

## THE ENUMERATION OF DOUBLY ALTERNATING BAXTER PERMUTATIONS

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ABSTRACT. In this paper, we give an alternative proof that the number of doubly alternating Baxter permutations is Catalan.

### 1. Introduction

A Baxter permutation is exactly a permutation  $\pi = a_1 a_2 a_3 \cdots a_n$  (or we notate  $\pi = \pi(1)\pi(2)\pi(3) \cdots \pi(n)$ ) in  $S_n$ , the symmetric group on  $[n] := \{1, 2, \dots, n\}$ , that satisfies the following two conditions: for every  $1 \leq i < j < k < l \leq n$ ,

1. if  $a_i + 1 = a_l$  and  $a_l < a_j$  then  $a_k > a_l$ ,
2. if  $a_l + 1 = a_i$  and  $a_i < a_k$  then  $a_j > a_i$ .

For example, for  $n = 4$ , 2413 and 3142 are the only non-Baxter permutations; 263154 and 5143762 are Baxter permutations. This class of permutations was introduced by Baxter [1] in the context of fixed points of the composite of commuting functions. These permutations can be regarded as permutations with forbidden subsequences [11], say  $S_n(25\bar{3}14, 41\bar{3}52)$ . For example, the set  $S_n(25\bar{3}14, 41\bar{3}52)$  is a set of the permutations of length  $n$  avoiding the patterns 2413 and 3142, each of them being allowed in the case where it is itself a subsequence of the pattern 25314 or 41352 in the permutation, respectively.

Chung, Graham, Hoggatt and Kleiman [2] have analytically showed that the number of Baxter permutations of length  $n$  is given by the formula

$$\binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{r=1}^n \binom{n+1}{r-1} \binom{n+1}{r} \binom{n+1}{r+1}.$$

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Received January 9, 2005.

2000 Mathematics Subject Classification: Primary 05A15, 05A05.

Key words and phrases: doubly alternating Baxter permutation, maximal-inversion-descent set, essential set.

Later, Mallows [13] has found a more precise interpretation according to the number of rises (indices  $i$  such that  $a_i < a_{i+1}$ ). Moreover, he gives a new formula for these permutations where the number of rises  $r$  is the only parameter which has a simple interpretation. Viennot [16] (see also [5]) has given a combinatorial proof of the formula obtained by Chung et al. [2] by establishing a one-to-one correspondence between Baxter permutations and some column-strict Young tableaux for which a formula is known. This correspondence is based on some classical bijections such as between permutations and Laguerre history, between two-colored Motzkin words and parallelogram polyominoes [4], and finally between non-intersecting paths and column-strict Young tableaux [10].

A permutation is said to be alternating if the permutation starts with a rise and then descents (indices  $i$  such that  $a_i > a_{i+1}$ ) and rises come in turn. More precisely, an alternating permutation  $\pi = a_1 a_2 \cdots a_n$  is such that  $a_{2i-1} < a_{2i}$  and  $a_{2i+1} < a_{2i}$  for  $1 \leq i \leq \lfloor n/2 \rfloor$ , that is to say, its descent (respectively rise) set denoted by  $Des(\pi)$  (respectively,  $Rise(\pi)$ ) happens at an even (respectively, odd) index. Cori, Dulucq and Viennot [3] have established a one-to-one correspondence which has proved that alternating Baxter permutations of length  $2n$  and  $2n+1$  are enumerated by  $c_n c_n$  and  $c_n c_{n+1}$ , respectively, where  $c_n = \frac{1}{n+1} \binom{2n}{n}$  is the  $n$ th Catalan number. More recently, Dulucq and Guibert [6] have found a new bijection which unifies [16] and [3], and have given a combinatorial interpretation of Mallows's formula [13].

A permutation is said to be doubly alternating if it is alternating and its inverse is alternating. We will use

REMARK 1.1. (Guibert and Linusson [12], Remark 1) For enumeration of doubly alternating Baxter permutations it does not make any difference if we define alternating to start with a rise or with a descent.

Now we consider another way to regard permutations. An  $n$ -by- $n$  permutation matrix can be represented by an  $n$ -by- $n$  array of squares with one dot in each row and column and all the other squares empty. The diagram of a permutation matrix (defined in 1800 by Rothe) is obtained by shading every row from the dot eastwards and shading every column from the dot southwards. We call a white square a white corner if it has no white neighbor either to the east or to the south. The essential set  $\mathcal{E}(\pi)$  of a permutation  $\pi$  is defined to be the set of white corners of the diagram of  $\pi$ . In other words, it is the set of southeast corners of the connected components of the diagram. For every white corner of  $\pi$ , its rank is defined by the number of dots northwest of the square.

The essential set, together with a rank function, has been introduced by Fulton [9]; it has been further studied by Eriksson and Linusson [7], [8].

Guibert and Linusson [12] have proved that the number of doubly alternating Baxter permutations is Catalan in 2000. They have proved it by the construction of the Rothe diagram and of the essential set. In this paper, we give an alternative proof of their result by the maximal-inversion-descent set which is defined in the next section. There is a difference between a maximal-inversion-descent set and an essential set; the essential set should be found by the diagram of the permutation while the maximal-inversion-descent set could be known from the permutation instantly. Guibert and Linusson [12] used the diagram of a permutation to prove it, but in this paper we prove it without using the diagram. In Section 3 we provide several properties of the maximal-inversion-descent set. In Section 4 we state the main theorem.

## 2. The maximal-inversion-descent set

DEFINITION 2.1. A *maximal-inversion* is a pair  $(i, b_i)$  of the permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & n \\ a_1 & a_2 & \cdots & a_i & \cdots & a_n \end{pmatrix},$$

where  $b_i$  is the maximum of  $a_k$ 's such that  $a_k < a_i$  for all  $k > i$ , that is to say,  $b_i = \max\{a_k \mid a_k < a_i, k > i\}$ . The set of all maximal-inversions of  $\pi$ , denoted by  $MI(\pi)$ , is called the *maximal-inversion set*.

EXAMPLE 2.2. Consider  $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 5 & 1 & 3 \end{pmatrix} \in S_6$ . Then the maximal-inversion set is

$$MI(\pi) = \{(1, 3), (2, 5), (3, 1), (4, 3)\}.$$

A somewhat more general result was proved by S. Min [14] and S. Min and S. Park [15].

DEFINITION 2.3. A *maximal-inversion-descent* of a permutation  $\pi$  in  $S_n$  is an element  $(i, b_i)$  in  $MI(\pi)$  with descent in position  $i$ . The *maximal-inversion-descent set* of  $\pi$ ,  $MID(\pi)$ , is the set of maximal-inversion-descents:  $MID(\pi) = \{(i, b_i) \in MI(\pi) \mid \pi(i) > \pi(i+1)\}$ .

EXAMPLE 2.4.

1. Let  $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 2 & 5 & 1 & 3 \end{pmatrix} \in S_6$ . Since  $\text{Des}(\pi) = \{2, 4\}$ , we know that  $MID(\pi) = \{(2, 5), (4, 3)\}$  by Example 2.2.

2. Let  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} \in S_5$ . Then  $MID(\sigma) = \{(2, 2), (4, 4)\}$ , since  $MI(\sigma) = \{(2, 2), (4, 4)\}$  and  $\text{Des}(\sigma) = \{2, 4\}$ .

REMARK 2.5. The number of the elements in  $MID(\pi)$  for every permutation  $\pi \in S_n$  is equal to the number of elements in  $\text{Des}(\pi)$ .

### 3. The property of the maximal-inversion-descent set

DEFINITION 3.1. (Fulton [9]) The essential set  $\mathcal{E}(\pi)$  of a permutation  $\pi \in S_n$  is defined as following:

$$\mathcal{E}(\pi) = \{(i, j) \in [1, n-1] \times [1, n-1] \mid \pi(i) > j, \pi^{-1}(j) > i, \\ \pi(i+1) \leq j, \pi^{-1}(j+1) \leq i\}.$$

EXAMPLE 3.2. If  $\pi = 4271635 \in S_7$ , then

$$\mathcal{E}(\pi) = \{(1, 3), (3, 1), (3, 3), (3, 6), (5, 3), (5, 5)\}.$$

EXAMPLE 3.3. If  $\pi = 4271635 \in S_7$ , then

$$MID(\pi) = \{(1, 3), (3, 6), (5, 5)\} \subseteq \mathcal{E}(\pi).$$

It is not appropriate to find out the essential set using only the definition of the permutation; therefore, we use a diagram. Eriksson and Linusson [8] have characterized the essential sets of Baxter permutations. By the definition of the maximal-inversion-descent set, we see that it is closely related to the essential set. Example 3.3 provides an example. In general, it is true that for any permutation  $\pi \in S_n$ ,  $MID(\pi) \subseteq \mathcal{E}(\pi)$ . But the converse is not true. We note that for a given permutation  $\pi$ , it is not easy to find the essential set of  $\pi$ , but it is easy to know the maximal-inversion-descent set of  $\pi$ .

PROPOSITION 3.4. For a given permutation  $\pi \in S_n$ , the maximal-inversion-descent set of  $\pi$  is a subset of the essential set of  $\pi$ , that is to say,  $MID(\pi) \subseteq \mathcal{E}(\pi)$ .

*Proof.* It follows easily from Definition 2.3 and Definition 3.1. Let  $(i, b_i) \in MID(\pi)$ . Then  $\pi(i) > \pi(i+1)$  and  $b_i$  is the maximum of  $\pi(k)$ 's such that  $\pi(k) < \pi(i)$  for all  $k > i$ . We know the following facts: (a)  $\pi(i) > b_i$ , (b)  $\pi^{-1}(b_i) > i$ , (c)  $\pi(i+1) \leq b_i$  since the maximality of  $b_i$  and  $\pi(i) > \pi(i+1)$ , and (d)  $\pi^{-1}(b_i+1) \leq i$ . (d) is verified as the following: if  $\pi^{-1}(b_i+1) > i$  then  $\pi(i)$  must be less than or equal to  $b_i+1$  by the

maximality of  $b_i$ , i.e.,  $b_i < \pi(i) \leq b_i + 1$ . Therefore,  $\pi(i) = b_i + 1$ , but it is impossible since  $\pi^{-1}(b_i + 1) > i$ . Hence  $(i, b_i) \in \mathcal{E}(\pi)$ .  $\square$

We need the following theorem to prove that the number of doubly alternating Baxter permutations is Catalan. It explains the relation between the maximal-inversion-descent set and the essential set of a Baxter permutation.

**THEOREM 3.5.** *If a permutation  $\pi$  is a Baxter, then  $MID(\pi) = \mathcal{E}(\pi)$ .*

*Proof.* Suppose that  $\pi$  is a Baxter permutation. By Proposition 3.4, it suffices to prove that  $\mathcal{E}(\pi) \subseteq MID(\pi)$ . Let  $(i, a) \in \mathcal{E}(\pi)$ . Equivalently,  $\pi(i) > a$ ,  $\pi^{-1}(a) > i$ ,  $\pi(i + 1) \leq a$ , and  $\pi^{-1}(a + 1) \leq i$ . Since  $\pi(i) > a \geq \pi(i + 1)$  (i.e.  $\pi(i) > \pi(i + 1)$ ), there exists unique  $b_i$  such that  $(i, b_i) \in MID(\pi)$  and by the maximality of  $b_i$ ,  $b_i \geq a$ . It is enough to show that  $a = b_i$ . Now we assume that  $a < b_i$ .

By Proposition 3.4,  $(i, b_i) \in \mathcal{E}(\pi)$ . Equivalently,  $\pi(i) > b_i$ ,  $\pi^{-1}(b_i) > i$ ,  $\pi(i + 1) \leq a (< b_i)$ , and  $\pi^{-1}(b_i + 1) \leq i$ . If  $(i, a), (i, b_i) \in \mathcal{E}(\pi)$  with  $a < b_i$ , then the permutation  $\pi$  is the form such that

$$\pi = \cdots (a + 1) \cdots \pi(i)\pi(i + 1) \cdots b_i \cdots ,$$

where it satisfies  $\pi(i) > b_i > a + 1 > \pi(i + 1)$ . It is proved as follows:

1.  $\pi^{-1}(b_i) > i + 1$ , since  $\pi^{-1}(b_i) > i$ , moreover, if  $\pi(i + 1) = b_i$ , then  $a \geq \pi(i + 1) = b_i$ , but it is impossible by the assumption,  $a < b_i$ ,
2.  $\pi^{-1}(a + 1) \leq i$ , since if  $\pi^{-1}(a + 1) = i$ , then  $a + 1 = \pi(i) > b_i$ , but it is impossible, and
3.  $a + 1 \leq b_i$ , since if  $a + 1 = b_i$ , then  $i < \pi^{-1}(b_i) = \pi^{-1}(a + 1) < i$ , but it is impossible.

Let  $b_i - a = l$ , where  $l = 2, 3, \dots$ . Now we get the following facts:

1.  $\pi(i), \pi(i + 1) \notin \{a + 1, a + 2, \dots, a + l\}$ , since  $\pi(i) > b_i = a + l$  and  $\pi(i + 1) < a + 1$ , and
2. there exists an integer  $k \in \{1, 2, \dots, l - 1\}$  such that  $\pi^{-1}(a + k) < i$  and  $\pi^{-1}(a + k + 1) > i$ , since consecutive numbers  $a + 1, a + 2, \dots, a + l$  are placed into the positions  $\pi(1), \dots, \pi(i - 1), \pi(i + 2), \dots, \pi(n)$  by 1,  $\pi^{-1}(a + 1) < i$ , and  $\pi^{-1}(a + l) (= \pi^{-1}(b_i)) > i$ .

Therefore we can construct a permutation  $\pi$ :

$$\pi = \cdots (a + k) \cdots \pi(i)\pi(i + 1) \cdots (a + k + 1) \cdots .$$

It satisfies  $\pi(i) > b_i \geq a + k + 1$ . But  $\pi(i + 1) < a + 1 < a + k + 1$ . Therefore,  $\pi$  is not a Baxter. It is a contradiction since  $\pi$  is a Baxter.  $\square$

Guibert and Linusson [12] have characterized the essential sets of doubly alternating Baxter permutations:

**PROPOSITION 3.6.** (Guibert and Linusson [12], Corollary 5) *A permutation is a doubly alternating Baxter permutation if and only if its essential set has exactly one square in each even row that is not the last row and in each even column that is not the last column.*

Now by Theorem 3.5 and Proposition 3.6, we characterize the maximal-inversion-descent set of doubly alternating Baxter permutations.

**COROLLARY 3.7.** *A permutation  $\pi$  is a doubly alternating Baxter permutation if and only if  $MID(\pi) = \{(i, j) | j = \sigma(i)\}$ , where  $\sigma$  is a bijection from  $\{2, 4, \dots, n - 1 - \lceil \frac{(-1)^n}{2} \rceil\}$  onto itself.*

#### 4. The main theorem

**THEOREM 4.1.** *If  $\pi$  is a Baxter permutation of length  $2n$  with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{2, 4, 6, \dots, 2n - 2\}$ , then*

$$\pi = (2k + 1) a_2 a_3 \cdots a_{2n-2k-1} (2n) a_{2n-2k+1} \cdots a_{2n}$$

such that  $\{a_{2n-2k+1}, \dots, a_{2n}\} = \{1, 2, \dots, 2k\}$ , where  $0 \leq k < n$ .

*Proof.* We will count the number of Baxter permutations of length  $2n$  with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{2, 4, \dots, 2n - 2\}$ . From Corollary 3.7, it follows that  $a_1$  is odd, say  $2k + 1$ .

Assume  $k = 0$ . Then it is enough to show that  $a_{2n} = 2n$ . If not, that is,  $a_i = 2n$  for some  $i = 2, 3, \dots, 2n - 1$ . Then, since  $2n = a_i > a_{i+1}$ , there exists an element  $(i, a_j) \in MID(\pi)$ , where  $a_j = \max\{a_{i+1}, a_{i+2}, \dots, a_{2n}\}$  and  $i$  and  $a_j$  are even by Corollary 3.7. By definition of  $a_j$ , we can write that  $a_j + 1 = a_m$  for some  $m = 2, 3, \dots, i - 1$ . Clearly,  $a_m$  is odd. First, if  $m$  is even, there must exist  $a_{m+1}$  which is less than  $a_m$  by Corollary 3.7, since  $m$  is even and  $a_m$  is odd. Thus there is a subsequence  $a_m (= a_j + 1), a_{m+1}, 2n, a_j$  which does not satisfy a Baxter condition. Second, if  $m$  is odd, the number of elements between  $a_m$  and  $2n$  is zero or more than 2. In case of zero,  $(i - 2, a_{i-1}) \in MID(\pi)$  since  $a_j$  is the maximum among the numbers  $a_{i+1}, a_{i+2}, \dots, a_{2n}$  and  $a_{i-2}$  is less than  $a_m = a_{i-1} = a_j + 1$ . Then by Corollary 3.7, it is a contradiction since  $a_{i-1}$  is odd. Otherwise,  $a_{m+1}, a_{m+2}, \dots, a_{i-1}$  must be larger than  $a_m$  since  $\pi$  is a Baxter. Thus,  $(m - 1, a_j + 1) \in MID(\pi)$ , but it is also a contradiction since  $a_m (= a_j + 1)$  is odd. Hence we conclude that  $a_{2n} = 2n$ .

Now, suppose that  $k \neq 0$ . Choose  $(i, 2k) \in MID(\pi)$ . Then by Corollary 3.7,  $i$  is even. Consider a permutation  $\pi$ :

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots & \cdot & \cdots \\ 2k+1 & a_2 & \cdots & a_i & \cdots & 2k & \cdots \end{pmatrix}.$$

Since  $(i, 2k) \in MID(\pi)$ , by Definition 2.3,  $a_i > 2k$ . By the Baxter condition,  $a_1 (= 2k + 1), a_2, \dots, a_i > 2k$ . Then there is a one-to-one correspondence between  $a_1 a_2 \cdots a_i$  and  $1 b_2 \cdots b_i$ , where  $b_j = a_j - 2k$  for all  $j = 2, 3, \dots, i$ . Thus  $1 b_2 \cdots b_i$  is also a doubly alternating Baxter permutation. Applying the similar arguments used in the proof of case  $k = 0$ ,  $i = b_i$  must be  $2n - 2k$ . Hence,  $a_{2n-2k} = 2n$  and the result follows.  $\square$

EXAMPLE 4.2. A Baxter permutation  $\pi = 5\ 7\ 6\ (15)\ (13)\ (14)\ 9\ (11)\ (10)\ (12)\ 8\ (17)\ (16)\ (18)\ 1\ 3\ 2\ 4$  of with  $Des(\pi) = Des(\pi^{-1}) = \{2, 4, \dots, 16\}$  length 18 is decomposed into two parts;  $5\ 7\ 6\ (15)\ (13)\ (14)\ 9\ (11)\ (10)\ (12)\ 8\ (17)\ (16)\ (18)$  and  $1\ 3\ 2\ 4$ , where  $5\ 7\ 6\ (15)\ (13)\ (14)\ 9\ (11)\ (10)\ (12)\ 8\ (17)\ (16)\ (18)$  is a one-to-one correspondence to the permutation  $1\ 3\ 2\ (11)\ 9\ (10)\ 5\ 7\ 6\ 8\ 4\ (13)\ (12)\ (14)$ .

Considering the definition, it is clear that if  $\pi = a_1 a_2 \cdots a_n$  of length  $n$  is a Baxter permutation then its complement  $\pi^*$  defined by  $\pi^* = (n + 1 - a_1)(n + 1 - a_2) \cdots (n + 1 - a_n)$  is also a Baxter permutation. Now, we know that  $a_2 a_3 \cdots a_{2n-2k-1}$  is isomorphic to any permutation of the mirrored complement of a doubly alternating Baxter permutation of length  $2n - 2k - 2$ , that is to say, with descents in odd rows and columns. The following corollary derives directly from the above explanation and Remark 1.1.

COROLLARY 4.3. (Guibert and Linusson [12], Corollary 8) *If  $\pi$  is a Baxter permutation of length  $2n + 1$  with  $Des(\pi) = Des(\pi^{-1}) = \{1, 3, \dots, 2n - 1\}$ , then  $\pi(2n + 1) = 2n + 1$ . In particular, we have  $d_{2n} = d_{2n+1}$ , where  $d_{2n+\epsilon}$  is the number of doubly alternating Baxter permutations of length  $2n + \epsilon$  ( $\epsilon = 0$  or  $1$ ).*

COROLLARY 4.4. (Guibert and Linusson [12], Theorem 3)  *$d_{2n+\epsilon}$  is the Catalan number  $c_n = \frac{1}{n+1} \binom{2n}{n}$ .*

*Proof.* By Theorem 4.1 and Corollary 4.3, we get the recursion

$$d_{2n} = \sum_{k=0}^{n-1} d_{2n-2k-2} \cdot d_{2k} = \sum_{k=0}^{n-1} d_{2n-2k-1} \cdot d_{2k}.$$

This is the well-known recursion for the Catalan number.  $\square$

The next corollary follows directly from the definition of the complement of a permutation of length  $2n$ . For example, 4231 and 2143 are the only permutations which are the Baxter permutations of length 4 with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{1, 3\}$ .

**COROLLARY 4.5.** *If  $\pi = a_1 a_2 \cdots a_{2n}$  is a Baxter permutation with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{1, 3, \dots, 2n - 1\}$ , then  $\pi = (2t) a_2 \cdots a_{2t-1} 1 a_{2t+1} \cdots a_{2n}$  such that  $\{1, a_{2t-1}, \dots, a_2, 2t\} = \{1, 2, \dots, 2t\}$ , where  $1 \leq t \leq n$ .*

**EXAMPLE 4.6.** A Baxter permutation  $\pi = (14)(12)(13)465(10)897(11)231(18)(16)(17)(15)$  of length 18 with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{1, 3, \dots, 17\}$  is decomposed into two parts;  $(14)(12)(13)465(10)897(11)231$  and  $(18)(16)(17)(15)$ , where  $(18)(16)(17)(15)$  is a one-to-one correspondence to the permutation 4231.

The next corollary follows directly from Corollary 4.3 and Corollary 4.5.

**COROLLARY 4.7.** *If  $\pi = a_1 a_2 \cdots a_{2n+1}$  is a Baxter permutation with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{2, 4, \dots, 2n\}$ . Then  $a_1 = 1$  and  $(a_2 - 1)(a_3 - 1) \cdots (a_{2n+1} - 1)$  is a Baxter permutation with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{1, 3, \dots, 2n - 1\}$ .*

**EXAMPLE 4.8.**  $\pi = 1(15)(13)(14)576(11)9(10)8(12)342(19)(17)(18)(16)$  is a Baxter permutation with  $\text{Des}(\pi) = \text{Des}(\pi^{-1}) = \{2, 4, \dots, 18\}$ .

**EXAMPLE 4.9.** 15342 and 13254 are the only permutations of length 5 satisfying Corollary 4.7.

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