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NOTES ON A NON-ASSOCIATIVE ALGEBRAS WITH EXPONENTIAL FUNCTIONS I

SEUL HEE CHOI

Abstract. For the evaluation algebra $\mathbf{F}[e^{\pm x}]_M$, if $M = \{\partial\}$, the automorphism group $Aut_{non}(\mathbf{F}[e^{\pm x}]_M)$ and $Der_{non}(\mathbf{F}[e^{\pm x}]_M)$ of the evaluation algebra $\mathbf{F}[e^{\pm x}]_M$ are found in the paper [12]. For $M = \{\partial^n\}$, we find $Aut_{non}(\mathbf{F}[e^{\pm x}]_M))$ and $Der_{non}(\mathbf{F}[e^{\pm x}]_M))$ of the evaluation algebra $\mathbf{F}[e^{\pm x}]_M$ in this paper. We show that a derivation of some non-associative algebra is not inner.

1. Preliminaries

Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbf{N}$ and $\mathbf{Z}$ will denote the non-negative integers and the integers, respectively. Let A be an associative algebra and $M = \{\delta_u | \delta_u$ is a mapping from A to itself $, u \in I\}$ where I is an index set. The evaluation algebra $A_M = \{a\delta | a \in A, \delta \in M\}$ [1-3] with the obvious addition and the multiplication $*$ is defined as follows:

$$a_1\delta_1 * a_2\delta_2 = a_1\delta_1(a_2)\delta_2$$

for any $a_1\delta_1, a_2\delta_2 \in A_M$ [11]. For A_M, if $M = \{id\}$, then the ring $A_M = A$ where id is the identity map of A. Note that $A_M =$ is not an associative ring generally. Using the commutator $[,]$ of $A_{M[,]}$, we can define the semi-Lie ring. If the Jacobi identity holds in $A_{M[,]}$, then

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$A_{M[,]}$ is a Lie ring. Let $\mathbf{F}[e^{x_1}, e^{x_2}, ..., e^{x_n}]$ be a ring in the formal power series ring $\mathbf{F}[[x_1, x_2, ..., x_n]]$. If we take the algebra $\mathbf{F}[e^{\pm x}]$ [5-6] and the map $M = \{\partial^n\}$, the we have the simple evaluation algebra $\mathbf{F}[e^x]_M$. Throughout the paper, we will take $M = \{\partial^n\}$. It is well known that the non-associative algebra $\mathbf{F}[e^x]_M$ is simple [1-3].

2. Derivations of $\mathbf{F}[e^{\pm x}]_M$

The non-associative algebra $\mathbf{F}[e^x]_M$ has the standard basis $\{e^{ax}\partial^n | a \in \mathbf{F}\}$. For any basis element $e^{ax}\partial^n$ of $\mathbf{F}[e^{\pm x}]_M$, we define degree $deg(e^{ax}\partial^n)$ as a. Using the degree, for any element l in $\mathbf{F}[e^{\pm x}]_M$, throughout the paper, l can be written as follows:

$$l = C(a_1)e^{a_1 x}\partial^n + C(a_2)e^{a_2 x}\partial^n + \cdots + C(a_s)e^{a_s x}\partial^n$$

such that $a_1 > \cdots > a_s$ with appropriate coefficients. Thus we can define the order of elements of $\mathbf{F}[e^{\pm x}]_M$ obviously.

Note 1. For any basis element $e^{mx}\partial^n$ of $\mathbf{F}[e^{\pm x}]_M$ of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$, if we define $\mathbf{F}$-additive linear map D_d of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ as follows:

$$D_d(e^{mx}\partial^n) = dme^{mx}\partial^n \tag{1}$$

then D_d can be linearly extended to a derivation of the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ where $d \in \mathbf{F}$ [4], [8-10]. □

Lemma 2.1. *Let D be a derivation of $\mathbf{F}[e^{\pm x}]_M$ such that $D(\partial^n) = 0$. For any basis element $e^{mx}\partial^n$ of $\mathbf{F}[e^{\pm x}]_M$, we have that*

$$D(e^{px}\partial^n) = cpe^{px}\partial^n$$

for $c \in \mathbf{F}$.

Proof. Let D be the derivation of $\mathbf{F}[e^{\pm x}]_M$ in the lemma. Since ∂^n is a left identity of $e^x\partial^n$, we have that $\partial^n * D(e^x\partial^n) = D(e^x\partial^n)$. This implies that

$$D(e^x\partial^n) = \sum_{1\leq r\leq u} c_r e^{a_r x}\partial^n \tag{2}$$

where $u \in \mathbf{N}$ with appropriate coefficients. Since ∂^n is a left identity of $e^x\partial^n$, we can prove that $a_r^n = 1$. This implies that we have the following two cases, Case I: n is odd and Case II: n is even.

Case I. Let us assume that n is odd. Since a_r is an integer and $a_r^n = 1$, we have that $a_r = 1$. Thus we can easily prove that $D(e^x\partial^n) = ce^x\partial^n$ with $c \in \mathbf{F}$. Thus we have proven the lemma in this case.

Case II. Let us assume that n is even. We have that $a_r \in \{1, -1\}$. If $a_q = -1$, for some $1 \leq q \leq u$, then $D(e^x\partial^n) = \sum_{r\neq q} c_r e^x\partial^n + c_q e^{-x}\partial^n$. By $D(e^x\partial^n * e^x\partial^n) = D(e^{2x}\partial^n)$, we have that $D(e^{2x}\partial^n) = \sum_{r\neq q} 2c_r e^{2x}\partial^n + 2c_q\partial^n$. If $c_q \neq 0$, then by $D(\partial^n * e^{2x}\partial^n) = 2^n D(e^{2x}\partial^n)$, we have a contradiction. So we have that $a_r = 1$, for $1 \leq r \leq u$. This implies that $D(e^x\partial^n) = ce^x\partial^n$, $c \in \mathbf{F}$. By $D(e^{-x}\partial^n * e^x\partial^n) = 0$, we have that $D(e^{-x}\partial^n) = -ce^{-x}\partial^n$. By induction on p of $x^{px}\partial^n$, we can prove that

$$D(e^{px}\partial^n) = cpe^{px}\partial^n \tag{3}$$

This completes the proof of the lemma. □

Lemma 2.2. *For any D in $Der_{non}(\mathbf{F}[e^{\pm x}]_M)$, $D(\partial^n) = 0$ holds.*

Proof. Let D be a derivation of $\mathbf{F}[e^{\pm x}]_M$. Since ∂^n annihilates itself, we may put $D(\partial^n) = a\partial^n$ with $a \in \mathbf{F}$. Since ∂^n is a left identity of $e^x\partial^n$, we have that

$$(4) \qquad a\partial^n * e^x\partial^n + \partial^n * D(e^x\partial^n) = D(e^x\partial^n)$$

This implies that $a = 0$, i.e., $D(\partial^n) = 0$. This completes the proof of the lemma. □

Lemma 2.3. *For any D in $Der_{non}(\mathbf{F}[e^{\pm x}]_M)$, D is the derivation D_d which is defined in Note 1.*

Proof. Let D be a derivation of $\mathbf{F}[e^{\pm x}]_M$. By Lemma 2.2, we have that $D(\partial^n) = 0$. So by Lemma 2.1, by taking appropriate scalars d, we have that $D = D_d$ which is defined in Note 1. Thus we have proven the lemma. □

The following theorem generalizes Theorem 4 in the paper [12].

Theorem 2.1. *If $n \geq 1$, then $Der_{non}(\mathbf{F}[e^{\pm x}]_M)$ is generated by the derivation D_d which is defined in Note I with appropriate scalar.*

Proof. The proof of the theorem is straightforward by Lemmas 2.1-3 and Note 1. □

The results of Theorem 2.1 show that if $n \neq 1$ the derivation of $\mathbf{F}[e^{\pm x}]_M$ is not inner, i.e., it is an outer derivation.

Corollary 2.1. *$Dim(Der_{non}(\mathbf{F}[e^{\pm x}]_M))$ is one.*

Proof. The proof of the corollary is straightforward by Theorem 2.1.
□

3. Automorphisms of $\mathbf{F}[e^{\pm x}]_M$

Note 2. For $n = even$, for any basis element $e^{px}\partial^n$ of $\mathbf{F}[e^{\pm x}]_M$, and for $d_1, d_2 \in \mathbf{F}^\bullet$, if we define $\mathbf{F}$-maps θ_1^+ and θ_2^- of $\mathbf{F}[e^{\pm x}]_M$ respectively as follows:

$$\theta_1^+(e^{px}\partial^n) = d_1^p e^{px}\partial^n$$
$$\theta_2^-(e^{px}\partial^n) = d_2^p e^{-px}\partial^n$$

(5)

then θ_1^+ and θ_2^- can be linearly extended to non-associative algebra automorphisms of $\mathbf{F}[e^{\pm x}]_M$. Similarly, if n is odd, we can define its automorphisms θ_3^+ and θ_4^- of $\mathbf{F}[e^{\pm x}]_M$ respectively as follows:

$$\theta_3^+(e^{px}\partial^n) = d_3^p e^{px}\partial^n$$
$$\theta_4^-(e^{px}\partial^n) = (-1)^{p-1} d_4^p e^{-px}\partial^n$$

(6)

where $d_3, d_4 \in \mathbf{F}^\bullet$ [4], [6]. □

Lemma 3.1. *For any automorphism θ of $\mathbf{F}[e^{\pm x}]_M$, $\theta(\partial^n) = c\partial^n$ where $c \in \mathbf{F}$.*

Proof. The proof of the lemma is standard, so it is omitted. □

Lemma 3.2. *For any automorphism θ of $\mathbf{F}[e^{\pm x}]_M$, if n is even, then θ is either θ_1^+ or θ_2^- and if n is odd, then θ is either θ_3^+ or θ_4^- which are defined in Note 2.*

Proof. Let θ be any automorphism of $\mathbf{F}[e^{\pm x}]_M$. By Lemma 3.1, $\theta(\partial^n) = c\partial^n$ where $c \in \mathbf{F}^\bullet$. Since ∂^n is a left (multiplicative) identity of $e^x\partial^n$, we have that

$$c\partial^n * \theta(e^x\partial^n) = \theta(e^x\partial^n) \tag{7}$$

We have the following two cases: Case I: n is even and Case II: n is odd.

Case I. Let us assume that n is even. By (7), we have the following two subcases, Subcase I: $\theta(e^x\partial^n) = d_1 e^x \partial^n$ and Subcase II: $\theta(e^x\partial^n) = d_2 e^{-x}\partial^n$ for $d_1, d_2 \in \mathbf{F}^\bullet$.

Subcase I. Let us assume that $\theta(e^x\partial^n) = d_1 e^x\partial^n$. By

$$(8) \qquad \theta(e^{-x}\partial^n) * \theta(e^x\partial^n) = c\partial^n$$

we have that $\theta(e^{-x}\partial^n) = cd_1^{-1}e^{-x}\partial^n$. By $\theta(e^x\partial^n * e^x\partial^n) = \theta(e^{2x}\partial^n)$, we have that $\theta(e^{2x}\partial^n) = d_1^2 e^{2x}\partial^n$. By $theta(e^{-x}\partial^n * e^{2x}\partial^n) = 2^n\theta(e^x\partial^n)$, we have that $C = 1$. by induction on p of $e^{px}\partial^n$, we can prove that $\theta(e^{px}\partial^n) = d_1^p e^{px}\partial^n$. This implies that θ is the automorphism θ_1^+ in Note 2.

Subcase II. Let us assume that $\theta(e^x\partial^n) = d_2e^{-x}\partial^n$. By (8), we have that $\theta(e^{-x}\partial^n) = cd_2^{-1}e^x\partial^n$. Easily, we have that $\theta(e^{2x}\partial^n) = d_2^2e^{-2x}\partial^n$. Similarly to Subcase I, we can prove that $c = 1$. By appropriate induction, we can prove that θ is θ_2^- in Note 2. This completes the proof of the lemma.

Case II. Let us assume that n is odd. Since $n \neq 1$, by (7), we have the following two subcases, Subcase III: $\theta(e^x\partial^n) = d_3e^x\partial^n$ and Subcase IV: $\theta(e^x\partial^n) = d_4e^{-x}\partial^n$ for $d_3, d_4 \in \mathbf{F}^\bullet$.

Subcase III. Let us assume that $\theta(e^x\partial^n) = d_3e^x\partial^n$. This implies that by (8), we have that $\theta(e^{-x}\partial^n) = cd_3^{-1}e^{-x}\partial^n$. Easily, we have that $\theta(e^{2x}\partial^n) = d_3^2e^{2x}\partial^n$. Similarly to Subcase I, we can prove that $c = 1$. Thus by induction on p of $e^{px}\partial^n$, we can prove that $\theta(e^{px}\partial^n) = d_3^pe^{px}\partial^n$. This implies that θ is the automorphism θ_3^+ in Note 2.

Subcase IV. Let us assume that $\theta(e^{x}\partial^{n}) = d_4 e^{-x}\partial^{n}$. By (8), we can prove that $\theta(e^{-x}\partial^{n}) = -cd_4^{-1}e^{x}\partial^{n}$. Easily we can prove that $\theta(e^{2x}\partial^{n}) = -d_4^2 e^{-2x}\partial^{n}$. By $\theta(e^{-x}\partial^{n} * e^{2x}\partial^{n}) = 2^{n}\theta(e^{x}\partial^{n})$, we have that $c = -1$. Thus by induction on p of $e^{px}\partial^{n}$, we can prove that $\theta(e^{px}\partial^{n}) = (-1)^{p-1}d_4^{p}e^{px}\partial^{n}$. This implies that θ is the automorphism θ_4^- in Note 2.

Thus by Cases I-II, we have proven the lemma. □

Theorem 3.1. *If n is even, then the automorphism group $Aut_{non}(\mathbf{F}[e^{\pm x}]_M)$ of $\mathbf{F}[e^{\pm x}]_M$ is generated by θ_1^+ and θ_2^- and if n is odd, then the automorphism group $Aut_{non}(\mathbf{F}[e^{\pm x}]_M)$ of $\mathbf{F}[e^{\pm x}]_M$ is generated by θ_3^+ and θ_4^- which are defined in Note 2 appropriately.*

Proof. The proof of the theorem is straightforward by Lemma 3.2. Let us omit the details of the proof. □

The following theorem generalizes Theorem 3 in the paper [12].

Proposition 3.1. *For $M = \{\partial^{n}\}$, $1 \leq n \in \mathbf{N}$, the non-associative algebra $\mathbf{F}[e^{\pm x}]_M$ is not isomorphic to the non-associative algebra $\mathbf{F}[x^{\pm 1}]_M$ as non-associative algebras.*

Proof. The proof of the proposition is standard. Let us omit it. □

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Seul Hee Choi
Dept. of Mathematics,
Univ. of Jeonju,
hon-ju 560-759,
South Korea
Email : chois@jj.ac.kr