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MATRIX REALIZATION AND ITS APPLICATION OF THE LIE ALGEBRA OF TYPE F_4

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Abstract. The Lie algebra of type F_4 has the 26 dimensional representation. Its matrix realization can be obtained via 26 by 26 matrices and has a direct useful application to degenerate principal series for p-adic groups of type F_4.

1. Introduction

The problem of classifying the unitary dual of G, a connected reductive group over a field F, has been studied using normalized inductions. Among normalized inductions from parabolic subgroups of G, we will look into degenerate principal series. Degenerate principal series are representations obtained by inducing a one-dimensional representation of a maximal parabolic subgroup. Jantzen [2, 3, 4] determined reducibility points of degenerate principal series for orthogonal groups and symplectic groups using their matrix realizations. The Lie algebra of type G_2 was shown to be the Lie algebra of derivations of the Cayley algebra in [6] and [7]. This gives the seven dimensional representation of the Lie algebra of type G_2 and its matrix realization is explicitly shown in [5]. In this paper, we derive a matrix realization of the Lie algebra of

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type F_4 and apply this realization to degenerate principal series for p-adic groups of type F_4. This matrix realization will be useful in other research topics involving the Lie algebra of type F_4.

2. Preliminaries and Computation

The Lie algebra of type F_4 over a field F is the derivation algebra $\mathfrak{D}$ of the exceptional Jordan algebra $\mathfrak{J}$ of dimension 27 over F in [1] and [8]. The matrix realization of type F_4 via 26 by 26 matrices can be obtained as follows.

Let $\mathfrak{C}$ be the split Cayley algebra over F of characteristic $\neq 2$. Let $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ be the usual basis for the space of triples of elements of F.
Set

$$u_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, u_{4+i} = -2\begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix} (i = 1,2,3)$$

$$u_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u_8 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\{u_1, ..., u_8\}$ is a basis for $\mathfrak{C}$.

Let $\mathfrak{J}$ be the 27 dimensional space over F of all 3 by 3 matrices of the form

$$\alpha = \begin{pmatrix} \alpha_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & \alpha_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & \alpha_{33} \end{pmatrix}$$

$$= diag(\alpha_{11}, \alpha_{22}, \alpha_{33}) + a_{12}(1,2) + a_{13}(1,3) + a_{23}(2,3),$$

where $\alpha_{ii} \in F$ and $a_{ij} \in \mathfrak{C}$.

Let $\mathcal{L}$ be the Lie algebra of all derivations of $\mathfrak{J}$. In [8], Seligman showed that $\mathcal{L} = \mathfrak{C} \oplus \mathfrak{C} \oplus \mathfrak{C} \oplus D_4$ where the Lie algebra D_4 consists of all skew transformations of $\mathfrak{J}$. For $a_{12}, a_{13}, a_{23} \in \mathfrak{C}$ and $T \in D_4$, we define a linear transformation D of $\mathfrak{J}$ by

$$
\begin{aligned}
&(diag(\beta_{11}, \beta_{22}, \beta_{33}) + b_{12}(1,2) + b_{13}(1,3) + b_{23}(2,3))D \\
&= \frac{1}{2} [\ diag(-2(b_{12}, a_{12}) - 2(b_{13}, a_{13}), 2(b_{12}, a_{12}) - 2(b_{23}, a_{23}), \\
&\qquad 2(b_{13}, a_{13}) + 2(b_{23}, a_{23})) \\
&\quad +((\beta_{11} - \beta_{22})a_{12} - b_{13}\overline{a_{23}} - a_{13}\overline{b_{23}} + b_{12}T)(1,2) \\
&\quad +((\beta_{11} - \beta_{33})a_{13} + b_{12}a_{23} - a_{12}b_{23} + b_{13}T^{\psi})(1,3) \\
&\quad +((\beta_{22} - \beta_{33})a_{23} + \overline{b_{12}}a_{13} + \overline{a_{12}}b_{13} + b_{23}T^{\phi})(2,3)\ ]
\end{aligned}
$$

where the symmetric bilinear form (x, y) is defined by $(x, y)I = (x\bar{y} + y\bar{x})/2$ for $x, y \in \mathfrak{C}$ and T^{ψ}, T^{ϕ} are defined by the principal of triviality. The principal of triviality says that if T is a linear transformation of $\mathfrak{C}$ which is skew w.r.t. (x, y), there are uniquely determined skew transformations T^{ψ}, T^{ϕ} such that $(xy)T^{\psi} = (xT)y + x(yT^{\phi})\ \forall\ x, y \in \mathfrak{C}$.

If E_{ij} are the unit matrices relative to the basis $\{u_1, ..., u_8\}$ of $\mathfrak{C}$, then $H_i = E_{ii} - E_{i+4,i+4}$ $(1 \le i \le 4)$ spans the Cartan subalgebra of D_4. Let $\mathcal{S}$ be the subspace spanned by $2(0,0,0,H_i) = h_i$ $(1 \le i \le 4)$. For $h \in \mathcal{S}$ and $1 \le i \le 8$,

$$
\begin{aligned}
[(u_i, 0, 0, 0), h] &= \beta(h)(u_i, 0, 0, 0) \\
[(0, u_i, 0, 0), h] &= \beta(h)(0, u_i, 0, 0) \\
[(0, 0, u_i, 0), h] &= \beta(h)(0, 0, u_i, 0)
\end{aligned}
$$

where β is one of the 24 short roots of F_4.

For $h \in \mathcal{S}$ and $T \in \{E_{ij} - E_{j+4,i+4} \mid 1 \le i,j \le 4\} \cup \{E_{i,j+4} - E_{j,i+4}, E_{i+4,j} - E_{j+4,i} \mid 1 \le i < j \le 4\}$,

$$[(0,0,0,T), h] = \beta(h)(0,0,0,T)$$

where β is one of the 24 long roots of F_4.

Let $\mathfrak{J}'$ be the space of matrices of trace zero. Set $(1 \leq i \leq 8)$

$$v_1 = diag(1,-1,0) + 0(1,2) + 0(1,3) + 0(2,3)$$

$$v_2 = diag(0,1,-1) + 0(1,2) + 0(1,3) + 0(2,3)$$

$$v_{i+2} = diag(0,0,0) + u_i(1,2) + 0(1,3) + 0(2,3)$$

$$v_{i+10} = diag(0,0,0) + 0(1,2) + u_i(1,3) + 0(2,3)$$

$$v_{i+18} = diag(0,0,0) + 0(1,2) + 0(1,3) + u_i(2,3).$$

Then $\{v_i \mid 1 \leq i \leq 26\}$ is a basis for $\mathfrak{J}'$. By computing the matrix for $\{h_i \mid 1 \leq i \leq 4\}$ and each element of $\mathcal{L}$ corresponding to 24 short roots and 24 long roots w.r.t. the basis $\{v_i \mid 1 \leq i \leq 26\}$, we obtain the matrix realization via 26 by 26 matrices.

3. Matrix Realization

Let $\mathfrak{g}$ be the Lie algebra of G, a split group of type F_4. In the following realization, the set $\mathfrak{a}$ of diagonal matrices in $\mathfrak{g}$ is a Cartan subalgebra corresponding to a maximal split torus A of G.

Denote by $E_{i,j}$ the 26×26 matrix whose r,s entry is $\delta_{r,i}\delta_{s,j}$ and abbreviate $E_{i,i}$ to E_i. The Cartan subalgebra $\mathfrak{a}$ is spanned by the four vectors

$$E_{Y1} = E_1 - E_{26} + (E_2 + E_3 + E_4 + E_5 + E_6 + E_8 + E_{10} + E_{12})/2$$
$$-(E_{15} + E_{17} + E_{19} + E_{21} + E_{22} + E_{23} + E_{24} + E_{25})/2,$$
$$E_{Y2} = E_7 - E_{20} + (E_2 + E_3 + E_4 + E_6 + E_{15} + E_{17} + E_{19} + E_{22})/2$$
$$-(E_5 + E_8 + E_{10} + E_{12} + E_{21} + E_{23} + E_{24} + E_{25})/2,$$
$$E_{Y3} = E_9 - E_{18} + (E_2 + E_3 + E_5 + E_8 + E_{15} + E_{17} + E_{21} + E_{23})/2$$

$$-(E_4 + E_6 + E_{10} + E_{12} + E_{19} + E_{22} + E_{24} + E_{25})/2,$$
$$E_{Y4} = E_{11} - E_{16} + (E_2 + E_4 + E_5 + E_{10} + E_{15} + E_{19} + E_{21} + E_{24})/2$$
$$-(E_3 + E_6 + E_8 + E_{12} + E_{17} + E_{22} + E_{23} + E_{25})/2.$$

Define linear functionals α,β,γ,δ on $\mathfrak{a}$ by

$$\alpha(pE_{Y1} + qE_{Y2} + rE_{Y3} + sE_{Y4}) = (p - q - r - s)/2,$$
$$\beta(pE_{Y1} + qE_{Y2} + rE_{Y3} + sE_{Y4}) = s,$$
$$\gamma(pE_{Y1} + qE_{Y2} + rE_{Y3} + sE_{Y4}) = r - s,$$
$$\delta(pE_{Y1} + qE_{Y2} + rE_{Y3} + sE_{Y4}) = q - r.$$

We set as follows for 24 positive roots and use lowercase letters for 24 negative roots,

$$E_H = E_\alpha,\ \ E_D = E_\beta,\ \ E_R = E_\gamma,\ \ E_P = E_\delta,$$
$$E_L = E_{\alpha+\beta},\ \ E_C = E_{\beta+\gamma},\ \ E_X = E_{2\beta+\gamma},\ \ E_Q = E_{\gamma+\delta},$$
$$E_I = E_{\alpha+\beta+\gamma},\ \ E_B = E_{\beta+\gamma+\delta},\ \ E_M = E_{2\alpha+2\beta+\gamma},\ \ E_W = E_{2\beta+\gamma+\delta},$$
$$E_E = E_{\alpha+2\beta+\gamma},\ \ E_J = E_{\alpha+\beta+\gamma+\delta},\ \ E_N = E_{2\alpha+2\beta+\gamma+\delta},$$
$$E_V = E_{2\beta+2\gamma+\delta},\ \ E_F = E_{\alpha+2\beta+\gamma+\delta},\ \ E_O = E_{2\alpha+2\beta+2\gamma+\delta},$$
$$E_G = E_{\alpha+2\beta+2\gamma+\delta},\ \ E_U = E_{2\alpha+4\beta+2\gamma+\delta},\ \ E_K = E_{\alpha+3\beta+2\gamma+\delta},$$
$$E_T = E_{2\alpha+4\beta+3\gamma+\delta},\ \ E_A = E_{2\alpha+3\beta+2\gamma+\delta},\ \ E_S = E_{2\alpha+4\beta+3\gamma+2\delta}.$$

Then the collection Φ of forty eight linear functionals forms a root system of type F_4 and the Lie algebra of type F_4 is generated by $\{E_\omega \mid \omega \in \Phi\}$.

In this way, we have the following matrix realization

$$\begin{bmatrix}
Y1 & H & L & I & J & E & 4M & F & 4N & 2G & 4O & 2K & 4A & 0 & 0 & 2U & 0 & 2T & 0 & 2S & 0 & 0 & 0 & 0 & 0 & 0 \\
h & Y2 & D & C & B & X & 4E & W & -4F & 2V & 4G & 0 & 0 & -4K & 4A & 0 & 4U & 0 & -8T & 0 & 8S & 0 & 0 & 0 & 0 & 0 \\
l & d & Y3 & R & Q & C & 4I & B & -4J & 0 & 0 & 2V & 4G & 4G & -4O & 2K & 4A & 0 & 0 & 0 & 0 & -8T & 8S & 0 & 0 & 0 \\
i & c & r & Y4 & P & -D & -4L & 0 & 0 & 2B & 4J & -2W & -4F & -4F & -4N & 0 & 0 & -2K & -8A & 0 & 0 & 8U & 0 & -8S & 0 & 0 \\
j & b & q & p & Y5 & 0 & 0 & -D & 4L & -2C & -4I & 2X & 4E & 4E & -4M & 0 & 0 & 0 & 0 & -2K & 8A & 0 & -8U & 8T & 0 & 0 \\
e & x & c & -d & 0 & Y6 & -4H & P & 0 & -2Q & 0 & 2B & 0 & 4J & 0 & -2F & -4N & -2G & -8O & 0 & 0 & -8A & 0 & 0 & -8S & 0 \\
m/4 & e/4 & i/4 & -l/4 & 0 & -h/4 & Y7 & 0 & -P & 0 & -Q & 0 & B & 0 & -J & W/2 & -F & V/2 & 2G & 0 & 0 & 2K & 0 & 0 & 0 & -2S \\
f & w & b & 0 & -d & p & 0 & Y8 & 4H & 2R & 0 & -2C & 0 & -4I & 0 & 2E & -4M & 0 & 0 & -2G & 8O & 0 & 8A & 0 & 8T & 0 \\
n/4 & -f/4 & -j/4 & 0 & l/4 & 0 & -p & h/4 & Y9 & 0 & -R & 0 & C & 0 & -I & X/2 & -E & 0 & 0 & -V/2 & 2G & 0 & 2K & 0 & 0 & -2T \\
g/2 & v/2 & 0 & b/2 & -c/2 & -q/2 & 0 & r/2 & 0 & Y10 & -2H & D & 0 & 2L & 0 & 0 & 0 & -2E & 4M & F & 4N & 0 & 0 & -4A & -2U & 0 \\
o/4 & g/4 & 0 & j/4 & -i/4 & 0 & -q & 0 & -r & -h/2 & Y11 & 0 & D & 0 & -L & 0 & 0 & -X/2 & E & -W/2 & 2F & 0 & 0 & 2K & 0 & -U \\
k/2 & 0 & v/2 & -w/2 & x/2 & b/2 & 0 & -c/2 & 0 & d & 0 & Y12 & -2H & -2H & 0 & L & 0 & I & 0 & J & 0 & 4M & 4N & 4O & -4A & 0 \\
a/2 & k/4 & g/4 & -f/4 & e/4 & -j/4 & 2b & i/4 & 2c & -l/2 & 2d & -h/2 & 0 & 0 & H & -D & L & -C & -2I & -B & 2J & -2E & 2F & -2G & -2K & -4A \\
-a/4 & -k/2 & g/4 & -f/4 & e/4 & j/2 & -b & -i/2 & -c & l & -d & -h/2 & 0 & 0 & H & D/2 & -2L & C/2 & 4I & B/2 & -4J & -2E & 2F & -2G & 4K & 2A \\
0 & a/4 & -o/4 & -n/4 & -m/4 & 0 & -j & 0 & -i & 0 & -l & 0 & h & h & -Y12 & 0 & -D & 0 & 2C & 0 & -2B & -2X & 2W & -2V & 0 & -2K \\
u/2 & 0 & -k/2 & 0 & 0 & -f/2 & 2w & e/2 & 2x & 0 & 0 & l & -2d & 0 & 0 & -Y11 & 2H & R & 0 & Q & 0 & 4I & -4J & 0 & -4G & -4O \\
0 & u/4 & a/4 & 0 & 0 & -n/4 & -f & -m/4 & -e & 0 & 0 & 0 & 0 & -l & -d & h/2 & -Y10 & 0 & -2R & 0 & 2Q & 2C & -2B & 0 & -2V & -2G \\
t/2 & 0 & 0 & -k/2 & 0 & -g/2 & 2v & 0 & 0 & e & -2x & i & -2c & 0 & 0 & r & 0 & -Y9 & -4H & P & 0 & -4L & 0 & 4J & 4F & -4N \\
0 & -t/8 & 0 & -a/8 & 0 & -o/8 & g/2 & 0 & 0 & m/4 & -e/2 & 0 & 0 & i/2 & c/2 & 0 & -r/2 & -h/4 & -Y8 & 0 & -P & D & 0 & -B & -W & -F \\
s/2 & 0 & 0 & 0 & -k/2 & 0 & 0 & -g/2 & -2v & f & -2w & j & -2b & 0 & 0 & q & 0 & p & 0 & -Y7 & 4H & 0 & 4L & -4I & -4E & -4M \\
0 & s/8 & 0 & 0 & a/8 & 0 & 0 & o/8 & g/2 & n/4 & f/2 & 0 & 0 & -j/2 & -b/2 & 0 & q/2 & 0 & -p & h/4 & -Y6 & 0 & D & -C & -X & -E \\
0 & 0 & -t/8 & u/8 & 0 & -a/8 & k/2 & 0 & 0 & 0 & 0 & m/4 & -e/2 & -e/2 & -x/2 & i/4 & c/2 & -l/4 & d & 0 & 0 & -Y5 & -P & -Q & -B & -J \\
0 & 0 & s/8 & 0 & -u/8 & 0 & 0 & a/8 & k/2 & 0 & 0 & n/4 & f/2 & f/2 & w/2 & -j/4 & -b/2 & 0 & 0 & l/4 & d & -p & -Y4 & -R & -C & -I \\
0 & 0 & 0 & -s/8 & t/8 & 0 & 0 & 0 & 0 & -a/4 & k/2 & o/4 & -g/2 & -g/2 & -v/2 & 0 & 0 & j/4 & -b & -i/4 & -c & -q & -r & -Y3 & -D & -L \\
0 & 0 & 0 & 0 & 0 & -s/8 & 0 & t/8 & 0 & -u/4 & 0 & -a/4 & 0 & k/2 & 0 & -g/4 & -v/2 & f/4 & -w & -e/4 & -x & -b & -c & -d & -Y2 & -H \\
0 & 0 & 0 & 0 & 0 & 0 & -s/2 & 0 & -t/2 & 0 & -u/2 & 0 & -a/2 & 0 & -k/2 & -o/4 & -g/2 & -n/4 & -f & -m/4 & -e & -j & -i & -l & -h & -Y1
\end{bmatrix}.$$

4. Application

To determine reducibility points of degenerate principal series in regular cases, we will use a criterion developed by Jantzen in [2]. And we will use matrix realization derived in the previous section.

For a root $\omega \in \Phi$, we will write $x_\omega(t)$ for $exp(tX_\omega)$. Set $w_\omega(t) = x_\omega(t)x_{-\omega}(-t^{-1})x_\omega(t)$ and $h_\omega(t) = w_\omega(t)w_\omega(1)^{-1}$.

The group A, generated by $\{h_\omega(t) \mid \omega \in \Phi, t \in F^\times\}$, is equal to the set of 26×26 matrices of the form

$$diag(p,q,r,s) = pE_1 + qE_2 + rE_3 + sE_4 + pq/rsE_5 + rs/qE_6 + rs/pE_7$$
$$+p/sE_8 + q/sE_9 + p/rE_{10} + q/rE_{11} + p/qE_{12} + E_{13} + E14 + q/pE_{15}$$
$$+r/qE_{16} + r/pE_{17} + s/qE_{18} + s/pE_{19} + p/rsE_{20} + q/rsE_{21} + rs/pqE_{22}$$
$$+1/sE_{23} + 1/rE_{24} + 1/qE_{25} + 1/pE_{26},\ p,q,r,s \in F^\times.$$

The $w_\omega(t)$'s and A generate the group N. The Weyl group $W = N/A$ is generated by the reflections

$$s_1 = w_H(1),\ \ s_2 = w_D(1)\ \ s_3 = w_R(1)\ s_4 = w_P(1).$$

Let $B = AU_{min}$ be the minimal parabolic and set $N_i = \langle B, s_i\rangle \ \ (i = 1,2,3,4)$, which is the Levi factor of a larger parabolic subgroup. To use Jantzen's criterion, we need to understand the structure of N_i and the Weyl group actions on A.

Using matrix realization, we can identify the structure of N_i which are isomorphic to $GL(1,F)\times GL(1,F)\times GL(2,F)$. And we can compute the Weyl group action on A with the help of a computational software such as Maple and Matlab.

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