morphism and h is an element of $\mathcal{E}(X \times Y)$. Thus by Theorem 4.1, the matrix $i^{\#}(h)$

$$\begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix}$$

is an invertible matrix. That is, there exists a 2×2 matrix

$$\begin{bmatrix} k_{XX} & k_{XY} \\ k_{YX} & k_{YY} \end{bmatrix}$$

such that

$$\begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_{XX} & k_{XY} \\ k_{YX} & k_{YY} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_{XX} & k_{XY} \\ k_{YX} & k_{YY} \end{bmatrix} \begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix}.$$

So we have

$$k_{XX} \circ h_{XX} = 1 = h_{XX} \circ k_{XX}, k_{XY} = 0, k_{YX} = 0 \text{ and } k_{YY} = 1.$$

That is, $[h_{XX} \circ k_{XX}] = [id_X] = [k_{XX} \circ h_{XX}]$. Thus $h_{XX} \in \mathcal{E}(X)$. □

The sequence (5) can be extended to the pointed sets of pair homotopy classes. That is, we have the following exact sequence:

$$(6) \qquad [(X \wedge Y, CY), (X \times Y, Y)] \xrightarrow{q^{\#}} [(X \times Y, Y), (X \times Y, Y)]$$
$$\xrightarrow{i^{\#}} [[(X \vee Y, Y), (X \times Y, Y)]],$$

where CY is the reduced cone of Y. Actually, the pointed sets are groups, since the H-group structure on $X \times Y$ is induced by the H-group structures on X and Y coordinate-wisely. $i^{\#}$ is said to be a *multiplicative monomorphism pair-wisely* if it is a monomorphism as a map in the sequences (5) and (6) simultaneously.

In Proposition 3.7, we showed that there is a monomorphism $\Gamma : \mathcal{E}(X) \to \mathcal{E}(X \times Y, Y; id_Y)$ given by $\Gamma[f] = [f \times id_Y, id_Y]$. Now we show that the monomorphism Γ is also onto and so an isomorphism if $i^{\#} : [X \times Y, X \times Y] \to [X \vee Y, X \times Y]$ is a multiplicative monomorphism pair-wisely and $[X, Y] = 0$.

THEOREM 4.4 *Let $i : X \vee Y \to X \times Y$ be the inclusion such that the induced homomorphism $i^{\#} : [X \times Y, X \times Y] \to [X \vee Y, X \times Y]$ is a multiplicative monomorphism pair-wisely. Then the map $\Gamma : \mathcal{E}(X) \to \mathcal{E}(X \times Y, Y; id_Y)$ given by $\Gamma[f] = [id_Y, f \times id_Y]$ is onto provided that $[X, Y] = 0$.*

Proof. Let $[id_Y, h]$ be an element of $\mathcal{E}(X \times Y, Y; id_Y)$. By Proposition 4.3, there is an element $[h_{XX}] \in \mathcal{E}(X)$ such that

$$i^{\#}(h) = \begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix}.$$

Moreover,

$$\begin{aligned} & i^{\#}(h_{XX} \times id_Y) \\ = & \begin{bmatrix} p_X \circ (h_{XX} \times id_Y) \circ i \circ i_X & p_Y \circ (h_{XX} \times id_Y) \circ i \circ i_X \\ p_X \circ (h_{XX} \times id_Y) \circ i \circ i_Y & p_Y \circ (h_{XX} \times id_Y) \circ i \circ i_Y \end{bmatrix} \\ = & \begin{bmatrix} h_{XX} & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Thus $i^{\#}(h) = i^{\#}(h_{XX} \times id_Y)$ in $[(X \vee Y, Y), (X \times Y, Y)]$, since $[X, Y] = 0$. So by the hypothesis, we have $[id_Y, h] = [id_Y, h_{XX} \times id_Y]$ in $\mathcal{E}(X \times Y, Y; id_Y)$. Therefore, we conclude that for each $[id_Y, h] \in \mathcal{E}(X \times Y, Y; id_Y)$, there exists an element $[h_{XX}] \in \mathcal{E}(X)$ such that

$$\Gamma[h_{XX}] = [id_Y, h_{XX} \times id_Y] = [id_Y, h].$$

□

COROLLARY 4.5. *The group $\mathcal{E}(X \times Y, Y; id_Y)$ is isomorphic to $\mathcal{E}(X)$ provided that $[X \wedge Y, X \times Y] = 0$ and $[X, Y] = 0$.*

Proof. The fact $[X \wedge Y, X \times Y] = 0$ implies $[(X \wedge Y, CY), (X \times Y, Y)] = 0$, since the reduced cone CY is contractible. Thus the result follows from Theorems 4.2 and 4.4 and the mapping cone sequences (5) and (6) of the inclusion $i : (X \vee Y, Y) \to (X \times Y, Y)$. □

THEOREM 4.6. *Let X and Y be CW homotopy associative and inversive H-spaces such that the two sets $[X \wedge Y, X \times Y]$ and $[X, Y]$ are trivial. Then there exists a split short exact sequence*

$$1 \to \mathcal{E}(X) \to \mathcal{E}(X \times Y, Y) \to \mathcal{E}(Y) \to 1.$$

Especially, we have $\mathcal{E}(X \times Y, Y) \cong \mathcal{E}(X) \oplus \mathcal{E}(Y)$ if $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are abelian groups.

Proof. It follows from Corollaries 3.6 and 4.5. □

For integers $n > m \geq 1$ and abelian groups G and H, consider the Eilenberg-Maclane spaces $K(G, n)$ and $K(H, m)$. Then we have

$$[K(G, n) \wedge K(H, m), K(G, n) \times K(H, m)] = 0.$$

Moreover,

$$[K(G, n), K(H, m)] = H^m(G, n; H) = 0$$

and

$$[S_1, K(G,n)] = \pi_1(K(G,n)) = 0.$$

So by Theorem 4.6, Corollary 9 in [10, p796] and the fact that $\mathcal{E}(K(G,n))$ is the group $\mathrm{Aut}(G)$ of automorphisms on G, we have

COROLLARY 4.7. *For two cyclic groups H and G, let $K(G,n)$ and $K(H,m)$ be Eilenberg-Maclane spaces with $n > m \geq 1$. Then we have*

$$\mathcal{E}(K(G,n) \times K(H,m), K(H,m)) \cong \mathrm{Aut}(G) \oplus \mathrm{Aut}(H),$$

where $\mathrm{Aut}(G)$ is the group of automorphisms on G.

Moreover,

$$\begin{aligned}\mathcal{E}(S^1 \times K(G,n), K(G,n)) &\cong \mathcal{E}(K(G,n) \times S^1, S^1) \\ &\cong \mathcal{E}(K(G,n) \times S^1) \cong \mathbb{Z}_2 \oplus \mathrm{Aut}(G).\end{aligned}$$

EXAMPLE 4.8. Let CP^∞ and RP^∞ be infinite complex projective space and infinite real projective space respectively. Then we have

$$\mathcal{E}(CP^\infty \times RP^\infty, RP^\infty) \cong \mathrm{Auto}(\mathbb{Z}) \oplus \mathrm{Auto}(\mathbb{Z}_2) \cong \mathbb{Z}_2$$

and

$$\mathcal{E}(CP^\infty \times S^1, S^1) \cong \mathcal{E}(S^1 \times CP^\infty, CP^\infty) \cong \mathcal{E}(S^1 \times CP^\infty) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

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