Honam Mathematical J. **28** (2006), No. 2, pp. 261–277

ON πgs-CLOSED SETS IN TOPOLOGICAL SPACES

BU YOUNG LEE, SU-YOUNG SHIN AND JIN HAN PARK*

Abstract. In this paper a new class of sets called πgs-closed sets is introduced and its properties are studied. Further the notions of πgs-$T_{1/2}$ spaces and πgs-continuity are introduced.

1. Introduction and preliminaries

Levine [18] initiated the investigation of so-called g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by many authors. Arya and Nour [2] defined gs-closed sets and studied some of their properties and obtained characterizations of s-normal spaces due to Maheswari and Prasad [19]. This notion is generalization of semiclosed sets which were further studied by Devi, Maki and Balachandran [9,10,21], Park [27] and Caldas [5]. On the other hand, Zaitsev [31] introduced the concept of π-closed sets and a class of topological spaces called quasi-normal spaces. Recently, Dontchev and Noiri [13] defined the concept of πg-closed sets as a weak form of g-closed sets and used this notion to obtain a characterization and some preservation theorems for quasi-normal spaces. In this paper, we introduce the concept of πgs-closed sets which implied by both that of πg-closed sets and gs-closed sets and study its basic properties.

Received March 6, 2006. Revised June 15, 2006.

2000 Mathematics Subject Classification : 54C08, 54D10.

Key words and phrases : πgs-closed sets, πgs-$T_{1/2}$ spaces, πgs-continuous functions.

† This research was supported by the Dong-A University Research Grant in 2003.

*Corresponding author.

We introduce a new class of topological spaces called πgs-$T_{1/2}$ spaces and show that the relationships among πgs-$T_{1/2}$ spaces, preregular $T_{1/2}$ spaces due to Gnanambal [14] and semi-pre-$T_{1/2}$ spaces due to Dontchev [12]. Moreover, the notions of πgs-continuity and πgs-irresoluteness are introduced and studied.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = \mathrm{int}(\mathrm{cl}(A))$ (resp. $A = \mathrm{cl}(\mathrm{int}(A))$). A point $x \in X$ is called a δ-cluster point [30] of A if $A \cap U \neq \emptyset$ for every regular open set containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\mathrm{cl}_\delta(A)$. If $\mathrm{cl}_\delta(A) = A$, then A is called δ-closed [30]. The complement of a δ-closed set is said to be δ-open [30]. The finite union of regular open sets is said to be π-open [31]. The complement of a π-open set is said to be π-closed [31].

A subset A is to be semiopen [17] (resp. α-open [24], preopen[23], semi-preopen [1]) if $A \subset \mathrm{cl}(\mathrm{int}(A))$ (resp. $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, $A \subset \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$) and the complement of a semiopen (resp. α-open, preopen, semi-preopen) set is called semiclosed (resp. α-closed, preclosed, semi-preclosed). The intersection of all semiclosed (resp. preclosed, semi-preclosed) sets containing A is called the semi-closure [7] (resp. preclosure [23], semi-preclosure [1]) of A and is denoted by $\mathrm{scl}(A)$ (resp. $\mathrm{pcl}(A)$, $\mathrm{spcl}(A)$). The semi-interior [7] (resp. preinterior[23]) of A is defined to be the union of all semiopen (resp. preopen) sets contained in A and is denoted by $\mathrm{sint}(A)$ (resp. $\mathrm{pint}(A)$). Note that $\mathrm{scl}(A) = A \cup \mathrm{int}(\mathrm{cl}(A))$ and $\mathrm{sint}(A) = A \cap \mathrm{cl}(\mathrm{int}(A))$.

We recall the following definitions used in sequel.

Definition 1.1 A subset A of a space X is said to be:

(a) g-closed [18] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X;

(b) πg-closed [13] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X;

(c) gs-closed [2] if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is open in X;

(d) gsp-closed [12] if $\mathrm{spcl}(A) \subset U$ whenever $A \subset U$ and U is open in X;

(e) rg-closed [26] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is regular open in X;

(f) gpr-closed [14] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is regular open in X;

(g) πgp-closed [28] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X;

(h) g-open (resp. πg-open, gs-open, gsp-open, rg-open, πgp-open) if the complement of A is g-closed (resp. πg-closed, gs-closed, gsp-closed, rg-closed, πgp-closed).

Definition 1.2 A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) semi-continuous [17] (resp. irresolute [8]) if $f^{-1}(V)$ is semiclosed in X for every closed (resp. semiclosed) set V of Y;

(b) g-continuous [3] (resp. rg-continuous [26]) if $f^{-1}(V)$ is g-closed (resp. rg-closed) in X for every closed set V of Y;

(c) gs-continuous [10] (resp. gs-irresolute [10]) if $f^{-1}(V)$ is gs-closed in X for every closed (resp. gs-closed) set V of Y;

(d) πg-continuous [13] (resp. almost π-continuous [13]) if $f^{-1}(V)$ is πg-closed (resp. π-closed) in X for every closed (resp. regular closed) set V of Y;

(e) presemiclosed [8] (resp. presemiopen [8], rc-preserving [15]) if $f(F)$ is semiclosed (resp. semiopen, regular closed) in Y for every semiclosed (resp. semiopen, regular closed) set F of X.

Definition 1.3 A space (X, τ) is called:

(a) $T_{1/2}$ [18] if every g-closed set is closed;

(b) semi-pre-$T_{1/2}$ [12] if every gsp-closed set is semi-preclosed;

(c) preregular $T_{1/2}$ [14] if every gpr-closed set is preclosed;

(d) πgp-$T_{1/2}$ [28] if every πgp-closed set is preclosed.

2. Basic properties of πgs-closed sets

Definition 2.1 A subset A of a space (X, τ) is said to be πgs-closed if $\mathrm{scl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

Remark 2.2 Since, for a subset of a space, we have the implications: regular open $\Rightarrow$ π-open $\Rightarrow$ δ-open $\Rightarrow$ open, from definitions stated above, we have the following diagram of implications:

$$\begin{array}{ccccccc} \text{closed} & \Rightarrow & g\text{-closed} & \Rightarrow & \pi g\text{-closed} & \Rightarrow & rg\text{-closed} \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ \text{semiclosed} & \Rightarrow & gs\text{-closed} & \Rightarrow & \pi gs\text{-closed} & & \end{array}$$

where none of these implications is reversible as the following example shows.

Example 2.3 (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Put $A = \{b\}$. Then A is gs-closed in (X, τ) but it is neither closed nor g-closed. Put $B = \{a\}$. Then B is πgs-closed but not gs-closed in (X, τ).

(b) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Put $A = \{a\}$. Then A is πgs-closed but not πg-closed in (X, τ). Put $B = \{a, b\}$. Then B is rg-closed but not πgs-closed in (X, τ).

Recall that a space (X, τ) is called extremally disconnected [4] if every open subset of X has open closure or equivalently if every regular closed set is open.

Lemma 2.4 [11,6] *For a space (X, τ) the following hold:*

(a) *X is extremally disconnected if and only if $cl(A) = scl(A)$ for every semiopen set A of X.*

(b) *X is extremally disconnected if and only if every semiclosed set of X is α-closed.*

Theorem 2.5 *For a subset A of a space (X, τ), the following hold:*

(a) *If A is π-open and πgs-closed in X, then it is semiclosed and hence clopen.*

(b) *If A is semiopen and πgs-closed in an extremally disconnected space X, then it is πg-closed.*

Proof. (a) If A is π-open and πgs-closed, then $\mathrm{scl}(A) \subset A$ and so A is semiclosed. Hence A is clopen, since π-open set is open and semiclosed open set is closed.

(b) Let $A \subset U$ where U is π-open in X. Since A is πgs-closed, $\mathrm{scl}(A) \subset U$. By Lemma 2.4, $\mathrm{cl}(A) = \mathrm{scl}(A) \subset U$. Hence A is πg-closed. □

Theorem 2.6 *If A is a πgs-closed subset of a space (X, τ), then $scl(A) \setminus A$ does not contain any non-empty π-closed set.*

Proof. Let F be any π-closed set such that $F \subset \mathrm{scl}(A) \setminus A$. Then $A \subset X \setminus F$. Since A is πgs-closed and $X \setminus F$ is π-open, we have $\mathrm{scl}(A) \subset X \setminus F$, i.e. $F \subset X \setminus \mathrm{scl}(A)$. Hence $F \subset \mathrm{scl}(A) \cap (X \setminus \mathrm{scl}(A)) = \emptyset$. This shows that $F = \emptyset$. □

Corollary 2.7 *Let A be a πgs-closed subset of a space (X, τ). Then A is semiclosed if and only if $scl(A) \setminus A$ is π-closed if and only if $A = sint(scl(A))$.*

Remark 2.8 (a) Every finite union of πgs-closed sets may fail to be a πgs-closed set.

(b) Every finite intersection of πgs-closed sets may fail to be a πgs-closed set.

Example 2.9

Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then we have

(a) Let $A = \{a\}$ and $B = \{b\}$. Then A and B are πgs-closed but $A \cup B = \{a, b\}$ is not πgs-closed in (X, τ), since $\{a, b\}$ is π-open and $\mathrm{scl}(A \cup B) = X$.

(b) Let $A = \{a, b, c\}$ and $B = \{a, b, d\}$. Then A and B is πgs-closed but $A \cap B = \{a, b\}$ is not πgs-closed in (X, τ).

However, we have the following.

Theorem 2.10 *If A and B are semiopen πgs-closed sets in extremally disconnected space (X, τ), then $A \cup B$ is πgs-closed.*

Proof. Let $A \cup B \subset U$ where U is π-open in X. Since A and B are πgs-closed, $\mathrm{scl}(A) \subset U$ and $\mathrm{scl}(B) \subset U$. Since X is extremally disconnected, by Lemma 2.4 we have $\mathrm{scl}(F) = \mathrm{cl}(F)$ for any semiopen set F of X. Hence we obtain $\mathrm{scl}(A \cup B) = \mathrm{scl}(A) \cup \mathrm{scl}(B) \subset U$. This shows that $A \cup B$ is πgs-closed. □

Lemma 2.11 [25] *If $A \subset Y \subset X$ and Y is a preopen subset of X, then $\mathrm{scl}_Y(A) = \mathrm{scl}(A) \cap Y$, where $\mathrm{scl}_Y(A)$ is semiclosure of A in subspace Y.*

Lemma 2.12 [28] *Let Y is a preopen subset of a space X. Then we have:*

(a) *If A is π-open in Y, then there exists a π-open set B of X such that $A = B \cap Y$.*

(b) *If A is π-open in X, then $A \cap Y$ is π-open in Y.*

Theorem 2.13 *Let $A \subset Y \subset X$. Then:*

(a) *If Y is preopen in X and A is πgs-closed in X, then A is πgs-closed in Y.*

(b) *If Y is πgs-closed and regular open in X and A is πgs-closed in Y, then A is πgs-closed in X.*

Proof. (a) Let $A \subset U$ where U is π-open in Y. By Lemma 2.12, $U = V \cap Y$ for some π-open V in X. Since A is πgs-closed in X, we have $\mathrm{scl}(A) \subset V$ and by Lemma 2.11, $\mathrm{scl}_Y(A) = \mathrm{scl}(A) \cap Y \subset V \cap Y = U$. Hence A is πgs-closed in Y.

(b) Let $A \subset U$ where U is π-open in X. By Lemma 2.12, $U \cap Y$ is π-open in Y and since A is πgs-closed in Y, $\mathrm{scl}_Y(A) \subset U \cap Y$. By Lemma

2.11 and Theorem 2.5 (a), we have $\mathrm{scl}(A) = \mathrm{scl}_Y(A) \cap Y = \mathrm{scl}_Y(A) \subset U$. Hence A is πgs-closed in X. □

Corollary 2.14 *If A is πgs-closed and regular open subset and B is semiclosed subset of a space X, then $A \cap B$ is πgs-closed.*

Proof. Let $A \cap B \subset U$ where U is π-open in A. Since B is semiclosed in X, $A \cap B$ is semiclosed in A and thus $scl_A(A \cap B) = A \cap B$. That is $\mathrm{scl}_A(A \cap B) \subset U$. Then $A \cap B$ is πgs-closed in the πgs-closed and regular open set A and hence by above theorem $A \cap B$ is πgs-closed in X. □

Theorem 2.15 *If A is πgs-closed in a space X and $A \subset B \subset scl(A)$, then B is πgs-closed.*

Proof. Let $B \subset U$ where U is π-open in X. Since $A \subset U$ and A is πgs-closed, $\mathrm{scl}(A) \subset U$ and then $\mathrm{scl}(B) = \mathrm{scl}(A) \subset U$. Hence B is πgs-closed. □

3. On πgs-open sets

Definition 3.1 A subset A of a space (X, τ) is called πgs-open if its complement $X \setminus A$ is πgs-closed.

Theorem 3.2 *A subset A of a space X is πgs-open if and only if $F \subset sint(A)$ whenever F is π-closed and $F \subset A$.*

Proof. Let $F \subset A$ where F be π-closed in X. Then $X \setminus A \subset X \setminus F$ and $X \setminus F$ is π-open in X. Since $X \setminus A$ is πgs-closed, $\mathrm{scl}(X \setminus A) \subset X \setminus F$. By Theorem 1.6 of [7], we have $\mathrm{scl}(X \setminus A) = X \setminus \mathrm{sint}(A) \subset X \setminus F$, i.e. $F \subset \mathrm{sint}(A)$.

Conversely, let $X \setminus A \subset U$ where U is π-open in X. Then $X \setminus U$ is π-closed and $X \setminus U \subset A$. By hypothesis, we have $X \setminus U \subset \mathrm{sint}(A)$, i.e. $\mathrm{scl}(X \setminus A) = X \setminus \mathrm{sint}(A) \subset U$. This implies $X \setminus A$ is πgs-closed and thus A is πgs-open. □

Theorem 3.3 *If A is a πgs-open subset of X, then $U = X$ whenever U is π-open and sint$(A) \cup (X \setminus A) \subset U$.*

Proof. Let U be a π-open set and $\text{sint}(A) \cup (X \setminus A) \subset U$. Then $X \setminus U \subset (X \setminus \text{sint}(A)) \cap A$, i.e. $(X \setminus U) \subset \text{scl}(X \setminus A) \cap A$. By Theorem 2.6, $X \setminus U = \emptyset$ and hence $U = X$. □

Theorem 3.4 *Let $A \subset Y \subset X$ and Y be π-open and closed in X. If A is πgs-open in Y, then A is πgs-open in X.*

Proof. Let F be any π-closed set and $F \subset A$. Since F is π-closed in Y and A is πgs-open in Y, $F \subset \text{sint}_Y(A)$, where $\text{sint}_Y(A)$ is semi-interior of A in subspace Y, and hence $F \subset \text{sint}(A) \cap Y \subset \text{sint}(A)$. This shows that A is πgs-open in X. □

Theorem 3.5 *If A is πgs-open in X and sint$(A) \subset B \subset A$, then B is πgs-open.*

Proof. Let $F \subset B$ and F be π-closed in X. Since A is πgs-open and $F \subset A$, we have $F \subset \text{sint}(A)$ and thus $F \subset \text{sint}(B)$. Hence B is πgs-open. □

Theorem 3.6 *If A is πgs-closed in X, then scl$(A) \setminus A$ is πgs-open.*

Proof. Let $F \subset \text{scl}(A) \setminus A$ and F be π-closed in X. Then by Theorem 2.6, we have $F = \emptyset$ and hence $F \subset \text{sint}(\text{scl}(A) \setminus A)$. This shows that $\text{scl}(A) \setminus A$ is πgs-open. □

4. On πgs-$T_{1/2}$ spaces

Definition 4.1 A space (X, τ) is called πgs-$T_{1/2}$ if every πgs-closed set is semiclosed.

Next we have a characterization of πgs-$T_{1/2}$ spaces.

Theorem 4.2 *For a space (X, τ) the following conditions are equivalent:*

(a) *X is πgs-$T_{1/2}$.*

(b) *Every singleton of X is either π-closed or semiopen.*

(c) *Every singleton of X is either π-closed or open.*

Proof. (a)$\Rightarrow$(b) Let $x \in X$ and assume that $\{x\}$ is not π-closed. Then clearly $X \setminus \{x\}$ is not π-open. Since X is the only π-open set containing $X \setminus \{x\}$, $X \setminus \{x\}$ is πgs-closed. By (a), it is semiclosed and thus $\{x\}$ is semiopen.

(b)$\Rightarrow$(a) Let A be πgs-closed. Let $x \in \mathrm{scl}(A)$. We consider the following two cases:

Case I. Let $\{x\}$ be π-closed. By Theorem 2.6, $\mathrm{scl}(A) \setminus A$ does not contain $\{x\}$. Since $x \in \mathrm{scl}(A)$, then $x \in A$.

Case II. Let $\{x\}$ be semiopen. Since $x \in \mathrm{scl}(A)$, then $\{x\} \cap A \neq \emptyset$. Thus $x \in A$.

So, in both case, $x \in A$. This show that $\mathrm{scl}(A) \subset A$ or equivalently A is semiclosed.

(b)$\Leftrightarrow$(c) Note that a singleton is semiopen if and only if it is open.□

Dontchev [12] showed that a space (X, τ) is semi-pre-$T_{1/2}$ if and only if every singleton of X is closed or preopen and obtained the following implication: $T_{1/2} \Rightarrow$ semi-pre-$T_{1/2}$ but not conversely. Recently, Park [28] showed that a space (X, τ) is πgp-$T_{1/2}$ if and only if every singleton of X is π-closed or preopen and obtained the following implication: preregular $T_{1/2} \Rightarrow \pi gp$-$T_{1/2} \Rightarrow$ semi-pre-$T_{1/2}$. Since, for subset of a space, regular closed $\Rightarrow$ π-closed $\Rightarrow$ closed, we have the following result:

Remark 4.3 For a space (X, τ) the following implications hold:

$$
\begin{array}{ccccc}
 & & \text{preregular } T_{1/2} & & \\
 & & \Downarrow & & \\
\pi gs\text{-}T_{1/2} & \Rightarrow & \pi gp\text{-}T_{1/2} & \Rightarrow & \text{semi-pre-}T_{1/2}
\end{array}
$$

However, the reverses of the above implications are not always true and the notions of preregular $T_{\frac{1}{2}}$ and πgs-$T_{1/2}$ are independent of each other as the following example shows.

Example 4.4 (a) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is πgs-$T_{1/2}$ but not preregular $T_{1/2}$.

(b) Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}, \{c\}\}$. Then (X, τ) is preregular $T_{1/2}$ but not πgs-$T_{1/2}$.

(c) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then (X, τ) is semi-pre-$T_{1/2}$ (even $T_{1/2}$, α-space) but neither πgs-$T_{1/2}$ nor πgp-$T_{1/2}$.

Dontchev [12] proved that every α-space is semi-pre-$T_{1/2}$ but not conversely. An *α-space* [11] is a space in which every α-closed set is closed, i.e. $\tau^\alpha = \tau$. Example 4.4 and the following example show that the concepts of α-spaces and πgs-$T_{1/2}$ spaces are independent of each other.

Example 4.5 Let X be the real numbers with the usual topology. Then X is cleanly πgs-$T_{1/2}$ space. Set $A = \{\frac{1}{i} : i = 1, 2, 3, \cdots\}$. Then A is nowhere dense but obviously A is not closed. By Lemma 1.2 of [11], X is not an α-space.

The next theorem gives further characterization of πgs-$T_{1/2}$ spaces.

Theorem 4.6 *For a space (X, τ) the following conditions are equivalent:*

(a) *X is πgs-$T_{1/2}$.*

(b) *Every preclosed singleton of X is π-closed.*

(c) *Every non-open singleton is π-closed.*

Proof. (a)$\Rightarrow$(b) Let $x \in X$ and assume that $\{x\}$ is preclosed. Since every singleton in any space is either open or preclosed [22, Lemma 2.3], $\{x\}$ is not open and hence by Theorem 4.2, $\{x\}$ is π-closed.

(b)$\Rightarrow$(a) If for some $x \in X$, $\{x\}$ is not open, then $\{x\}$ is preclosed. Then by (b), it is π-closed and hence by Theorem 4.2, X is πgs-$T_{1/2}$.

(b)⇔(c) It is obvious. □

Recall that a space (X, τ) is called submaximal [4] if every dense subset of X is open. Reilly and Vamanamurthy [29] showed that (X, τ) is submaximal if and only if every preopen subset of X is open.

Theorem 4.7 *Let (X, τ) be a submaximal space in which every closed set is π-closed. Then the following conditions are equivalent:*

(a) *X is semi-pre-$T_{1/2}$.*

(b) *X is πgs-$T_{1/2}$.*

Proof. Obvious. □

Theorem 4.8 *Let A be open in (X, τ). If (X, τ) is a πgs-$T_{1/2}$ space, then the subspace $(A, \tau|A)$ is also πgs-$T_{1/2}$.*

Proof. Using Theorem 4.2, it is enough to show that every singleton of $(A, \tau|A)$ is π-closed or semiopen. Let $x \in A$. If $\{x\}$ is π-closed in X, then it is π-closed in $(A, \tau|A)$. Assume that $\{x\}$ is semiopen in X. Since the intersection of a semiopen set and a open set is semiopen in open set and since A is open, then $\{x\}$ is semiopen in $(A, \tau|A)$. □

5. πgs-continuous and πgs-irresolute functions

Definition 5.1 A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(a) πgs-continuous if $f^{-1}(V)$ is πgs-closed in X for every closed set V of Y;

(b) πgs-irresolute if $f^{-1}(V)$ is πgs-closed in X for every πgs-closed set V of Y.

Remark 5.2 From Definitions 1.2 and 5.1, we obtain the following diagram:

continuity	⇒	g-continuity	⇒	πg-continuity	⇒	rg-continuit
⇓		⇓		⇓		
semi-continuity	⇒	gs-continuity	⇒	πgs-continuity		

(a) None of these implications is reversible as shown by Example 1 of Balachandran et al. [3] and the following Example 5.3.

(b) The notions of πgs-continuity and rg-continuity are independent of each other.

(c) The notions of πg-continuity and gs-continuity are independent of each other.

(d) Every πgs-irresolute function is πgs-continuous, but not conversely.

(e) The notions of πgs-irresoluteness and gs-irresoluteness are independent of each other.

Example 5.3 (a) Let $X = \{a,b,c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{X, \emptyset, \{c\}, \{a,b\}\}$. Let $f : (X,\tau) \rightarrow (X,\sigma)$ be the identity. Then f is rg-continuous but not πgs-continuous, since $\{a,b\}$ is closed in (X,σ) and $f^{-1}(\{a,b\})$ is not πgs-closed in (X,τ).

(b) Let $X = \{a,b,c\}, \tau = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{X, \emptyset, \{c\}\}$. Let $f : (X,\tau) \rightarrow (X,\sigma)$ be the identity. Then f is πgs-continuous (even πg-continuous) but not gs-continuous, since $\{a,b\}$ is closed in (X,σ) and $f^{-1}(\{a,b\})$ is not gs-closed in (X,τ).

(c) Let $X = \{a,b,c,d\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ and $\sigma = \{X, \emptyset, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$. Let $f : (X,\tau) \rightarrow (X,\sigma)$ be a function defined by $f(a) = c$, $f(b) = a$, $f(c) = d$ and $f(d) = b$. Then f is gs-continuous but not rg-continuous, since $\{d\}$ is closed in (X,σ) and $f^{-1}(\{d\})$ is not rg-closed in (X,τ). Moreover, since X is the only nonempty regular open set in (X,σ), every subset of X is πgs-closed in (X,σ). Hence f is not πgs-irresolute, since $\{a,c\}$ is πgs-closed in (X,σ) and $f^{-1}(\{a,c\})$ is not πgs-closed in (X,τ).

Example 5.4 (a) Let $X = \{a,b,c\}$, $\tau = \{X, \emptyset, \{a,b\}\}$ and $\sigma = \{X, \emptyset, \{c\}, \{b,c\}\}$. Let $f : (X,\tau) \rightarrow (X,\sigma)$ be a function defined by $f(a) = c$, $f(b) = b$, and $f(c) = a$. Then f is πgs-irresolute but not gs-irresolute.

(b) Let $X = \{a,b,c,d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $\sigma = \{X, \emptyset,$

$\{a\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be a function defined by $f(a) = a$, $f(b) = f(c) = c$ and $f(d) = d$. Then f is gs-irresolute but not πgs-irresolute.

Theorem 5.5 *For a function $f : (X, \tau) \to (Y, \sigma)$, the following hold:*

(a) *If f is πgs-irresolute and X is πgs-$T_{1/2}$, then f is irresolute.*

(b) *If f is πgs-continuous and X is πgs-$T_{1/2}$, then f is semi-continuous.*

(c) *If f is πgs-continuous and X is an extremally disconnected α-space, then f is πg-continuous.*

Proof. (a) Let V be a semiclosed subset of Y. Then V is πgs-closed in Y and since f is πgs-irresolute, then $f^{-1}(V)$ is πgs-closed in X. Since X is πgs-$T_{1/2}$, $f^{-1}(V)$ is semiclosed in X. Hence f is irresolute.

(b) Similar to (a).

(c) Let V be any closed subset of Y. Let $f^{-1}(V) \subset U$, where U is π-open in X. Then $f^{-1}(V)$ is πgs-closed in X. Since X is extremally disconnected α-space, $\text{cl}(f^{-1}(V)) = \text{scl}(f^{-1}(V)) \subset U$, i.e. $f^{-1}(V)$ is πg-closed in X. Hence f is πg-continuous. □

Theorem 5.6 *If $f : (X, \tau) \to (Y, \sigma)$ is an almost π-continuous and presemiclosed function, then $f(A)$ is πgs-closed in Y for every πgs-closed set A of X.*

Proof. Let A be any πgs-closed set of X. Let $f(A) \subset V$, where V is regular open in Y. Then V is π-open. Since f is almost π-continuous, $f^{-1}(V)$ is π-open in X and $A \subset f^{-1}(V)$. Then we have $\text{scl}(A) \subset f^{-1}(V)$ and hence $f(\text{scl}(A)) \subset V$. Since f is presemiclosed, $f(\text{scl}(A))$ is semiclosed in Y and hence $\text{scl}(f(A)) \subset \text{scl}(f(\text{scl}(A))) \subset V$. This shows that $f(A)$ is πgs-closed in Y. □

The composition of two πgs-continuous function need not be πgs-continuous. For consider the following example:

Example 5.7 Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$, $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\rho = \{X, \emptyset, \{c, d\}\}$. Let $f : (X, \tau) \to$

(X, σ) be the identity and $g : (X, \sigma) \to (X, \rho)$ be a function defined by $g(a) = a$, $g(b) = c$, $g(c) = d$ and $g(d) = d$. Then f and g are πgs-continuous but the composition $g \circ f$ is not πgs-continuous, since $\{a, b\}$ is closed in (X, ρ) and $(g \circ f)^{-1}(\{a, b\})$ is not πgs-closed in (X, τ).

However, the following theorem holds:

Theorem 5.8 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ be two functions. Then:*

(a) *If f is πgs-continuous and g is continuous, then $g \circ f$ is πgs-continuous.*

(b) *If f is πgs-irresolute and g is πgs-irresolute, then $g \circ f$ is πgs-irresolute.*

(c) *If f is πgs-irresolute and g is πgs-continuous, then $g \circ f$ is πgs-continuous.*

(d) *Let Y be a πgs-$T_{1/2}$ space. If f is irresolute and g is πgs-continuous, then $g \circ f$ is semi-continuous.*

Proof. Obvious. □

Theorem 5.9 *Let $f : (X, \tau) \to (Y, \sigma)$ be a πgs-irresolute and pre-semiclosed surjection. If X is a πgs-$T_{1/2}$ space, then Y is also πgs-$T_{1/2}$.*

Proof. Let A be a πgs-closed subset of Y. Since f is πgs-irresolute, then $f^{-1}(A)$ is πgs-closed in X. Since X is πgs-$T_{1/2}$, then $f^{-1}(A)$ is semiclosed in X. By the rest of the assumption it follows that A is semiclosed in Y. Hence Y is πgs-$T_{1/2}$. □

Theorem 5.10 *Let $f : (X, \tau) \to (Y, \sigma)$ be a rc-preserving and pre-semiopen (open) bijection. If X is a πgs-$T_{1/2}$ space, then Y is also πgs-$T_{1/2}$.*

Proof. Let $y \in Y$. Since X is πgs-$T_{1/2}$ and f is bijective, then by Theorem 4.2 for some $x \in X$ with $f(x) = y$, we have $\{x\}$ is π-closed or semiopen (open). If $\{x\}$ is π-closed, then $\{y\} = f(\{x\})$ is π-closed since f is rc-preserving and bijective. If $\{x\}$ is semiopen (open), then $\{y\}$ is semiopen (open) since f is presemiopen (open). Hence by Theorem 4.2, Y is πgs-$T_{1/2}$. □

References

[1] D. Andrijevic, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.

[2] S.P. Arya and T. Nour, Characterizations of s-normal spaces, Indian J. Pure Appl. Math., 21 (1990), 717-719.

[3] K. Balachandran, P. Sundaram and H. Maki, On generalized continuous maps in topological spaces, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 12 (1991), 5-13.

[4] N. Bourbaki, General Topology, Part I, Addison Wesley, Reading, Mass., 1966.

[5] M.C. Caldas, Semi-generalized continuous maps in topological spaces, Portugal. Math. 52(4) (1995), 399-407.

[6] J. Cao, M. Ganster and I. Reilly, Submaximality, extremal disconnectedness and generalized closed sets, Houston J. Math., 24 (1998), 681-688.

[7] S.G. Crossely and S.K. Hildebrand, Semi-closure, Texas J. Sci., 22 (1971), 99-112.

[8] S.G. Crossely and S.K. Hildebrand, Semi-topological properties, Fund. Math., 74 (1972), 233-254.

[9] R. Devi, K. Balachandran and H. Maki, Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces, Indian J. Pure Appl. Math., 26 (1995), 271-284.

[10] R. Devi, H. Maki and K. Balachandran, Semi-generalized closed maps and generalized semi-closed maps, Mem. Fac. Sci. Kochi Univ. Math., 14 (1993), 41-54.

[11] J. Dontchev, An answer to a question of Mrseic and Reilly, Questions Answers Gen. Topology, 12 (2) (1994), 205-207.

[12] J. Dontchev, On generalizing semi-preopen sets, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 16 (1995), 35-48.

[13] J. Dontchev and T. Noiri, Quasi-normal spaces and πg-closed sets, Acta Math. Hungar., 89(3) (2000), 211-219.

[14] Y. Gnanambal, On generalized preregualar closed sets in topological spaces, Indian J. Pure Appl. Math., 28(3) (1997), 351-360.

[15] Y. Gnanambal and K. Balachandran, On gpr-continuous function in topological spaces, Indian J. Pure Appl. Math., 30(6) (1999), 581-593.

[16] D.S. Jankovic and I.L. Reilly, On semiseparation properties, Indian J. Pure Appl. Math., 16(9) (1985), 957-964.

[17] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70 (1963), 36-41.

[18] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19 (1970), 89-96.

[19] S.N. Maheswari and R. Prasad, On s-normal spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie, 22 (1978), 27-29.

[20] D. Maio and T. Noiri, On s-closed space, Indian J. Pure Appl. Math., 18(3) (1987), 226-233.

[21] H. Maki, K. Balachandran and R. Devi, Remarks on semi-generalized closed sets and generalized semi-closed sets, Kyungpook Math. J., 36 (1996), 155-163.

[22] H. Maki, J. Umehara and T. Noiri, Every topological space is pre-$T_{1/2}$, Mer. Fac. Sci. Kochi Univ. Ser. A. Math., 17 (1996), 33-42.

[23] A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and weak precontinuous functions, Proc. Math. Phys. Soc. Egypt, 53 (1982), 47-53.

[24] O. Njåstad, On some classes of nearly open sets, Pacific J. Math. 15 (1965), 961-970.

[25] T. Noiri and B. Ahmad, A note on semi-open functions, Math. Sem. Notes Kobe Univ., 10 (1982), 437-441.

[26] N. Palaniappan and K.C. Rao, Regular generalized closed sets, Kyungpook Math. J., 33 (1993), 211-219.

[27] J.H. Park, On s-normal spaces and some functions, Indian J. Pure

Appl. Math. 30(6) (1999), 575-580.

[28] J.H. Park, On πgp-closed sets in topological spaces, to appear.

[29] I.L. Reilly and M.K. Vamanamurthy, On some questions concerning preopen sets, Kyungpook Math. J., 30(1) (1990), 87-93.

[30] N.V. Veličko, H-closed topological spaces, Amer. Math. Soc. Transl., 78 (1968), 103-118.

[31] V. Zaitsav, On certain classes of topological spaces and their bicompactifications, Dokl. Akad. Nauk SSSR 178 (1968), 778-779.

Bu Young Lee, Su-Young Shin
Department of Mathematics,
Dong-A University,
Pusan 604-714, Korea(south)

Jin Han Park
Division of Mathematical Sciences,
Pukyong National University,
Pusan 608–739, Korea(south)
Email : jihpark@pknu.ac.kr