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JORDAN AUTOMORPHIC GENERATORS OF EUCLIDEAN JORDAN ALGEBRAS

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ABSTRACT. In this paper we show that the Koecher's Jordan automorphic generators of one variable on an irreducible symmetric cone are enough to determine the elements of scalar multiple of the Jordan identity on the attached simple Euclidean Jordan algebra. Its various geometric, Jordan and Lie theoretic interpretations associated to the Cartan-Hadamard metric and Cartan decomposition of the linear automorphisms group of a symmetric cone are given with validity on infinite-dimensional spin factors

1. Introduction

Let V be a simple Euclidean Jordan algebra with an identity e and let Ω be the associated symmetric cone, the cone of invertible squares. In [11], Koecher introduced a remarkable family of Jordan automorphisms

$$V(a,b) := P(P(a^{1/2})b^{-1})^{1/2}P(a^{-1/2})P(b^{1/2}),$$

where P stands for the quadratic representation of V and a, b vary over the symmetric cone Ω (more generally in a neighborhood of the Jordan identity e in any semisimple Jordan algebra) and proved that these Jordan automorphisms, called *Koecher's Jordan automorphic generators*, generate the connected component K of the identity in the Jordan automorphism group $\mathrm{Aut}(V)$. In the simple Euclidean Jordan algebra $\mathrm{Sym}(n,\mathbb{R})$ of $n \times n$ symmetric matrices, the Jordan automorphism $V(a,b)$ with $a, b \in \Omega$ can be realized as the special orthogonal transformation

$$(A^{-1/2}BA^{-1/2})^{-1/2}A^{-1/2}B^{1/2} \in \mathrm{SO}(n)$$

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with positive definite A, B and hence the Lie group $\mathrm{SO}(n)$ is generated by

$$\{(NN^T)^{-1/2}N : N \in \mathcal{N}\},$$

where $\mathcal{N} := \{AB : A, B \text{ are positive definite}\}$. This result was rediscovered by R. Hauser [6] independently.

Koecher's Jordan automorphic generators that are closely related to the Cartan decomposition of the linear automorphism group $G = P(\Omega) \cdot K$ and to a Cartan-Hadamard metric on the symmetric cone (generally this kind of generators appears in a semisimple Lie group with a Cartan-decomposition realizing a symmetric space of non-positive curvature, see Remark 4.3), has recently played a key role in semidefinite and symmetric programming. The transitivity of K on Jordan frames of V implies that the elements fixed by K (or by a generating set of K) are exactly $\mathbb{R} \cdot e$, the scalar multiples of the Jordan identity e of V. Using Koecher's Jordan automorphic generators, an element $a \in V$ is a scalar multiple of the Jordan identity e if $V(x, y)(a) = a$ for all $x, y \in \Omega$. This result has played a crucial role for classifying self-scaled barriers for semidefinite and symmetric programming (the Hessian of a self-scaled barrier is a positive scalar multiple of that of the standard logarithmic barrier $F(x) = -\log\det(x)$, see [6], [7], and [8]).

One of main interest of this paper is to find *optimal* Jordan automorphic generators determining elements of scalar multiple of the Jordan identity. The following main result of this paper shows particularly that the Koecher's Jordan automorphic generators of 'one variable' are enough to determine the elements of scalar multiple of the identity. A compact version of this main result appears in Theorem 5.1 related to involutive elements, hyperbolic spaces and the Cartan-Hadamard metric on Ω.

THEOREM 1.1. *Let a be an element of Ω such that $V(a, x)(a) = a$ for all $x \in \Omega$. Then $a = \lambda e$ for some positive real number λ.*

We also have the following theorem which plays a key role for the proof of Theorem 1.1 together with the decomposition theorem (Theorem 3.3) of symmetric cones.

THEOREM 1.2. *Let x be an element of V. If $[L(x), L(\sigma(x))] = 0$ for any involutive Jordan automorphism of the form $\sigma = P(w)$, then $x = \lambda e$ for some real number λ, where $L(x)$ denotes the Jordan multiplication operator.*

In the last section we revisit the class of Euclidean Jordan algebras of rank 2 and show how the main results are reflected to Lorentz cones in a direct way. We pay particular attention to infinite-dimensional spin factors due to the recent development of polynomial-time primal-dual algorithm for infinite-dimensional second-order cone programs [4]. Using the classification of JB-algebras (Jordan Banach algebras) of finite rank [10] and explicit calculation of scaling points (geometric means) by Faybusovich and Tsuchiya (Corollary 4.1, [4]) we conclude that Theorem 1.1 and Theorem 1.2 remain valid for this class of infinite-dimensional Jordan Banach algebras.

2. Simultaneous diagonalization

We recall certain basic notions and well-known facts concerning Jordan algebras from the book [2] by J. Faraut and A. Korányi. A *Jordan algebra* V over the field $\mathbb{R}$ or $\mathbb{C}$ is a commutative algebra satisfying $x^2(xy) = x(x^2y)$ for all $x, y \in V$. Denote $L(x)$ by the multiplication operator $L(x)y = xy$, and set $P(x) = 2L(x)^2 - L(x^2)$ for $x \in V$. An element $x \in V$ is said to be invertible if there exists an element y in the subalgebra generated by x and e such that $xy = e$. It is known that an element x in V is invertible if and only if $P(x)$ is invertible. In this case, $P(x)^{-1} = P(x^{-1})$. If x and y are invertible, then $P(x)y$ is invertible and $(P(x)y)^{-1} = P(x^{-1})y^{-1}$. Furthermore, the fundamental formula $P(P(x)y) = P(x)P(y)P(x)$ holds true for any elements x and y.

A finite-dimensional real Jordan algebra V is called a *Euclidean Jordan algebra* if it carries an associative inner product $\langle \cdot | \cdot \rangle$ on V, namely $\langle xy|z\rangle = \langle y|xz\rangle$ for all $x, y, z \in V$. The spectral theorem (Theorem III.1.2 of [2]) of a Euclidean Jordan algebra V states that for $x \in V$ there exist a Jordan frame (a complete system of orthogonal primitive idempotents) $c_1, \ldots, c_r$ (r is the rank of V) and real numbers $\lambda_1, \ldots, \lambda_r$ (eigenvalues of x) such that $x = \sum_{i=1}^r \lambda_i c_i$. We note that $P(k(x)) = kP(x)k^{-1} = kP(x)k^t$ for any Jordan automorphism k of V, where k^t denotes the adjoint of k with respect to the inner product $\langle \cdot, \cdot \rangle$ on the Euclidean Jordan algebra V. The trace inner product $\mathrm{tr}(xy)$ is associative and in this case every Jordan automorphism is an orthogonal transformation with respect to the trace inner product. Let Ω be the open convex cone of invertible squares of V. Then Ω is a symmetric cone, that is, the group $G(\Omega) := \{g \in \mathrm{GL}(V) : g(\Omega) = \Omega\}$ acts transitively on it and Ω is a self-dual cone with respect to the associative inner product. For an element $g \in G(\Omega)$, g is a Jordan automorphism if and only if $g(e) = e$ (Proposition VIII.2.4, [2]).

Throughout this paper we will always assume that V is a simple Euclidean Jordan algebra equipped with the trace inner product and $\mathbf{c} := \{c_1, \dots, c_r\}$ *is a fixed Jordan frame.* For any idempotent c of V we denote the corresponding eigenspaces by $V(c, \lambda) := \{x \in V : cx = \lambda x\}$ and the Peirce decomposition $V = V(c, 0) \oplus V(c, 1/2) \oplus V(c, 1)$. If we set $V_{ii} = V(c_i, 1), V_{ij} = V(c_i, 1/2) \cap V(c_j, 1/2)$ for $1 \leq i \neq j \leq r$, then $V = \bigoplus_{i \leq j} V_{ij}$. The following multiplication property will be useful (see Theorem IV.2.1, [2])

$$\text{(2.1)} \quad V = \bigoplus_{i \leq j} V_{ij} \quad \text{and} \quad \begin{cases} V_{ij}V_{ji} \subset V_{ii} + V_{jj}, & \\ V_{ij}V_{jk} \subset V_{ik} & \text{if } i \neq k, \\ V_{ij}V_{kl} = \{0\} & \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \end{cases}$$

We note that $V_{ii} = \mathbb{R}c_i$, $V_{ij} \neq \{0\}$ and $\dim V_{ij} = \dim V_{kl}$ for any $i < j$ and $k < l$. Let d denote this dimension. Then $r + \frac{d}{2}r(r-1) = n := \dim(V)$. We further note that

$$\text{(2.2)} \qquad L(c_k)x = \frac{1}{2}(\delta_{ik} + \delta_{jk})x, \ x \in V_{ij}.$$

Corresponding to the Peirce decomposition, for $x \in V$ there exist unique real numbers $d_i(x)$ and $x_{ij} \in V_{ij} = V(c_i, 1/2) \cap V(c_j, 1/2)$ $(i < j)$ such that

$$x = \sum_{i=1}^{r} d_i(x)c_i + \sum_{i<j} x_{ij}.$$

An element x of V is said to be diagonal (with respect to the Jordan frame $\mathbf{c}$) if $x \in \sum_{i=1}^{r} \mathbb{R} \cdot c_i$. Two elements x and y are said to be simultaneously diagonalizable if there is a Jordan automorphism $k \in \mathrm{Aut}(V)$ such that both $k(x)$ and $k(y)$ are diagonal elements. We note that this notion is independent of the choice of Jordan frames because of the transitivity of the group $\mathrm{Aut}(V)$ on Jordan frames of V (see Theorem IV.2.5, [2]).

PROPOSITION 2.1. *Let $x \in V$. Then* $\mathrm{tr}(x) = \sum_{i=1}^{r} d_i(x)$. *If x is a diagonal element then* $\mathrm{tr}(xy) = \sum_{i=1}^{r} d_i(x)d_i(y)$ *for all $y \in V$.*

Proof. Let $x = \sum_{i=1}^{r} d_i(x)c_i + \sum_{i<j} x_{ij}$. By (2.2), $L(x)$ has the eigenvalue $\frac{1}{2}(d_i(x) + d_j(x))$ on V_{ij}. Since $d = \dim(V_{ij})$ for any $i < j$, we have

$$\begin{aligned}\mathrm{Tr}(L(x)) &= \sum_{i=1}^{r} d_i(x) + d\sum_{i<j}\frac{d_i(x)+d_j(x)}{2}\\ &= \sum_{i=1}^{r}(1+\frac{d}{2}(r-1))d_i(x)\\ &= \frac{n}{r}\sum_{i=1}^{r} d_i(x), \quad (n = r+\frac{d}{2}r(r-1)).\end{aligned}$$

It then follows from Proposition III.4.2 of [2] that $\mathrm{tr}(x) = \frac{r}{n}\mathrm{Tr}(L(x)) = \sum_{i=1}^{r} d_i(x)$. Suppose that x is diagonal and let $y = \sum_{i=1}^{r} d_i(y)c_i + \sum_{i<j} y_{ij}$. Then by (2.1) and (2.2), $xy = \sum_{i=1}^{r} d_i(x)d_i(y)c_i + \sum_{i<j} z_{ij}$ for some $z_{ij} \in V_{ij}$ and hence $\mathrm{tr}(xy) = \sum_{i=1}^{r} d_i(x)d_i(y)$. □

Let $\mathrm{Der}(V)$ be the Lie algebra of the Jordan automorphism group $\mathrm{Aut}(V)$. Then $\mathrm{Der}(V)$ consists of all Jordan derivations of V (see page 36, [2]). Since V is simple, all derivations are inner ([2], Proposition VI.1.2) and hence each derivation D is a sum of derivations of the form $[L(a), L(b)] = L(a)L(b) - L(b)L(a)$.

The following Jordan-Lie algebraic interpretations of simultaneous diagonalization will be useful for our purposes.

THEOREM 2.2. *Let x and y be elements of V and let $a = \exp(x), b = \exp(y)$. Then the following are equivalent:*

(1) *x and y are simultaneously diagonalizable,*
(2) $y \perp \mathrm{Der}(V).x$,
(3) $[L(x), L(y)] = 0$,
(4) *a and b are simultaneously diagonalizable,*
(5) $[P(a), P(b)] = 0$,
(6) (Fuglede – Putnam) $P(a)b^2 = P(b)a^2$.

In particular, if $x = \sum_{i=1}^{r} \lambda_i c_i$ is a diagonal element. Then

$$(\mathrm{Der}(V).x)^{\perp} = \sum_{i=1}^{r} \mathbb{R}c_i \bigoplus \sum_{i<j,\ \lambda_i=\lambda_j} V_{ij}.$$

Proof. The equivalences of (1)–(3) are established in [20]. Observe that $L(x)$ and $L(y)$ are symmetric with respect to the trace metric and $P(a) = \exp 2L(x), P(b) = \exp 2L(y)$ by Proposition II.3.4 of [2]. Then the equivalences of (3)–(5) follow from injectivity of the exponential mappings on V and on symmetric transformations. Suppose that

$P(a)b^2 = P(b)a^2$. Then $A := P(a), B := P(b)$ are positive definite transformations on V satisfying $AB^2A = BA^2B$ (the fundamental formula). Setting $R = AB$ so that $RR^T = AB^2A = BA^2B = R^TR$. By applying the Fuglede-Putnam theorem (Theorem 12.16, [22]), we have that R is a symmetric transformation and therefore $P(a)P(b) = P(b)P(a)$. The commutativity of $P(a)$ and $P(b)$, $P(a)P(b) = P(b)P(a)$, is obeyed by the single element e, the Jordan identity: $P(a)P(b)e = P(a)b^2 = P(b)a^2 = P(b)P(a)e$.

Next, suppose that $x = \sum_{i=1}^r \lambda_i c_i$ is a diagonal element. Let $y \in (\mathrm{Der}(V).x)^{\perp}$ and let $y = \sum_{i=1}^r d_i(y)c_i + \sum_{i<j} y_{ij}$ be the Peirce decomposition of y. Suppose that $\lambda_i \neq \lambda_j$. Setting $D = [L(c_i), L(y_{ij})]$ so that

$$0 = \langle D(x), y\rangle = \frac{\lambda_j - \lambda_i}{4}\langle y_{ij}, y\rangle.$$

From (2.1), one may see that $y_{ij}y_{ij}$ is the only diagonal factor of $y_{ij}y$. Therefore by Proposition 2.1, $0 = \mathrm{tr}(y_{ij}y) = \mathrm{tr}(y_{ij}^2)$ and hence $y_{ij} = 0$.

Conversely, suppose that $y_{ij} = 0$ for $\lambda_i \neq \lambda_j$. To prove that $y \in (\mathrm{Der}(V).x)^{\perp}$, it is enough to show that $[L(x), L(y)] = 0$ by Theorem 2.2. Since $[L(c_i), L(c_j)] = 0$ for all $1 \leq i, j \leq r$, it suffices to show that $[L(x), L(y_{ij})] = 0$ for any $i < j$ satisfying $\lambda_i = \lambda_j$. Suppose that $\lambda_i = \lambda_j$. Let us consider an equivalence relation on $\{1, 2, \ldots, r\}$ defined by $l \sim k$ if and only if $\lambda_l = \lambda_k$, and let $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l\}$ be the corresponding partition of $\{1, 2, \ldots, r\}$. Let $e_{\mathcal{P}_k} = \sum_{\mu \in \mathcal{P}_k} c_\mu$. Then $L(x)$ can be written as

$$L(x) = \sum_{k=1}^{l} a_k L(e_{\mathcal{P}_k}),$$

where a_k is the eigenvalue of x representing the class $\mathcal{P}_k$. From $\lambda_i = \lambda_j$, we have either $\{i, j\} \subset \mathcal{P}_k$ or $\{i, j\} \cap \mathcal{P}_k = \emptyset$ and thus $e_{\mathcal{P}_k} y_{ij} = 0$ or $e_{\mathcal{P}_k} y_{ij} = y_{ij}$ for each $1 \leq k \leq l$. Applying Proposition II.1.1(i) of [2], we see that $[L(e_{\mathcal{P}_k}), L(y_{ij})] = 0$ for both cases. This shows that

$$[L(x), L(y_{ij})] = \sum_{k=1}^{l} a_k [L(e_{\mathcal{P}_k}), L(y_{ij})] = 0$$

and completes the proof. □

3. A decomposition theorem on symmetric cones

The symmetric cone Ω carries a $G(\Omega)$-invariant Riemannian metric defined by

$$\gamma_x(u,v) = \langle P(x^{-1})u|v\rangle, x \in \Omega, u, v \in V$$

for which the Jordan inversion $x \to x^{-1}$ on Ω is an involutive isometry fixing e. The symmetric space Ω is then a Cartan-Hadamard manifold and the unique geodesic passing two points a and b is given by

$$\gamma(t) = P(a^{1/2})(P(a^{-1/2})b)^t.$$

The geometric mean $a\#b$ of elements a and b in Ω is defined by

$$a\#b = P(a^{1/2})(P(a^{-1/2})b)^{1/2}$$

and it coincides with the geodesic middle $\gamma(1/2)$ of a and b for the Riemannian metric distance. In the Euclidean Jordan algebra $\mathrm{Sym}(n,\mathbb{R})$ of $n \times n$ real symmetric matrices, the geometric mean of positive definite matrices A and B is given by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

PROPOSITION 3.1 ([15], [17], [18]). *Let a and b be elements of Ω. Then*

(1) *the quadratic equation $P(x)a^{-1} = b$ has a unique solution in Ω given by $a\#b$,*
(2) *$a\#b = b\#a$ (commutativity property),*
(3) *$(a\#b)^{-1} = a^{-1}\#b^{-1}$ (inversion property),*
(4) *$P(a\#b) = P(a)\#P(b)$*
$= P(a^{1/2})\left(P(a^{-1/2})P(b)P(a^{-1/2})\right)^{1/2} P(a^{1/2})$,
(5) *$g(a\#b) = g(a)\#g(b)$ for all $g \in G(\Omega)$ (transformation property).*

For $a, b \in \Omega$, the element $F(a,b) := P(a^{-1}\#b)^{1/2}a$, which can be viewed as a unique solution belonging to Ω of the equation

$$(a^{-1}\#b)^{1/2} = a^{-1}\#x,$$

is known as the *spectral geometric mean* of a and b.

PROPOSITION 3.2. *Let $a, b \in \Omega$. Then*

(1) *$F(a,b) = F(b,a)$ (commutativity property),*
(2) *$F(a,b)^{-1} = F(a^{-1}, b^{-1})$ (inversion property),*
(3) *$P(F(a,b)) = F(P(a), P(b))$,*
(4) *$F(\sigma(a), \sigma(b)) = \sigma(F(a,b))$ for any Jordan automorphism σ of V,*
(5) *$F(P(a)b, P(a^{-1})b) = b$,*

(6) $F(a,b) = a\#b$ *if and only if* a *and* b (hence $\log(a)$ and $\log(b)$) *are simultaneously diagonalizable,*

(7) (Spectral mean) $\lambda_i(F(a,b)) = \lambda_i(P(a^{1/2})b)^{1/2} = \lambda_i(P(b^{1/2})a)^{1/2}$, *where* $\lambda_i(x)$ *are the eigenvalues of* x *in non-decreasing order.*

Proof. (1) By Proposition 3.1 (1), $P(a^{-1}\#b)a = b$ and hence from the commutativity and the inversion properties of geometric means

$$F(a,b) = P(a^{-1}\#b)^{1/2}a = P(a^{-1}\#b)^{-1/2}b = P(b^{-1}\#a)^{1/2}b = F(b,a).$$

(2) It follows from the inversion property of the geometric mean that

$$F(a,b)^{-1} = (P(a^{-1}\#b)^{1/2}a)^{-1} = P(a\#b^{-1})^{1/2}a^{-1} = F(a^{-1},b^{-1}).$$

(3) From the fundamental formula $P(P(x)y) = P(x)P(y)P(x)$ and Proposition 3.1 (4) we have that

$$\begin{aligned} P(F(a,b)) &= P(P(a^{-1}\#b)^{1/2}a) \\ &= P(a^{-1}\#b)^{1/2}P(a)P(a^{-1}\#b)^{1/2} \\ &= (P(a)^{-1}\#P(b))^{1/2}P(a)(P(a)^{-1}\#P(b))^{1/2} \\ &= F(P(a),P(b)). \end{aligned}$$

(4) It follows from $\sigma(x\#y) = \sigma(x)\#\sigma(y)$ and $P(\sigma(x)) = \sigma P(x)\sigma^{-1}$ that

$$\begin{aligned} F(\sigma(a),\sigma(b)) &= P(\sigma(a)^{-1}\#\sigma(b))^{1/2}\sigma(a) \\ &= P(\sigma(a^{-1}\#b))^{1/2}\sigma(a) \\ &= \sigma P(a^{-1}\#b)^{1/2}\sigma^{-1}\sigma(a) \\ &= \sigma(F(a,b)). \end{aligned}$$

(5) It follows that

$$\begin{aligned} F(P(a)b, P(a^{-1})b) &= \Big(P(P(a^{-1})b^{-1}\#P(a^{-1})b)\Big)^{1/2}P(a)b \\ &= \Big(P(P(a^{-1})(b^{-1}\#b)\Big)^{1/2}P(a)b \\ &= P(a^{-1})(P(a)b) = b. \end{aligned}$$

(6) It turns out (Theorem 5.2 of [5]) that two positive definite matrices A and B commute if and only if the geometric mean $A\#B$ coincides with the spectral geometric mean $F(A,B)$. Using this fact and Proposition 3.1, we have that for any elements a and b in Ω, $a\#b = F(a,b)$ if and only if $P(a\#b) = P(F(a,b))$ if and only if $P(a)\#P(b) = F(P(a),P(b))$ if and only if $P(a)P(b) = P(b)P(a)$ where we used the fact that the

quadratic representation P is injective from the symmetric cone Ω into the cone of positive definite operators on V [17]. By Theorem 2.2, this is equivalent to that a and b are simultaneously diagonalizable.

(7) By Lemma XIV.1.2 of [2], there exists a Jordan automorphism k such that $F(a,b) = P((a^{-1}\#b)^{1/2})a = kP(a^{1/2})(a^{-1}\#b)$ and is equal to $k(e\#P(a^{1/2})b) = k((P(a^{1/2})b)^{1/2})$ from the homogeneous property of the geometric mean. $\square$

Let σ be an involutive Jordan automorphism of V. We define

$$\Omega_\sigma^+ = \{x \in \Omega : \sigma(x) = x\},$$
$$\Omega_\sigma^- = \{x \in \Omega : \sigma(x) = x^{-1}\}.$$

THEOREM 3.3 (cf. [19]). *The maps*

$$T : \Omega_\sigma^+ \times \Omega_\sigma^- \to \Omega, \quad T(a,b) = P(a^{1/2})b,$$
$$S : \Omega_\sigma^- \times \Omega_\sigma^+ \to \Omega, \quad S(a,b) = P(a^{1/2})b,$$

are differentiable diffeomorphisms with their inverses

$$T^{-1}(x) = (x\#\sigma(x), F(x, \sigma(x^{-1}))),$$
$$S^{-1}(x) = (x\#\sigma(x^{-1}), F(x, \sigma(x))),$$

respectively.

Proof. Let $x \in \Omega$. By the commutativity, inversion and transformation properties of the geometric mean (Proposition 3.1),

$$\sigma(x\#\sigma(x)) = \sigma(x)\#\sigma^2(x) = \sigma(x)\#x = x\#\sigma(x),$$
$$\sigma(x\#\sigma(x^{-1})) = \sigma(x)\#\sigma^2(x^{-1}) = \sigma(x)\#x^{-1} = (x\#\sigma(x^{-1}))^{-1},$$

which implies that $x\#\sigma(x) \in \Omega_\sigma^+$ and $x\#\sigma(x^{-1}) \in \Omega_\sigma^-$. By Proposition 3.2, we have

$$F(x, \sigma(x^{-1}))^{-1} = F(x^{-1}, \sigma(x)) = F(\sigma(x), \sigma^2(x^{-1})) = \sigma(F(x, \sigma(x^{-1}))),$$

which shows that $F(x, \sigma(x^{-1})) \in \Omega_\sigma^-$. Similarly,

$$\sigma(F(x, \sigma(x))) = F(\sigma(x), \sigma^2(x)) = F(x, \sigma(x))$$

implies that $F(x, \sigma(x)) \in \Omega_\sigma^+$. Therefore, the maps T^{-1} and S^{-1} are well-defined. To see that S^{-1} is the inverse function of S (the case T is similar), let $(a,b) \in \Omega_\sigma^- \times \Omega_\sigma^+$. Using $\sigma(P(x)y) = P(\sigma(x))\sigma(y)$ and by

Propositions 3.1-3.2, one can show directly that

$$\begin{aligned} S^{-1}(S(a,b)) &= S^{-1}(P(a^{1/2})b) \\ &= (P(a^{1/2})b\#\sigma(P(a^{-1/2})b^{-1}), F(P(a^{1/2})b, \sigma(P(a^{1/2})b))) \\ &= (P(a^{1/2})b\#P(a^{1/2})b^{-1}, F(P(a^{1/2})b, P(a^{-1/2})b) \\ &= (P(a^{1/2})(b\#b^{-1}), b) \\ &= (a,b). \end{aligned}$$

Similarly $S \circ S^{-1} = \mathrm{id}_\Omega$. □

REMARK 3.4. We note that a typical involutive Jordan automorphisms arises in the form $\sigma = P(w)$ with $w^2 = e$ (eigenvalues of w are ± 1) by Proposition II.4.4 of [2] and every element w such that $w^2 = e$ arises from an idempotent for since $c := \frac{1}{2}(w + e)$ is an idempotent.

If we denote

$$V_\sigma^+ = \{x \in V : \sigma(x) = x\}, V_\sigma^- = \{x \in V : \sigma(x) = -x\},$$

then V_σ^+ is a Jordan subalgebra of V and V_σ^- is the orthogonal complement of V_σ^+ with respect to the trace inner product. Furthermore, $\Omega_\sigma^+ = \exp(V_\sigma^+) = \Omega \cap V_\sigma^+$ and $\Omega_\sigma^- = \exp(V_\sigma^-)$ are symmetric (geodesic) submanifolds of Ω and then Theorem 3.3 can be viewed as the global tubular neighborhood theorem for the geodesic submanifolds $\Omega_\sigma^\pm$ in the Cartan-Hadamard manifold Ω (cf. [14]). This decomposition theorem is studied for more general cases, involutive dyadic symmetric sets or uniquely 2-divisible Bruck loop ([16]). The geodesic submanifold Ω_σ^+ is the corresponding symmetric cone of the Euclidean Jordan algebra V_σ^+ and is known as a Helwig space [1].

Though Fiedler and Pták [5] have introduced and developed the spectral geometric mean of positive definite matrices as an another mean operation on the positive definite cone comparing that of geometric mean of positive definite matrices, the notions of geometric and spectral geometric means on the symmetric cones turn out very useful for understanding decompositions of symmetric cones with respect to the geodesic submanifolds $\Omega_\sigma^\pm$. Metric and spectral geometric means are used to a primal-dual potential-reduction algorithm of symmetric programming (Lemma 3.2 and Proposition 3.3, [3]). Proposition 3.2 (5) will play a certain role for establishing our main results.

4. Koecher's Jordan automorphic generators

Let A and B be $n \times n$ positive definite real matrices. Then

$$A^{1/2}BA^{1/2} = (A^{1/2}B^{1/2})(B^{1/2}AB^{1/2})(A^{1/2}B^{1/2})^{-1}$$

and thus $A^{1/2}BA^{1/2}$ is similar to $B^{1/2}AB^{1/2}$.

Setting $C := (A^{1/2}BA^{1/2})^{-1/2}A^{1/2}B^{1/2}$, we have $C^T = C^{-1}$, $\det(C) = 1$ and hence C is a special orthogonal transformation. Furthermore, $A^{1/2}BA^{1/2} = C(B^{1/2}AB^{1/2})C^{-1}$ and so $A^{1/2}BA^{1/2}$ is *orthogonally* similar to $B^{1/2}AB^{1/2}$. The Jordan-algebraic version of this result is appeared in Lemma XIV. 1.2 of [2] using the polar decomposition of $G(\Omega)_\circ$ (Theorem III.3.1 of [2]): For $a, b \in \Omega$, there exists $k \in K$, the identity component of the Jordan automorphism group $\mathrm{Aut}(V)$ of V, such that $P(a^{1/2})b = kP(b^{1/2})a$.

For $a, b \in \Omega$, we denote

$$V(a,b) := P(P(a^{-1/2})b)^{-1/2}P(a^{-1/2})P(b^{1/2}).$$

Then $V(a,b)$ is a Jordan automorphism since

$$V(a,b)(e) = P(P(a^{-1/2})b)^{-1/2}(P(a^{-1/2})b) = e$$

and belongs to K by continuity. The Jordan automorphism $V(a^{-1}, b)$ can be regarded as the orthogonal factor (on the other hand $P(P(a^{1/2})b)^{1/2}$ is the symmetric factor) of $P(a^{1/2})P(b^{1/2})$ in the Cartan decomposition $G(\Omega)_\circ = P(\Omega) \cdot K$:

$$P(a^{1/2})P(b^{1/2}) = P(P(a^{1/2})b)^{1/2} \cdot V(a^{-1}, b) \in P(\Omega) \cdot K. \tag{4.3}$$

We seek some basic properties on the Jordan automorphisms $V(a,b)$.

PROPOSITION 4.1. *Let $a, b \in \Omega$.*

(1) *$V(a^{-1}, b)$ is a unique element of $k \in \mathrm{Aut}(V)$ such that $P(a^{1/2})P(b^{1/2}) = kP(P(b^{1/2})a)^{1/2}$. In particular, $P(a^{1/2})b = V(a^{-1}, b)(P(b^{1/2})a)$.*
(2) *$V(a,b) = P(a^{1/2})P(a\#b)^{-1}P(b^{1/2})$.*
(3) *$V(a,b) = V(a^{-1}, b^{-1}) = V(b,a)^{-1} = V(b,a)^t$.*
(4) *$V(a,b) = \mathrm{id}_V$ if and only if a and b are simultaneously diagonalizable.*
(5) *$V(k(a), k(b)) = kV(a,b)k^{-1}$ for any $k \in \mathrm{Aut}(V)$.*
(6) *$V(P(x^{1/2})a, P(x^{1/2})b) = V(x^{-1}, a)V(a,b)V(b, x^{-1})$ for any $x \in \Omega$.*

Proof. (1) Since $V(a,b)^{-1} = V(a,b)^t$, we find that

$$\begin{aligned}
&V(a,b)^{-1}P(a^{-1/2})P(b^{1/2})\\
&= P(b^{1/2})P(a^{-1/2})P(P(a^{-1/2})b)^{-1/2}P(a^{-1/2})P(b^{1/2})\\
&= P(b^{1/2})\Big(P(a^{-1/2})P(P(a^{1/2})b^{-1})^{1/2}P(a^{-1/2})\Big)P(b^{1/2})\\
&= P(b^{1/2})P(a^{-1}\#b^{-1})P(b^{1/2}) = P(b^{1/2})P(b^{-1}\#a^{-1})P(b^{1/2})\\
&= P(b^{1/2})\Big(P(b^{-1/2})P(P(b^{1/2})a^{-1})^{1/2}P(b^{-1/2})\Big)P(b^{1/2})\\
&= P(P(b^{1/2})a^{-1})^{1/2}.
\end{aligned}$$

(2) It follows from the inversion property of the geometric mean that

$$\begin{aligned}
V(a,b) &= P\big(P(a^{-1/2})b\big)^{-1/2}P(a^{-1/2})P(b^{1/2})\\
&= P(a^{1/2})\Big(P(a^{-1/2})P\big(P(a^{1/2})b^{-1}\big)^{1/2}P(a^{-1/2})\Big)P(b^{1/2})\\
&= P(a^{1/2})P(a^{-1}\#b^{-1})P(b^{1/2})\\
&= P(a^{1/2})P(a\#b)^{-1}P(b^{1/2}).
\end{aligned}$$

(3) Since (the Riccati solution) $\Big(P(a)\#P(b)\Big)P(b)^{-1}\Big(P(a)\#P(b)\Big) = P(a)$, we find

$$P(a)\Big(P(a)\#P(b)\Big)^{-1} = \Big(P(a)\#P(b)\Big)P(b)^{-1}$$

and thus

$$\begin{aligned}
V(a,b) &= P(a^{1/2})P(a\#b)^{-1}P(b^{1/2})\\
&= P(a^{-1/2})P(a\#b)P(b^{-1/2})\\
&= P(a^{-1/2})P(a^{-1}\#b^{-1})^{-1}P(b^{-1/2})\\
&= V(a^{-1},b^{-1}).
\end{aligned}$$

Furthermore, from the commutativity property of the geometric mean we have

$$\begin{aligned}
V(a,b)^{-1} = V(a,b)^t &= P(b^{1/2})P(a\#b)^{-1}P(a^{1/2})\\
&= P(b^{1/2})P(b\#a)^{-1}P(a^{1/2})\\
&= V(b,a).
\end{aligned}$$

(4) If $V(a,b) = \mathrm{id}_\Omega$, then $P(a^{-1/2})b = P(b^{1/2})a^{-1}$. By Theorem 2.2, $a^{-1/2}$ and $b^{1/2}$ (and hence a and b) are simultaneously diagonalizable. Conversely, suppose that a and b are simultaneously diagonalizable. Then $P(a)$ and $P(b)$ commute by Theorem 2.2 and hence $P(a\#b) = P(a)\#P(b) = P(a)^{1/2}P(b)^{1/2}$. Therefore,

$$V(a,b) = P(a^{1/2})P(a\#b)^{-1}P(b^{1/2}) = \mathrm{id}_V.$$

(5) If $k \in \mathrm{Aut}(V)$, then $P(k(x)) = kP(x)k^{-1}$ and hence

$$\begin{aligned} V(k(a),k(b)) &= P(k(a^{1/2}))P(k(a\#b))^{-1}P(k(b^{1/2})) \\ &= kP(a^{1/2})k^{-1}kP(a\#b)^{-1}k^{-1}kP(b^{1/2})k^{-1} \\ &= kV(a,b)k^{-1}. \end{aligned}$$

(6) Using the fundamental formula $P(P(x)y) = P(x)P(y)P(x)$ and Proposition 3.1, we have

$$\begin{aligned} &V(P(x)a, P(x)b) \\ &= P(P(x)a)^{1/2}\Big(P(P(x)a\#P(x)b)^{-1}\Big)P(P(x)b)^{1/2} \\ &= [P(x)P(x)^{-1}P(P(x)a)^{1/2}] \circ [P(x)^{-1}P(a\#b)^{-1}P(x^{-1})] \\ &\quad \circ [P(P(x)b)^{1/2}P(x^{-1})P(x)] \\ &= [P(x)P(x^2\#a^{-1})^{-1}P(a^{-1/2})] \circ [P(a^{1/2})P(a\#b)^{-1}P(b^{1/2})] \\ &\quad \circ [P(b^{-1/2})P(x^2\#b^{-1})^{-1}P(x)] \\ &= V(x^2, a^{-1})V(a,b)V(b^{-1}, x^2) \\ &= V(x^{-2}, a)V(a,b)V(b, x^{-2}). \qquad \square \end{aligned}$$

Next, we state Koecher's result on the family of Jordan automorphisms $V(a,b)$. First of all, it is not difficult to see from the Lie algebra of K is by $\mathfrak{L}(K) = [L(V), L(V)]$ (Theorem III.5.1 and Proposition VI.1.2 of [2]) that group $G(\Omega)_\circ = P(\Omega)\cdot K$ is generated by $P(\Omega)$ ([13], p.27). Let $k \in K$. Then $k = P(a_n)P(a_{n-1})\cdots P(a_1)$ for some $a_i \in \Omega$. By the explicit polar decomposition (4.3) and by induction, $k = P(a_n)P(a_{n-1})\cdots P(a_1) = P(w)h$ where h is a product of elements of the form $V(a,b), a, b \in \Omega$. From the uniqueness of the decomposition, we have $k = h$. We have proved the following (see Theorem IV.9 of [11] for general case).

THEOREM 4.2. *The group K is generated by $\{V(a,b) : a, b \in \Omega\}$.*

REMARK 4.3. By Proposition 4.1 (2) and (5), the set $\{V(a,b) : a,b \in \Omega\}$ of Koecher's Jordan automorphic generators is invariant under inversion and adjoint action of K. We further note that the notion $V(a,b)$ arises naturally in a connected semisimple Lie group with a Cartan decomposition $G = P \cdot K$ where the uniqueness of the square root of an element of P is guaranteed (see [12], more generally, Banach-Lie groups associated to infinite-dimensional symmetric Finsler manifolds with seminegative curvature, Theorem V.5 of [21]). In this case, using the corresponding Cartan involution one may see that for any $a, b \in P$,

$$V(a^{-1}, b) := (a^{1/2}ba^{1/2})^{-1/2}a^{1/2}b^{1/2}$$

belongs to K, the (orthogonal) K-factor of the hyperbolic element $a^{1/2}$ $b^{1/2}$. On the other hand, $(a^{1/2}ba^{1/2})^{1/2}$ is the (symmetric) P-factor of $a^{1/2}b^{1/2}$.

5. Proofs of main theorems

Proof of Theorem 1.2. Suppose that $x \in V$ such that $[L(x), L(\sigma(x))] = 0$ for any Jordan involution $\sigma = P(w), w^2 = e$. Let $k \in \mathrm{aut}(V)$. Then from $L(k(x)) = kL(x)k^{-1}$ and $kP(w)k^{-1} = P(k(w))$, $0 = [L(k(x)), L(P(k(w))k(x))]$. Since $k(w)^2 = k(w^2) = k(e) = e$ for all $w^2 = e$, choosing $k \in \mathrm{Aut}(V)$ such that $k(x)$ is diagonal, we may assume that x is a diagonal element of V. Let $x = \sum_{i=1}^{r} \lambda_i c_i$. If $\lambda_i = \lambda_j$ for all i, j then $x = \lambda_1 e$. Suppose that $\lambda_i \neq \lambda_j$ for some $i \neq j$. We may assume that $i = 1$ and $j = 2$. To lead a contradiction, it is enough to construct an element w of order 2 such that $P(w)(x)$ contains a non-zero V_{12}-factor by Theorem 2.2.

Note that c_1 and c_2 are orthogonal primitive idempotents of V and hence of the subalgebra $V(c_1+c_2, 1)$ which is simple by Lemma 1.2 of [9]. We further note that the simple Jordan algebra $V(c_1 + c_2, 1)$ is rank 2 and hence is isomorphic to a spin factor $\mathbb{R} \times W$ for some Euclidean space $(W, (\cdot|\cdot))$ with $\dim(W) \geq 2$. The corresponding Jordan frame of $\{c_1, c_2\}$ in $\mathbb{R} \times W$ can be written as $c_1 = \frac{1}{2}(1, u)$, $c_2 = \frac{1}{2}(1, -u)$ for some $u \in W$ such that $(u|u) = 1$. Pick $v \in W$ such that $(v|v) = 1$ and $(u|v) = 0$. Then $c_1' := \frac{1}{2}(1, v), c_2' := \frac{1}{2}(1, -v)$ form a Jordan frame of $\mathbb{R} \times W$, and hence there exists an element $w_0 \in \mathbb{R} \times W$ such that $w_0^2 = c_1' + c_2' = (1, 0)$, the identity for $\mathbb{R} \times W$ and $P(w_0)(c_1) = c_1'$ (see Corollary IV.2.4, [2]). We note that $P(w_0)$ is an involutive Jordan automorphism (Proposition II.4.4 of [2]) of $\mathbb{R} \times W$ and $P(w_0)(c_2) = c_2'$. Then one can compute directly that

$$\begin{aligned} P(w_0)(\lambda_1 c_1 + \lambda_2 c_2) &= \lambda_1 P(w_0)(c_1) + \lambda_2 P(w_0)(c_2) \\ &= \lambda_1 c_1' + \lambda_2 c_2' \\ &= \frac{\lambda_1}{2}(1, v) + \frac{\lambda_2}{2}(1, -v) \\ &= \frac{\lambda_1 + \lambda_2}{2}(1, 0) + \frac{\lambda_1 - \lambda_2}{2}(0, v) \end{aligned}$$

and can immediately see that in the Jordan algebra $\mathbb{R} \times W$, the Peirce factor V_{12} from the Jordan frame $\{c_1, c_2\}$ is $V_{12} = \{(0, y) \in \mathbb{R} \times W : (u|y) = 0\}$. Therefore $z := \frac{\lambda_1 - \lambda_2}{2}(0, v)$ is a non-zero V_{12}-vector of $\mathbb{R} \times W$. We keep the same notation of the constructed two elements w_0 and z in the Jordan subalgebra $V(c_1 + c_2, 1)$.

Set $w = w_0 + c_3 + \cdots + c_r$ (in the case $r = 2$, we just proved in the previous step). Since $w_0 \in V(c_1 + c_2, 1) \subset V(c_k, 0)$ for all $k \neq 1, 2$, we find that

$$w^2 = w_0^2 + c_3 + \cdots + c_r = c_1 + c_2 + c_3 + \cdots + c_r = e$$

and hence $\sigma := P(w)$ is an involutive Jordan automorphism of V. Now

$$\begin{aligned} \sigma(x) &= P(w)(\lambda_1 c_1 + \lambda_2 c_2) + P(w)(\lambda_3 c_3 + \cdots + \lambda_r c_r) \\ &= \Big(\frac{\lambda_1 + \lambda_2}{2}(c_1 + c_2) + z\Big) + \sum_{i=3}^{r} \lambda_i c_i \\ &= \Big(\frac{\lambda_1 + \lambda_2}{2}(c_1 + c_2) + \sum_{i=3}^{r} \lambda_i c_i\Big) + z \end{aligned}$$

has the non-zero V_{12}-factor $z \equiv \frac{\lambda_1 - \lambda_2}{2}(0, v)$, where in the second equality we have used the fact that $w_0 c_1, w_0 c_2 \in V(c_1 + c_2, 1) \subset V(c_k, 0)$ for $k \neq 1, 2$. This gives us a contradiction and completes the proof of Theorem 1.2. □

Next, we will prove Theorem 1.1. For $a \in \Omega$, we consider

$$M(a) := \{a \# P(w) a : w^2 = e\}.$$

Then

(i) $M(a)$ is a compact subset of Ω containing a,
(ii) $M(a^{-1}) = M(a)^{-1}$,
(iii) $M(a) = \{a\}$ if and only if $a = \lambda e$ for some $\lambda > 0$,
(iv) $k.M(a) = M(k(a))$ for all $k \in \mathrm{Aut}(V)$,
(v) $P(a^{-1/2}).M(a) = \{(P(a^{-1/2})P(w)P(a^{1/2})e)^{1/2} : w^2 = e\}$,

(vi) $\det(x) = \det(a)$ for all $x \in M(a)$, and
(vii) $M(\lambda a) = \lambda M(a)$ for all $\lambda > 0$.

Now, it is immediate from Proposition 4.1 (2) and (3) that $V(a,x)(a) = a$ if and only if $P(x^{1/2})a = P(a\#x)P(a^{-1/2})a$. Using $P(a^{-1/2})a = e$, this is equivalent to $P(x^{1/2})a = P(a\#x)e = (a\#x)^2$ or $(P(x^{1/2})a)^{1/2} = a\#x$. Therefore, Theorem 1.1 follows from the following:

THEOREM 5.1. *Let a be an element of the symmetric cone Ω. If*

$$a\#x = (P(x^{1/2})a)^{1/2}$$

for all $x \in M(a)$ then $a = \lambda e$ for some positive real number λ.

Proof. By Theorem 1.2, it is enough to show that

$$[L(\log a), L(\log \sigma(a))] = 0$$

for any involutive Jordan automorphism of the form $\sigma = P(w), w^2 = e$. Let $\sigma = P(w)$ be an involutive Jordan automorphism and let

$$a = P(x^{1/2})y,$$

the factorization of a from Theorem 3.3, where $x = a\#\sigma(a)$ and $y = F(a, \sigma(a^{-1}))$. Then by Proposition 3.1

$$\begin{aligned} P(x^{1/2})y^{1/2} &= P(x^{1/2})(y\#e) \\ &= (P(x^{1/2})y)\#(P(x^{1/2})e) \\ &= (P(x^{1/2})y)\#x \\ &= a\#x. \end{aligned}$$

Since $x = a\#\sigma(a)$ belongs to $M(a)$, $P(x^{1/2})y^{1/2} = a\#x = (P(x^{1/2})a)^{1/2}$ from the hypothesis. Therefore, $y^{1/2} = P(x^{-1/2})(P(x^{1/2})a)^{1/2} = x^{-1}\#a$, equivalently

$$P(y^{1/2})x = a$$

by Proposition 3.1 (1). Now since $y \in \Omega_\sigma^-$ and $x \in \Omega_\sigma^+$, we have that

$$(y,x) = S^{-1}(a) = (a\#\sigma(a)^{-1}, F(a, \sigma(a)))$$

from the decomposition theorem for the geodesic submanifold Ω_σ^-. Therefore the geometric and spectral geometric means of a and $\sigma(a)$ coincide

$$a\#\sigma(a) = x = F(a, \sigma(a))$$

which occurs only when $[L(\log a), L(\log \sigma(a))] = 0$ by Proposition 3.2 (5). This completes the proof. $\square$

For $a \in \Omega$, let $f_a : \Omega \to \Omega$ defined by $f_a(x) = P(x^{1/2})a$. Then it is not difficult to see that f_a is a differential diffeomorphism with the inverse function $f_a^{-1}(x) = (a^{-1}\#x)^2$ from Proposition 3.1. Thus Theorem 5.1 leads the following

COROLLARY 5.2. *$f_a^{-1} = f_{a^{-1}}$ if and only if $a = \lambda e$ for some positive real number of λ.*

For $a \in \Omega$, we let $\Omega_a := \{x \in \Omega : a\#x = (P(x^{1/2})a)^{1/2}\}$. Then Ω_a has the following interesting properties:

(i) $\Omega_a = \{x \in \Omega : a\#x = P(x^{1/2})(a\#x^{-1})\}$ (by Proposition 3.1 (5)),
(ii) $\{x \in \Omega : a$ and x are simultaneousely diagonalizable$\} \subseteq \Omega_a$,
(iii) Ω_a is a closed cone of Ω,
(iv) $\Omega_a = \Omega_{a^{-1}} = (\Omega_a)^{-1}$,
(v) $k.\Omega_a = \Omega_{k(a)}$ for any $k \in \mathrm{Aut}(V)$, and
(vi) $\Omega_{\lambda e} = \Omega$ for any $\lambda > 0$.

As an immediate consequence of Theorem 5.1, we have

COROLLARY 5.3. *Let $a \in \Omega$. Then $\Omega_a = \Omega$ if and only if $a = \lambda e$ for some positive scalar λ.*

Let α and β be functions from Ω into itself. Let $j : \Omega \to \Omega, j(x) = x^{-1}$ denote the Jordan inversion on Ω.

COROLLARY 5.4. *If $x\#y = \alpha(x)\#\beta(y)$ for all $x, y \in \Omega$ then $\alpha = \lambda \cdot \mathrm{id}_\Omega, \beta = j \circ \alpha \circ j$ for some positive real number λ.*

Proof. Suppose that $x\#y = \alpha(x)\#\beta(y)$ for all $x, y \in \Omega$. Then by Proposition 3.1, $P(x\#y)\alpha(x)^{-1} = \beta(y)$ and $P(x\#y)\beta(y)^{-1} = \alpha(x)$ for all $x, y \in \Omega$. Setting $y = x^{-1}$ and using $e = x\#x^{-1}$, we get $\alpha(x)^{-1} = \beta(x^{-1})$ for all $x \in \Omega$ and hence $\beta = j \circ \alpha \circ j$. In particular, $\beta(e) = \alpha(e)^{-1}$. Setting $x = e$ we get $\beta(y) = P(y^{1/2})\alpha(e)^{-1}$ and upon changing the roles of x and y we have $\alpha(x) = P(x^{1/2})\beta(e)^{-1}$. Let $a = \beta(e)^{-1}$. Then for any $x \in \Omega$,

$$\begin{aligned} P(x^{1/2})a = \alpha(x) &= P(x\#a)\beta(a)^{-1} \\ &= P(x\#a)P(a^{-1/2})\alpha(e) \\ &= P(x\#a)P(a^{-1/2})\beta(e)^{-1} \\ &= P(x\#a)P(a^{-1/2})a \\ &= P(x\#a)e \\ &= (x\#a)^2. \end{aligned}$$

By Theorem 5.1, we have that $a = \beta(e)^{-1} = \lambda e$ for some positive real number λ. Since $\alpha(x) = P(x^{1/2})\beta(e)^{-1} = P(x^{1/2})(\lambda e) = \lambda x$, we conclude that $\beta = j \circ \alpha \circ j$ and $\alpha = \lambda \cdot \mathrm{id}_\Omega$. □

6. Spin factors

In this section, we shall show that our main results Theorem 5.1 and Theorem 1.2 (and hence Corollaries 5.2-5.4) hold true for infinite-dimensional spin factors.

Let $V = \mathbb{R}\times Y$, where $(Y, (\cdot|\cdot))$ is a real Hilbert space with $\dim(Y) \geq 2$ (even infinite dimensional). Then V equipped with the product defined by

$$(t,x)(s,y) = (ts + (x|y), ty + sx)$$

becomes a JB-algebra, Jordan Banach algebra, with the norm from the inner product

$$\langle (t,x)|(s,y)\rangle = 2(ts + (x|y)).$$

Then $e := (1,0)$ is the identity for V and the corresponding symmetric cone of V is the Lorentz cone

$$\Omega = \{(s,y) : s > ||y||\}.$$

See [23] for details on JB-algebras and corresponding infinite-dimensional symmetric cones.

For $z = (s,y) \in V$, the multiplicative operator $L(z)$ admits the block partition

$$L(z) = \begin{pmatrix} s & l_y \\ l_y^T & s\cdot I_Y \end{pmatrix} \tag{6.4}$$

where I_Y denotes the identity operator on Y and $l_y : Y \to \mathbb{R}$ is the linear functional defined by $l_y(y') = (y|y')$, and the quadratic representation of $P(s,y)$ is given by

$$P(s,y) = \det(z)I_V + 2\begin{pmatrix} (y|y) & s\cdot l_y \\ s\cdot l_y^T & y\otimes y \end{pmatrix}, \tag{6.5}$$

where $\det(z) = s^2 - (y|y), y\otimes y(y') = (y|y')y$ for $y' \in Y$. It is shown by Faybusovich and Tsuchiya [4] that for $z_1, z_2 \in \Omega$, the quadratic equation

$$P(z)z_1^{-1} = z_2 \tag{6.6}$$

has a unique solution $z_3 \in \Omega$. In our notation $z_3 = z_1\#z_2$, the geometric mean of z_1 and z_2. Furthermore, they explicitly calculated the geometric mean $z_1 \# z_2$ (Corollary 4.2, [4]): Let $z_1 = (s, y), z_2 = (t, x)$, and let

$$a := \sqrt{\det(z_2)} = \sqrt{t^2 - (x|x)}, b := \sqrt{\det(z_1)} = \sqrt{s^2 - (y|y)}.$$

Then

$$z_1 \# z_2 = \frac{1}{\sqrt{2}\sqrt{ab + st - (x|y)}} \Big(as + bt, bx + ay\Big). \tag{6.7}$$

First, we shall show that Theorem 1.2 remains valid for V. Let $z = (s, y)$ be an element of V such that $[L(z), L(\sigma(z))] = 0$ for any Jordan involution of the form $\sigma = P(w), w^2 = e$. It is easy to see that

$$\{w \in V : w^2 = e\} = \{(0, x) \in V : (x|x) = 1\} \cup \{e\}.$$

Let $w = (0, x)$ with $(x|x) = 1$. Then from (6.5) and (6.4), we find that

$$P(w)z = (s, -y + 2(x|y)x) \tag{6.8}$$

and

$$L(P(w)z) = \begin{pmatrix} s & l_{(-y+2(x|y)x)} \\ l^T_{(-y+2(x|y)x)} & s \cdot I_Y \end{pmatrix}.$$

Thus $[L(z), L(P(w)z)] = 0$ implies that

$$l_y^T l_{(-y+2(x|y)x)} = l^T_{(-y+2(x|y)x)} l_y$$

and therefore $(-y + 2(x|y)x|u)y = (y|u)(-y + 2(x|y)x)$ for all $u \in V$. This is equivalent to

$$(x|y)(x|u)y = (x|y)(y|u)x$$

for all $u \in Y$. Because x varies on the unit sphere, this occurs only when $y = 0$. Therefore $z = (s, 0) = s(1, 0) \in \mathbb{R} \cdot e$.

Next, we describe the set $M(z) = \{z \# P(w)z : w^2 = e\}$. Let $z = (s, y)$ be a fixed element of Ω. From (6.8),

$$\{P(w)z : w^2 = e\} = \{(s, x) : ||x|| = ||y||\} \tag{6.9}$$

and therefore

$$M(z) = \{(s, y) \# (s, x) : ||x|| = ||y||\} \tag{6.10}$$

which is a subset of the hyperboloid $\{u \in \Omega : \det(u) = \det(z) = s^2 - ||y||^2\}$. On the other hand, observing from $\operatorname{rank}(V) = 2$ that for $a \in \Omega$,

$$\det(a) = 1 \text{ if and only if } a^{1/2} = \Big(\frac{1}{\det(e + a)}\Big)^{1/2}(e + a)$$

and from $\det(P(z^{-1/2})P(w)z) = 1$ for $w^2 = e$ that

$$z \# P(w)z = \Big(\frac{\det(z)}{\det(z + P(w)z)}\Big)^{1/2}(z + P(w)z)$$

(the geometric mean of z and $P(w)z$ is a scalar multiple of their sum $z+P(w)z$), we have that

$$M(z) = \left\{ \Big(\frac{\det(z)}{\det(z+P(w)z)}\Big)^{1/2}(z+P(w)z) : w^2 = e \right\} \tag{6.11}$$

$$= \left\{ \Big(\frac{s^2-||y||^2}{4s^2-||x+y||^2}\Big)^{1/2}(2s, x+y) : ||x|| = ||y|| \right\} \tag{6.12}$$

$$= \left\{ \Big(2+\langle z^{-1}|\tilde{T}(z)\rangle\Big)^{-1/2}(\mathrm{id}_V+\tilde{T})(z) : T \in \mathrm{O}(Y) \right\}, \tag{6.13}$$

where $\mathrm{O}(Y)$ denotes the orthogonal group on the Hilbert space Y and

$$\tilde{T} = \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}.$$

For instance, if $z = (1, 1/2, 1/2) \in \mathbb{R} \times \mathbb{R}^2$ then $M(z)$ represents the following simple closed curve on the hyperboloid $\{(x_1, x_2, x_3) : x_1^2 = \frac{1}{2} + x_2^2 + x_3^2, x_1 > 0\}$,

$$\Big(8 - [(1+\cos t)^2 + \sin^2 t]\Big)^{-1/2}\Big(2, \frac{1+\cos t + \sin t}{2}, \frac{1+\cos t - \sin t}{2}\Big)$$

passing through $(\frac{1}{\sqrt{2}}, 0, 0)$ and $z = (1, 1/2, 1/2)$.

Finally, suppose that $(z\#z_1)^2 = P(z_1^{1/2})z$ for all $z_1 \in M(z)$ or

$$[(s,y)\#(2s, x+y)]^2 = P((2s, x+y)^{1/2})(s,y), \forall ||x|| = ||y||$$

from (6.12). Suppose that $y \neq 0$. Then by multiplying $1/||y||^2$ we may assume that $||y|| = 1$. From (6.7) and (6.5), choosing x as $||x|| = 1$ and $(x|y) = 0$, the Y-coordinates of $[(s,y)\#(2s,x+y)]^2$ and $P((2s, x+y)^{1/2})(s,y)$ are

$$\frac{1}{a}(2sbx + 2(a+b)sy), \quad \frac{s(a+2s)+1}{a+2s}x + \frac{(a+s)(a+2s)+1}{a+2s}y,$$

respectively, where $a = \sqrt{(2s)^2 - ||x+y||^2} = \sqrt{4s^2-2}, b = \sqrt{s^2-1}$. Since x and y are linearly independent, we find that $a = 2s$ from

$$\frac{2sb}{a} = \frac{s(a+2s)+1}{a+2s}, \quad \frac{2(a+b)s}{a} = \frac{(a+s)(a+2s)+1}{a+2s}$$

which gives a contradiction. Therefore, $y = 0$ and hence $z = (s, 0) = s(1, 0) = s \cdot e$. We conclude that Theorem 5.1 remains valid for infinite dimensional spin factors.

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