J. Korean Math. Soc. **43** (2006), No. 3, pp. 579–591

A GENERALIZATION OF A RESULT OF CHOA ON ANALYTIC FUNCTIONS WITH HADAMARD GAPS

Stevo Stević

Abstract. In this paper we obtain a sufficient and necessary condition for an analytic function f on the unit ball B with Hadamard gaps, that is, for $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ (the homogeneous polynomial expansion of f) satisfying $n_{k+1}/n_k \geq \lambda > 1$ for all $k \in \mathbf{N}$, to belong to the weighted Bergman space

$$\mathcal{A}^p_\alpha(B) = \{f \mid \int_B |f(z)|^p(1-|z|^2)^\alpha dV(z) < \infty,\ f \in H(B)\}.$$

We find a growth estimate for the integral mean

$$\left(\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p},$$

and an estimate for the point evaluations in this class of functions. Similar results on the mixed norm space $H_{p,q,\alpha}(B)$ and weighted Bergman space on polydisc $\mathcal{A}^p_{\vec{\alpha}}(U^n)$ are also given.

1. Introduction and preliminaries

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in the complex vector space $\mathbf{C}^n$. By $\langle z, \omega\rangle = \sum_{k=1}^{n} z_k \bar{w}_k$ we denote the complex inner product of z and w, and $|z| = \sqrt{\langle z, z\rangle}$. Let U denote the unit disc in the complex plane, $dm(z) = r dr d\theta/\pi$ the normalized Lebesgue area measure on U, B the unit ball of $\mathbf{C}^n$, $B(a,r)$ the open unit ball centered at a of radius r, dV the normalized Lebesgue measure on B, $d\sigma$ the normalized surface measure on the boundary S of B, and $P(0, r_1, r_2) = \{w \,|\, r_1 < |w| < r_1\}$. By $H(B)$ we denote the class of all functions analytic in B.

Received February 7, 2005.

2000 Mathematics Subject Classification: Primary 32A35.

Key words and phrases: analytic functions, Hadamard gap, Bergman space, unit ball.

For an $f \in H(B)$, the radial derivative $\mathcal{R}f$ of f is defined by

$$\mathcal{R}f(z) = \sum_{k=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \sum_{k=0}^{\infty} kP_k(z)$$

if $f(z) = \sum_{k=0}^{\infty} P_k(z)$ is the homogeneous polynomial expansion of f.

As usual, we write

$$||f||_p = \left(\int_S |f(\zeta)|^p d\sigma(\zeta)\right)^{1/p}$$

if $p \in (0, \infty)$.

The expression $A \asymp B$ means that there are finite positive constants C and C' such that

$$CA \leq B \leq C'A.$$

In [6] J. Miao investigates analytic functions f with Hadamard gaps on the unit disk U, that is, those f such that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ where $n_{k+1}/n_k \geq \lambda > 1$ for all $k \in \mathbf{N}$, which belong to the space B^p defined as follows

$$B^p = \{f \in H(U) \mid ||f||_{B^p} < \infty\},$$

where

$$||f||_{B^p} = \sup_{a \in U} \left(\int_U |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2) dm(z)\right)^{1/p},$$

$\varphi_a(z) = (a - z)/(1 - \bar{a}z)$, or to its subspace B_0^p consisting of those f such that

$$\lim_{|a| \to 1-0} \int_U |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_a(z)|^2) dm(z) = 0.$$

He proves the following result.

THEOREM A. *Let $p \in (0, \infty)$. If $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is an analytic function on U such that $n_{k+1}/n_k \geq \lambda > 1$ for all $k \in \mathbf{N}$, then the following statements are equivalent:*

(i) $f \in B^p$;
(ii) $f \in B_0^p$;
(iii) $\sum_{k=1}^{\infty} |a_k|^p < \infty$.

Some other classical results of this type can be found, for example, in [4, 5, 13, 14].

An analytic function on B with the homogeneous expansion $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ (here, P_{n_k} is a homogeneous polynomial of degree n_k) is said to have Hadamard gaps if $n_{k+1}/n_k \geq \lambda > 1$ for all $k \in \mathbf{N}$. In [3], among others, J. S. Choa generalizes Theorem A proving the following result.

THEOREM B. *Let $p \in (0,\infty)$ and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ be an analytic function on B with Hadamard gaps. Then the following statements are equivalent*:

(i) $||f||_{X_p} = \left(\int_B |\mathcal{R}f(z)|^p(1-|z|^2)^{p-1}dV(z)\right)^{1/p} < \infty$;
(ii) $\sum_{k=1}^{\infty} ||P_{n_k}||_p^p < \infty$.

The weighted Bergman space $\mathcal{A}_\alpha^p = \mathcal{A}_\alpha^p(B)$, $\alpha > -1$, $p > 0$, is the space of all analytic functions f on B for which

$$||f||_{\mathcal{A}_\alpha^p} = \left(\int_B |f(z)|^p(1-|z|^2)^\alpha dV(z)\right)^{1/p} < \infty.$$

The weighted Bergman space on the unit disk, polydisc or on the unit ball has been investigated recently a great deal, see, for example, [1, 2, 7, 8, 9, 10, 11, 12] and the references in there.

Motivated by Theorems A and B, in this paper, we investigate analytic functions with Hadamard gaps, which belong to the weighted Bergman space $\mathcal{A}_\alpha^p(B)$, the mixed norm space (see, Section 3) and on the weighted Bergman space on U^n (Section 4). One of the main results is the following theorem.

THEOREM 1. *Let $p \in (0,\infty)$, $\alpha > -1$ and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ be an analytic function on B with Hadamard gaps. Then the following statements are equivalent*:

(i) $f \in \mathcal{A}_\alpha^p$;
(ii) $\sum_{k=0}^{\infty} \frac{||P_{n_k}||_p^p}{n_k^{\alpha+1}} < \infty$.

Now we gather auxiliary results which are used in the proof of the main result. The following two lemmas can be found, for example, in [6].

LEMMA 1. [14] *Let $p \in (0,\infty)$. If (n_k) is an increasing sequence of positive integers satisfying $n_{k+1}/n_k \geq \lambda > 1$ for all k, then there is a positive constant A depending only on p and λ such that*

$$\frac{1}{A}\left(\sum_{k=1}^{\infty}|a_k|^2\right)^{1/2} \leq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{\infty} a_k e^{in_k\theta}\right|^p d\theta\right)^{1/p} \leq A\left(\sum_{k=1}^{\infty}|a_k|^2\right)^{1/2}$$

for any number $a_k, k \in \mathbf{N}$.

LEMMA 2. *Let $\alpha > 0$, $p > 0$, $n \geq 0$, $a_n \geq 0$, $I_n = \{k \mid 2^n \leq k < 2^{n+1},\ k \in \mathbf{N}\}$, $t_n = \sum_{k\in I_n} a_k$ and $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Then there is a positive constant K depending only on p and α such that*

$$\frac{1}{K}\sum_{n=0}^{\infty}\frac{t_n^p}{2^{n\alpha}} \leq \int_0^1 (1-x)^{\alpha-1} f^p(x)dx \leq K\sum_{n=0}^{\infty}\frac{t_n^p}{2^{n\alpha}}.$$

The following lemma is well known.

LEMMA 3. *Let $a_n \geq 0$ and $n_0 \in \mathbf{N}$. Then for $p \in (0,1]$,*

$$\frac{1}{n_0^{1-p}}\left(\sum_{n=1}^{n_0} a_n^p\right) \leq \left(\sum_{n=1}^{n_0} a_n\right)^p \leq \left(\sum_{n=1}^{n_0} a_n^p\right)$$

and for $p \geq 1$,

$$\left(\sum_{n=1}^{n_0} a_n^p\right) \leq \left(\sum_{n=1}^{n_0} a_n\right)^p \leq {n_0}^{p-1}\left(\sum_{n=1}^{n_0} a_n^p\right).$$

2. Proofs of the main results

In this section first we prove Theorem 1.

Proof of Theorem 1. As in the proof of Proposition 2 in [3], first we use polar coordinates and Proposition 1.4.7 of [7]. We have

$$\begin{aligned}
&||f||^p_{\mathcal{A}^p_\alpha}\\
&= 2n\int_0^1\int_S |f(r\zeta)|^p d\sigma(\zeta)(1-r^2)^\alpha r^{2n-1}dr\\
&= 2n\int_0^1\int_S \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta}\zeta)|^p d\theta d\sigma(\zeta)(1-r^2)^\alpha r^{2n-1}dr\\
&= 2n\int_0^1\int_S \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^{\infty} P_{n_k}(\zeta) r^{n_k} e^{in_k\theta}\right|^p d\theta d\sigma(\zeta)(1-r^2)^\alpha r^{2n-1}dr.
\end{aligned}$$

By the second inequality in Lemma 1 and the change $\rho = r^2$, we get

$$\begin{aligned}
||f||^p_{\mathcal{A}^p_\alpha} &\leq 2nA^p\int_0^1\int_S \left(\sum_{k=1}^{\infty}|P_{n_k}(\zeta)|^2 r^{2n_k}\right)^{p/2} d\sigma(\zeta)(1-r^2)^\alpha r^{2n-1}dr\\
&= nA^p\int_0^1\int_S \left(\sum_{k=1}^{\infty}|P_{n_k}(\zeta)|^2 \rho^{n_k}\right)^{p/2} d\sigma(\zeta)(1-\rho)^\alpha \rho^{n-1}d\rho
\end{aligned}$$

$$\le nA^p \int_0^1 \int_S \left(\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k} \right)^{p/2} d\sigma(\zeta)(1-\rho)^\alpha d\rho.$$

By the first inequality in Lemma 1 and the change $\rho = r^{2n}$, it follows that

$$\begin{aligned} ||f||^p_{\mathcal{A}^p_\alpha} &\ge \frac{2n}{A^p} \int_0^1 \int_S \left(\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 r^{2n_k} \right)^{p/2} d\sigma(\zeta)(1-r^2)^\alpha r^{2n-1} dr \\ &= \frac{1}{A^p} \int_0^1 \int_S \left(\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k/n} \right)^{p/2} d\sigma(\zeta)(1-\rho^{1/n})^\alpha d\rho \\ &\ge \frac{C(\alpha)}{A^p} \int_0^1 \int_S \left(\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k} \right)^{p/2} d\sigma(\zeta)(1-\rho)^\alpha d\rho, \end{aligned}$$

since $\rho^{1/n} \ge \rho$ for $\rho \in [0,1]$, which implies

$$\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k/n} \ge \sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k}$$

and since

$$(1-\rho^{1/n})^\alpha = \frac{(1-\rho)^\alpha}{(1+\rho^{1/n}+\rho^{2/n}+\cdots+\rho^{(n-1)/n})^\alpha} \ge C(\alpha)(1-\rho)^\alpha,$$

where $C(\alpha) = 1/n^\alpha$ when $\alpha \ge 0$ and $C(\alpha) = 1$ when $\alpha \in (-1,0)$.

Thus

$$||f||^p_{\mathcal{A}^p_\alpha} \asymp \int_S \int_0^1 \left(\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k} \right)^{p/2} (1-\rho)^\alpha d\rho d\sigma(\zeta).$$

By Lemma 2 applied to the integral

$$\int_0^1 \left(\sum_{k=1}^{\infty} |P_{n_k}(\zeta)|^2 \rho^{n_k} \right)^{p/2} (1-\rho)^\alpha d\rho,$$

we obtain

$$||f||^p_{\mathcal{A}^p_\alpha} \asymp \int_S \left(\sum_{k=0}^{\infty} \frac{1}{2^{(\alpha+1)k}} \left(\sum_{n_m \in I_k} |P_{n_m}(\zeta)|^2 \right)^{p/2} \right) d\sigma(\zeta).$$

Because $n_{k+1}/n_k \ge \lambda > 1$ for all $k \in \mathbf{N}$, the number of P_{n_m} when $n_m \in I_k$ is at most $[\log_\lambda 2] + 1$. Using this fact and Lemma 3 it follows that

$$||f||^p_{\mathcal{A}^p_\alpha} \asymp \int_S \left(\sum_{k=0}^{\infty} \frac{1}{2^{(\alpha+1)k}} \sum_{n_m \in I_k} |P_{n_m}(\zeta)|^p \right) d\sigma(\zeta)$$

$$\asymp \sum_{k=0}^{\infty} \frac{1}{2^{(\alpha+1)k}} \sum_{n_m \in I_k} ||P_{n_m}||^p_p.$$

Form this and since $n_m \asymp 2^k$ when $n_m \in I_k$, we get

$$||f||^p_{\mathcal{A}^p_\alpha} \asymp \sum_{k=0}^{\infty} \frac{||P_{n_k}||^p_p}{n_k^{\alpha+1}}$$

as desired. □

COROLLARY 1. *Let $p \in (0, \infty)$, $\alpha > -1$ and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ be an analytic function on B with Hadamard gaps. Then the following statements are equivalent:*

(i) $\mathcal{R}^{(l)} f \in \mathcal{A}^p_\alpha$;
(ii) $\sum_{k=0}^{\infty} \frac{||P_{n_k}||^p_p}{n_k^{\alpha+1-lp}} < \infty$.

Proof. Since f has Hadamard gaps and $\mathcal{R}^{(l)} f(z) = \sum_{k=1}^{\infty} n_k^l P_{n_k}(z)$ it follows that $\mathcal{R}^{(l)} f$ has Hadamard gaps too. Applying Theorem 1 to the function $\mathcal{R}^{(l)} f$ we obtain that $\mathcal{R}^{(l)} f \in \mathcal{A}^p_\alpha$ if and only if

$$\sum_{k=0}^{\infty} \frac{||n_k^l P_{n_k}||^p_p}{n_k^{\alpha+1}} = \sum_{k=0}^{\infty} \frac{||P_{n_k}||^p_p}{n_k^{\alpha+1-lp}} < \infty,$$

finishing the proof. □

REMARK 1. Setting $l = 1$ and $\alpha = p - 1$ in Corollary 1, we obtain Theorem B.

Since for $n = 1$, $||P_{n_k}||^p_p = |a_k|^p$, $k \in \mathbf{N}$, from Theorem 1 it follows the following corollary:

COROLLARY 2. *Let $p \in (0, \infty)$, $\alpha > -1$ and let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be an analytic function on U with Hadamard gaps. Then the following statements are equivalent:*

(i) $f \in \mathcal{A}^p_\alpha(U)$;
(ii) $\sum_{k=0}^{\infty} \frac{|a_k|^p}{n_k^{\alpha+1}} < \infty$.

REMARK 2. Theorem 1 gives an estimation of the growth of the sequence $||P_{n_k}||_p$. Since the series

$$\sum_{k=0}^{\infty} \frac{||P_{n_k}||_p^p}{n_k^{\alpha+1}}$$

is convergent, we have

$$||P_{n_k}||_p = o\big(n_k^{(\alpha+1)/p}\big).$$

THEOREM 2. *Let $p \in (0, \infty)$, $\alpha > -1$ and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ be an analytic function on B with Hadamard gaps belonging to $\mathcal{A}_\alpha^p$. Then*

$$||f(z)||_p = o\left(\frac{1}{(1-|z|)^{(\alpha+1)/p}}\right) \quad \textit{as } |z| \to 1-0.$$

Proof. Let $z = r\zeta = |z|\zeta$ and $I_n = \{k \mid 2^n \le k < 2^{n+1}, k \in \mathbf{N}\}$. Let first $p \ge 1$. From Remark 2 we have that for every $\varepsilon > 0$ there is an $k_0 \in \mathbf{N}$ such that

$$(1) \qquad ||P_{n_k}||_p \le \varepsilon n_k^{(\alpha+1)/p} \quad \text{for} \quad k \ge k_0.$$

Without loss of generality we may assume that $k_0 = 1$, since for every $l \in \mathbf{N}$,

$$\lim_{|z|\to 1-0} (1-|z|)^{(\alpha+1)/p} \sum_{k=1}^{l} ||P_{n_k}(z)||_p = 0.$$

We have

$$\begin{aligned}
||f(z)||_p &\le \sum_{k=1}^{\infty} ||P_{n_k}(z)||_p = \sum_{k=1}^{\infty} ||P_{n_k}(\zeta)||_p |z|^{n_k} \\
&\le \varepsilon \sum_{k=1}^{\infty} n_k^{(\alpha+1)/p} |z|^{n_k} \\
(2) \qquad &\le \varepsilon \sum_{k=0}^{\infty} \sum_{n_m \in I_k} n_m^{(\alpha+1)/p} |z|^{n_m} \\
&\le \varepsilon \sum_{k=0}^{\infty} 2^{(k+1)(\alpha+1)/p} \sum_{n_m \in I_k} |z|^{n_m} \\
&= \varepsilon \sum_{k=0}^{\infty} 2^{(k+1)[(\alpha+1)/p-1]} 2^{k+1} \sum_{n_m \in I_k} |z|^{n_m}
\end{aligned}$$

$$\leq 4\varepsilon\left(C_1|z| + \sum_{k=1}^{\infty} 2^{(k+1)[(\alpha+1)/p-1]} \sum_{n_m\in I_k} 2^{k-1}|z|^{2^k}\right)$$

$$\leq 4\varepsilon C_1|z| + C_2\sum_{k=1}^{\infty} 2^{(k-1)[(\alpha+1)/p-1]} \sum_{m\in I_{k-1}} |z|^m$$

$$\leq 4\varepsilon C_1|z| + C_2\sum_{k=1}^{\infty} k^{[(\alpha+1)/p-1]}|z|^k,$$

where $C_1 = 2^{(\alpha+1)/p-2}\chi_{I_0}(n_1)$, $\chi_{I_0}(\cdot)$ the characteristic function of the set I_0, and $C_2 = 2^{2(\alpha+1)/p}\varepsilon([\log_\lambda 2]+1)$. From (2) it follows that there is a positive constant C such that

$$||f(z)||_p \leq \varepsilon C\sum_{k=0}^{\infty}(k+1)^{[(\alpha+1)/p-1]}|z|^k, \quad z\in B.$$

It is well known [14, p.77] that

$$\sum_{k=0}^{\infty}(k+1)^{[(\alpha+1)/p-1]}|z|^k \asymp (1-|z|)^{-(\alpha+1)/p}, \tag{3}$$

from which the result follows in this case.

Assume now that $p\in(0,1)$. As in the first case we may assume that (1) holds for every $k\in\mathbf{N}$. Then it holds

$$||f(z)||_p^p \leq \sum_{k=1}^{\infty}||P_{n_k}(z)||_p^p \leq \sum_{k=1}^{\infty}||P_{n_k}(\zeta)||_p^p|z|^{pn_k} \leq \varepsilon\sum_{k=1}^{\infty}n_k^{(\alpha+1)}|z|^{pn_k}.$$

As in the first case we obtain that there is a constant $M_1>0$ such that

$$||f(z)||_p^p \leq \varepsilon M_1\sum_{k=0}^{\infty}(k+1)^{\alpha}|z|^{kp}.$$

Using (3) and the inequality $(1-x)^p \leq 1-px$, for $x\in[0,1]$ and $p\in[0,1]$, we obtain

$$||f(z)||_p^p \leq \frac{\varepsilon M_2}{(1-|z|^p)^{\alpha+1}} \leq \frac{\varepsilon M_2}{(p(1-|z|))^{\alpha+1}},$$

as desired. □

THEOREM 3. *Let $p\in(0,\infty)$, $\alpha>-1$ and $f(z)=\sum_{k=1}^{\infty}P_{n_k}(z)$ be an analytic function on B with Hadamard gaps belonging to $\mathcal{A}_\alpha^p$. Then*

$$|f(z)| = o\left(\frac{1}{(1-|z|)^{(n+\alpha)/p}}\right) \quad \text{as } |z|\to 1-0. \tag{4}$$

Proof. By subharmonicity of $|f|^p$, $p > 0$, we obtain

$$|f(z)|^p \le \frac{2^n}{(1-|z|)^n} \int_{B(z,(1-|z|)/2)} |f(w)|^p dV(w). \tag{5}$$

For $w \in B(z,(1-|z|)/2)$, we have

$$\frac{1}{2}(1-|z|) < (1-|w|) < \frac{3}{2}(1-|z|). \tag{6}$$

By Theorem 2 it follows that for every $\varepsilon > 0$ there is a $\delta > 1/3$ such that

$$||f(z)||_p^p \le \frac{\varepsilon}{(1-|z|)^{\alpha+1}}, \quad |z| > \delta. \tag{7}$$

From (5)–(7) and by polar coordinates, we have

$$\begin{aligned}
|f(z)|^p &\le \frac{C}{(1-|z|)^{n+\alpha}} \int_{B(z,(1-|z|)/2)} |f(w)|^p (1-|w|)^\alpha dV(w) \\
&\le \frac{C}{(1-|z|)^{n+\alpha}} \int_{P(0,\frac{3|z|-1}{2},\frac{1+|z|}{2})} |f(w)|^p (1-|w|)^\alpha dV(w) \\
&= \frac{2nC}{(1-|z|)^{n+\alpha}} \int_{\frac{3|z|-1}{2}}^{\frac{1+|z|}{2}} ||f(r\zeta)||_p^p (1-r)^\alpha r^{2n-1} dr \\
&\le \frac{2nC\varepsilon}{(1-|z|)^{n+\alpha}} \int_{\frac{3|z|-1}{2}}^{\frac{1+|z|}{2}} \frac{dr}{1-r} \\
&= \frac{2nC\varepsilon \ln 3}{(1-|z|)^{n+\alpha}},
\end{aligned}$$

from which the result follows. □

Remark 3. The basic estimate for the point evaluations on the weighted Bergman space is the following (see, Corollary 3.5 in [1])

$$|f(z)| \le \frac{||f||_{\mathcal{A}^p_\alpha}}{(1-|z|)^{(n+\alpha)/p}}, \tag{8}$$

the equality holds at some $z \in B$ if and only if $f(w) = \lambda(1-\langle z,w\rangle)^{-2(n+\alpha)/p}$ for some $\lambda \in \mathbf{C}$ and every $w \in B$.

Thus, our Theorem 3 can be considered as an improvement of estimate (8) in the subclass of $\mathcal{A}^p_\alpha$ consisting of the functions having Hadamard gaps.

3. The case of mixed norm space

In this section we consider analytic functions with Hadamard gaps on the mixed norm space. The mixed norm space $H_{p,q,\alpha}(B)$, $p, q > 0$ and $\alpha \in (-1, \infty)$, consists of all $f \in H(B)$ such that

$$||f||_{p,q,\alpha} = \left(\int_0^1 ||f(r\zeta)||_p^q (1-r)^\alpha dr \right)^{1/p} < \infty.$$

Note that when $p = q$, $H_{p,q,\alpha}(B)$ is just weighted Bergman space. For $f \in H_{p,q,\alpha}(B)$, the following result holds:

THEOREM 4. *Let $f \in H_{p,q,\alpha}$ and $f(z) = \sum_{k=1}^\infty P_{n_k}(z)$ be an analytic function on B with Hadamard gaps. Then the following statements holds:*

(i) *If $p \in (0, 2]$, then $\sum_{k=0}^\infty \frac{||P_{n_k}||_p^q}{n_k^{\alpha+1}} < \infty$ implies $f \in H_{p,q,\alpha}(B)$;*

(ii) *If $p \geq 2$, then $f \in H_{p,q,\alpha}(B)$ implies $\sum_{k=0}^\infty \frac{||P_{n_k}||_p^q}{n_k^{\alpha+1}} < \infty$.*

Proof. (i) Similarly to the proof of Theorem 1 we have

$$\begin{aligned}
&||f||^p_{H_{p,q,\alpha}} \\
&= \int_0^1 \left(\int_S \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^\infty P_{n_k}(\zeta) r^{n_k} e^{in_k\theta} \right|^p d\theta d\sigma(\zeta) \right)^{q/p} (1-r)^\alpha dr \\
&\leq A^p \int_0^1 \left(\int_S \left(\sum_{k=1}^\infty |P_{n_k}(\zeta)|^2 r^{2n_k} \right)^{p/2} d\sigma(\zeta) \right)^{q/p} (1-r)^\alpha dr \\
&\leq A^p \int_0^1 \left(\int_S \sum_{k=1}^\infty |P_{n_k}(\zeta)|^p r^{pn_k} d\sigma(\zeta) \right)^{q/p} (1-r)^\alpha dr \\
&= A^p \int_0^1 \left(\sum_{k=1}^\infty ||P_{n_k}||_p^p r^{pn_k} \right)^{q/p} (1-r)^\alpha dr \\
&\asymp \int_0^1 \left(\sum_{k=1}^\infty ||P_{n_k}||_p^p \rho^{n_k} \right)^{q/p} (1-\rho)^\alpha d\rho \\
&\asymp \sum_{k=0}^\infty \frac{1}{2^{(\alpha+1)k}} \left(\sum_{n_m \in I_k} ||P_{n_m}||_p^p \right)^{q/p} \asymp \sum_{k=0}^\infty \frac{||P_{n_k}||_p^q}{n_k^{\alpha+1}},
\end{aligned}$$

where in the second inequality we used the condition $p/2 \le 1$ and in the first asymptotic relation "$\asymp$" the change $\rho = r^p$.

(ii) Since $p \ge 2$, in the above sequence of relations the reverse inequalities hold, from which the result follows.

Theorem 4, by a similar argument to the proof of Theorem 2, gives the next corollary.

COROLLARY 3. *Let $p \ge 2$, $\alpha > -1$ and $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ be an analytic function on B with Hadamard gaps belonging to $H_{p,q,\alpha}(B)$. Then*

$$||f(z)||_p = o\left(\frac{1}{(1-|z|)^{(\alpha+1)/q}}\right) \quad as\ |z| \to 1-0.$$

4. A version of Theorem 1 on the polydisc

Let $\mathcal{L}^p_{\vec{\alpha}}(U^n)$ denote the class of all measurable functions defined on the unit polydisc $U^n = \{z \in \mathbf{C}^n \mid |z_i| < 1, i = 1, \dots, n\}$ such that

$$||f||^p_{\mathcal{L}^p_{\vec{\alpha}}(U^n)} = \int_{U^n} |f(z)|^p \prod_{j=1}^{n} (1-|z_j|^2)^{\alpha_j} dm(z_j) < \infty,$$

where $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1, j = 1, \dots, n$. The weighted Bergman space on U^n is defined as $\mathcal{A}^p_{\vec{\alpha}}(U^n) = \mathcal{L}^p_{\vec{\alpha}}(U^n) \cap H(U^n)$.

THEOREM 5. *Let $p \in (0, \infty)$, $\alpha_j > -1$, $j = 1, \dots, n$, and*

$$f(z) = \sum_{\mathbf{k}} a_{\mathbf{k}} z^{\mathbf{m_k}} = \sum_{k_1,\dots,k_n \ge 1} a_{k_1,\dots,k_n} z_1^{m_{k_1}} \cdots z_n^{m_{k_n}}$$

be an analytic function on U^n, with Hadamard gaps in each variable, that is, there are $\lambda_j > 1$, $j = 1, \dots, n$, such that $m_{k_j+1}/m_{k_j} \ge \lambda_j$ for every $k_j \in \mathbf{N}$. Then the following statements are equivalent:

(i) $f \in \mathcal{A}^p_{\vec{\alpha}}(U^n)$;

(ii) $\sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{|a_{k_1,\dots,k_n}|^p}{\prod_{j=1}^{n} m_{k_j}^{\alpha_j+1}} < \infty$.

Proof. In order to avoid too much calculations we may assume that $n = 2$. It means that the function f can be written as follows

$$f(z_1, z_2) = \sum_{k1,k_2 \ge 1} a_{k_1,k_2} z_1^{m_{k_1}} z_2^{m_{k_2}} = \sum_{k_1=1}^{\infty} P_{k_1}(z_2) z_1^{m_{k_1}}, \tag{9}$$

where

$$P_{k_1}(z_2) = \sum_{k_2=1}^{\infty} a_{k_1,k_2} z_2^{m_{k_2}}. \tag{10}$$

Let $z_j = r_j e^{i\theta_j}$, $j = 1, 2$. Then from (9) and by Fubini's theorem we have

$$\begin{aligned} ||f||^p_{\mathcal{A}^p_{\vec{\alpha}}} &= \frac{1}{\pi^2} \times \int_0^1 \int_0^{2\pi} \left(\int_0^1 \int_0^{2\pi} \left| \sum_{k_1=1}^{\infty} P_{k_1}(z_2) z_1^{m_{k_1}} \right|^p \right. \\ &\qquad \left. d\theta_1 (1 - r_1^2)^{\alpha_1} r_1 dr_1 \right) d\theta_2 (1 - r_2^2)^{\alpha_2} r_2 dr_2. \end{aligned} \tag{11}$$

Fix z_2 for a moment. By Corollary 2 applied to the function

$$g_{z_2}(z_1) = \sum_{k_1=1}^{\infty} P_{k_1}(z_2) z_1^{m_{k_1}}$$

and to the integral

$$I(z_2) = \int_0^1 \int_0^{2\pi} \left| \sum_{k_1=1}^{\infty} P_{k_1}(z_2) z_1^{m_{k_1}} \right|^p d\theta_1 (1 - r_1^2)^{\alpha_1} r_1 dr_1$$

we obtain

$$I(z_2) \asymp \sum_{k_1=1}^{\infty} \frac{|P_{k_1}(z_2)|^p}{m_{k_1}^{\alpha_1+1}}. \tag{12}$$

Substituting (12) in (10) and applying Corollary 2 to the functions in (11), it follows that

$$\begin{aligned} &||f||^p_{\mathcal{A}^p_{\vec{\alpha}}} \\ &\asymp \sum_{k_1=1}^{\infty} \frac{1}{m_{k_1}^{\alpha_1+1}} \int_0^1 \int_0^{2\pi} |P_{k_1}(z_2)|^p d\theta_2 (1 - r_2^2)^{\alpha_2} r_2 dr_2 \\ &\asymp \sum_{k_1=1}^{\infty} \frac{1}{m_{k_1}^{\alpha_1+1}} \int_0^1 \int_0^{2\pi} \left| \sum_{k_2=1}^{\infty} a_{k_1,k_2} r_2^{m_{k_2}} e^{i\theta_2 m_{k_2}} \right|^p d\theta_2 (1 - r_2^2)^{\alpha_2} r_2 dr_2 \\ &\asymp \sum_{k_1=1}^{\infty} \frac{1}{m_{k_1}^{\alpha_1+1}} \sum_{k_2=1}^{\infty} \frac{|a_{k_1,k_2}|^p}{m_{k_2}^{\alpha_2+1}} = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{|a_{k_1,k_2}|^p}{m_{k_1}^{\alpha_1+1} m_{k_2}^{\alpha_2+1}}, \end{aligned}$$

as desired. □

References

[1] F. Beatrous and J. Burbea, *Holomorphic Sobolev spaces on the ball*, Dissertationes Math. **276** (1989), 1–57.

[2] G. Benke and D. C. Chang, *A note on weighted Bergman spaces and the Cesáro operator*, Nagoya Math. J. **159** (2000), 25–43.

[3] J. S. Choa, *Some properties of analytic functions on the unit ball with Hadamard gaps*, Complex Variables Theory Appl. **29** (1996), no. 3, 277–285.

[4] P. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics Vol. 38, Academic Press, New York, 1970.

[5] J. H. Mathews, *Coefficients of uniformly normal-Bloch functions*, Yokohama Math. J. **21** (1973), 29–31.

[6] J. Miao, *A property of analytic functions with Hadamard gaps*, Bull. Austral. Math. Soc. **45** (1992), no. 1, 105–112.

[7] W. Rudin, *Function theory in the unit ball of C^n*, Grundlehren der Mathematischen Wissenschaften, 241, Springer-Verlag, New York-Berlin, 1980.

[8] J. -H. Shi, *Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of $\mathbf{C}^n$*, Trans. Amer. Math. Soc. **328** (1991), no. 2, 619–637.

[9] A. Siskakis, *Weighted integrals of analytic functions*, Acta Sci. Math. **66** (2000), 651–664.

[10] S. Stević, *On an area inequality and weighted integrals of analytic functions*, Result Math. **41** (2002), no. 3-4, 386–393.

[11] ———, *Weighted integrals and conjugate functions in the unit disk*, Acta Sci. Math. **69** (2003), no. 1-2, 109–119.

[12] ———, *Weighted integrals of holomorphic functions on the polydisc*, Z. Anal. Anwendungen **23** (2004), no. 3, 577–587.

[13] S. Yamashita, *Gap series and α-Bloch functions*, Yokohama Math. J. **28** (1980), no. 1-2, 31–36.

[14] A. Zygmund, *Trigonometric series,* Cambridge University Press, London, 1959.

Mathematical Institute of Serbian Academy of Science
Knez Mihailova 35/I, 11000 Beograd, Serbia
E-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu