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A COUNTEREXAMPLE TO THE WIENER TYPE CRITERION FOR REGULAR BOUNDARY POINTS

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Abstract. We will investigate whether Jucha's result on the Zalcman type domain can be extended to an arbitrary domain in $\mathbb{C}$.

1. Introduction

Let E be the unit disk in $\mathbb{C}$, the complex plane. For a bounded domain $D \subset \mathbb{C}$ and $z_0 \in \partial D$, z_0 is *regular* (with respect to the Dirichlet problem) for D if there exists a neighbourhood U of z_0 and u, which is subharmonic on D, with $u < 0$ on $U \cap D$ such that $\lim_{U\cap D \ni z \to z_0} u(z) = 0$. This function u is called a *barrier* at z_0. If every $z_0 \in \partial D$ is regular, then D is called a *regular* domain.

We denote by $\mathcal{L}^2_h(D)$ the space of all square integrable functions on $D \subset \mathbb{C}^n$ which are holomorphic. Note that $\mathcal{L}^2_h(D)$ becomes a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_D$ induced from the $\mathcal{L}^2(D)$. Let $\|f\|_D$ denote the standard norm of the function $f \in \mathcal{L}^2(D)$. The point evaluation functional $\mathcal{L}^2_h(D) \ni f \mapsto f(w) \in \mathbb{C}(w \in D)$ is continuous. By Riesz representation theorem, there exists $K_D(\cdot, w) \in \mathcal{L}^2_h(D)$ such that

$$f(w) = \langle f, K_D(\cdot, w) \rangle_D, \quad w \in D,$$

for all $f \in \mathcal{L}^2_h(D)$. We call the function $k_D(z) := K_D(z, z)$ *the Bergman function* for D.

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Let D be a domain in $\mathbb{C}^n$ with $z_0 \in \partial D$. We say that D is *Bergman exhaustive at* z_0 if $\lim_{z\to z_0} k_D(z) = \infty$. We call D *Bergman exhaustive* if D is Bergman exhaustive at each $z_0 \in \partial D$. Recall that every regular domain in $\mathbb{C}$ is Bergman exhaustive([4],[2]).

Next we introduce some basic notions from the plane potential theory used in this paper. Let $\mathcal{P}(K)$ be the set of all probability Borel measures with their supports included in a compact set $K \subset \mathbb{C}^n$. We define *the logarithmic potential* p_μ of a measure $\mu \in \mathcal{P}(K)$

$$p_\mu(z) := \int_K \log|z-w|d\mu(w), \quad z \in \mathbb{C}.$$

Then p_μ is subharmonic on $\mathbb{C}$ and $p_\mu|_{\mathbb{C}\setminus K}$ is a harmonic function. To any such a μ, *its energy* is defined by

$$I(\mu) := \int_K p_\mu(z)d\mu(z) = \int_K\int_K \log|z-w|d\mu(z)d\mu(w).$$

A measure $\nu \in \mathcal{P}(K)$ is called *the equilibrium measure* of the compact set K if

$$I(\nu) = \sup\{I(\mu) : \mu \in \mathcal{P}(K)\}.$$

By $\mathcal{SH}$ we denoted the symbol for the set of all subharmonic functions. A set $F \subset \mathbb{C}$ is *polar* if there is $u \in \mathcal{SH}(\mathbb{C})$ such that $u \not\equiv -\infty$ and $F \subset \{u = -\infty\}$. It is well-known that the equilibrium measure exists and is unique if K is compact, not polar set([5]). *The logarithmic capacity* of a set $E \subset \mathbb{C}$ is the number

$$\text{cap}(E) := \exp(\sup\{I(\mu) : \mu \in \mathcal{P}(K), K \text{ is a compact subset of } E\}).$$

REMARK. (1) If $E_1 \subset E_2$ then $\text{cap}(E_1) \le \text{cap}(E_2)$.

(2) $\text{cap}(\Delta(\lambda, r)) = \text{cap}(\partial\Delta(\lambda, r)) = r$, where $\Delta(x, r) := \{z \in \mathbb{C} : |z - x| < r\}$; for any compact $K \subset \mathbb{C}$, we have $\text{cap}(K) \le \text{diam}(K)/2, \text{cap}(K) = \text{cap}(\partial K)$; moreover, if K is connected, then $\text{diam}(K)/4 \le \text{cap}(K)$.

2. The Zalcman type domains

Let us consider a following type of planar domains. Put

$$D := E \setminus \left(\bigcup_{k=1}^{\infty} \overline{\Delta}(x_k, r_k) \cup \{0\} \right), \tag{†}$$

where $\lim_{k\to\infty} x_k = 0, \overline{\Delta}(x_k, r_k) \subset E$ and $\overline{\Delta}(x_k, r_k) \cap \overline{\Delta}(x_l, r_l) = \emptyset$, for $k \neq l$. This domain is called a *Zalcman type* domain. Note that each boundary point $z_0 \neq 0$ of D is regular. So, if $0 \in D$ is regular, then D is a regular domain.

In $\mathbb{C}$, there is a necessary and sufficient condition for Bergman exhaustiveness. For this, we begin by defining the potential theoretic function. For a bounded domain $D \subset \mathbb{C}$, define $\gamma : \overline{D} \longrightarrow [0, \infty]$ by

$$\gamma_D(z) := \int_0^{\frac{1}{2}} \frac{d\delta}{\delta^3(-\log \operatorname{cap}(\overline{\Delta}(z,\delta) \setminus D))}.$$

γ_D is lower semicontinuous on $\overline{D}$([6]). In 2002, W. Zwonek discovered the following relation between γ_D and the Bergman exhaustiveness([6]).

Theorem 1 *Let D be a bounded domain in $\mathbb{C}$ and let $z_0 \in \partial D$. Then*

$$\lim_{D \ni z \to z_0} \gamma_D(z) = \infty \Leftrightarrow D \textit{ is Bergman exhaustive at } z_0.$$

Using the Theorem 1, Jucha gave a description of some Zalcman type domains, as follows([3]).

Theorem 2 *Let D be a domain given by (†). Assume that there exist $\theta_1, \theta_2 \in (0,1)$ such that*

$$\theta_1 \leq \frac{x_{k+1}}{x_k} \leq \theta_2, \quad k \geq 1.$$

Then D is Bergman exhaustive iff D is Bergman exhaustive at 0 iff $\gamma_D(0) = \infty$ iff

$$\sum_{k=1}^{\infty} \frac{-1}{x_k^2 \log r_k} = \infty.$$

3. Main Results

In this chapter we will consider a question naturally induced from Jucha's result(Theorem 2). More precisely, we will discuss whether Jucha's result on the Zalcman type domain can be extended to a general domain in $\mathbb{C}$. Here we shall give a partially affirmative answer.

Theorem 3 *Let D be a bounded domain in $\mathbb{C}$ and $z_0 \in \partial D$. Assume that there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus \overline{D}$ with $z_k \to z_0$ and $\theta_1, \theta_2 \in (0,1)$ such that*

$$\theta_1 \leq \frac{|z_{k+1} - z_0|}{|z_k - z_0|} \leq \theta_2, \quad k \geq 1.$$

If

$$(\ddagger) \qquad \sum_{k=1}^{\infty} \frac{1}{|z_k - z_0|^2(-\log \operatorname{dist}(z_k, \partial D))} = \infty,$$

where $\operatorname{dist}(z, \partial D) = \inf\{|z - x| : x \in \partial D\}$, then $\gamma_D(z_0) = \infty$. In particular, D is Bergman exhaustive at z_0.

Proof. Without loss of generality, we may assume that $D \Subset \frac{1}{2}E$, $z_0 = 0$ and $\{z_k\}_{k=1}^{\infty} \subset \frac{1}{2}E$. Let $t_k := |z_k|$ and $\delta_k = \operatorname{dist}(z_k, \partial D)$, for each $k \in \mathbb{N}$. Then

$$\begin{aligned}
\gamma_D(0) &= \int_0^{\frac{1}{2}} \frac{1}{\delta^3(-\log \operatorname{cap}(\overline{\Delta}(0,\delta) \setminus D))} d\delta \\
&\geq \sum_{k=1}^{\infty} \int_{t_{k+1}}^{t_k} \frac{1}{\delta^3(-\log \operatorname{cap}(\overline{\Delta}(0,\delta) \setminus D))} d\delta \\
&\geq \sum_{k=1}^{\infty} (t_k - t_{k+1}) \frac{1}{t_k^3(-\log \frac{1}{2}\delta_{k+1})} \\
&\geq \sum_{k=1}^{\infty} \frac{1}{2} \left(1 - \frac{t_{k+1}}{t_k}\right) \frac{1}{t_k^2(-\log \delta_{k+1})} \\
&\geq \sum_{k=1}^{\infty} \frac{1}{2}(1 - \theta_2) \frac{\theta_1^2}{t_{k+1}^2} \frac{1}{-\log \delta_{k+1}} \\
&= C \sum_{k=1}^{\infty} \frac{1}{t_{k+1}^2(-\log \delta_{k+1})}.
\end{aligned}$$

Here $C > 0$ is some constant. Second inequality follows from the fact that, for $\delta \in [x_{k+1}, x_k]$,

$$\Delta\left(z_{k+1} - \frac{1}{2}\delta_{k+1}, \frac{1}{2}\delta_{k+1}\right) \subset \overline{\Delta}(0,\delta) \setminus D$$

and the capacity of a disk is just the radius.

Since γ_D is lower semicontinuous on $\overline{D}$, we have

$$\lim_{D \ni z \to z_0} \gamma_D(z) = \infty.$$

Hence, in view of Theorem 1, D is Bergman exhaustive at z_0. □

Note that in the proof of Theorem 1 we used the following conventions:

$\frac{1}{2}E := \{\frac{1}{2}\zeta \mid \zeta \in E\}$ and we mean by $D \Subset \frac{1}{2}E$ that D is relatively compact subset of $\frac{1}{2}E$.

REMARK. Let D be a domain given by (†) and $z_0 = 0$. Then (‡) is equivalent to

$$\sum_{k=1}^{\infty} \frac{-1}{x_k^2 \log r_k} = \infty.$$

We thus obtain the necessary condition of Theorem 2.

But the next theorem implies that Theorem 2 cannot be extended to an arbitrary domain in $\mathbb{C}$.

Theorem 4 *For $n \in \mathbb{N}$*

$$F_n := \left\{ z \in \mathbb{C} : \frac{1}{2^{n+1}} \leq \operatorname{Re} z \leq \frac{1}{2^n}, \quad -\exp(-2^{2n}n^2) \leq \operatorname{Im} z \leq \exp(-2^{2n}n^2)\right.$$

Define

$$D := \frac{1}{2}E \setminus \left(\bigcup_{n=1}^{\infty} F_n \cup \{0\} \right).$$

Then D is Bergman exhaustive at 0 and for any sequence $\{z_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus \overline{D}$ with $z_k \to 0$ and $\frac{|z_{k+1}|}{|z_k|} \leq \theta < 1$, we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^2(-\log \operatorname{dist}(z_k, \partial D))} < \infty.$$

Furthermore, $\gamma_D(0) = \infty$.

Proof. First, to prove $\gamma_D(0) = \infty$, it suffices to show that there exists a barrier at 0. Let $0 < \epsilon \ll 1$. Consider a mapping $D \cap \Delta(0, \epsilon) \ni z \mapsto \log z \in \mathbb{C}$. Then $\log z \neq 0$ since ϵ is sufficiently small. So $\frac{1}{\log z} \in \mathcal{O}(D \cap \Delta(0, \epsilon))$. We can define a harmonic function h on $D \cap \Delta(0, \epsilon)$ by

$$h(z) := \operatorname{Re}\left(\frac{1}{\log z}\right) = \frac{\log |z|}{|\log z|^2} < 0.$$

Observe that

$$h(z) = \frac{\log |z|}{|\log z|^2} \geq \frac{1}{\log |z|} \longrightarrow 0,$$

as $z \to 0$. Thus the function h is a barrier at 0.

Without loss of generality, we may assume that $\{z_k\}_{k=1}^{\infty} \subset \cup F_n$. For each z_k, there is an $n \in \mathbb{N}$ with $z_k \in F_n$. From the definition of F_n, we have

$$\frac{1}{|z_k|^2(-\log \operatorname{dist}(z_k, \partial D))} \leq 4 \cdot 2^{2n} \frac{1}{-\log \exp(-2^{2n} n^2)} = \frac{4}{n^2}.$$

From this we obtain

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^2(-\log \operatorname{dist}(z_k, \partial D))} \leq C \sum_{n=1}^{\infty} \frac{4}{n^2} < \infty,$$

where C is a constant such that $C > \#\{m \in \mathbb{N} : \theta^{m-1} \in [1/2, 1]\}$.

Furthermore we have

$$\begin{aligned}
\gamma_D(0) &= \int_0^{\frac{1}{2}} \frac{1}{\delta^3(-\log \operatorname{cap} \overline{\Delta}(0,\delta) \setminus D)} d\delta \\
&= \sum_{k=1}^{\infty} \int_{\frac{1}{2^{k+1}}}^{\frac{1}{2^k}} \frac{1}{\delta^3(-\log \operatorname{cap} \overline{\Delta}(0,\delta) \setminus D)} d\delta \\
&\geq \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} 2^{3k} \frac{1}{-\log \operatorname{cap}(F_{k+2})} \\
&\geq \sum_{k=1}^{\infty} \frac{2^{3k}}{2^{k+1}} \frac{1}{-\log(C_1 2^{k+2})} = C_2 \sum_{k=1}^{\infty} \frac{2^{2k}}{k} = \infty.
\end{aligned}$$

Here C_1 and C_2 are some positive constants. □

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