Structural Dynamics Optimization by Second Order Sensitivity with respect to Finite Element Parameter

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유한요소 구조 인자의 2차 민감도에 의한 동적 구조 최적화

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Abstract

This paper discusses design sensitivity analysis and its application to a structural dynamics modification. Eigenvalue derivatives are determined with respect to the element parameters, which include intrinsic property parameters such as Young's modulus, density of the material, diameter of a beam element, thickness of a plate element, and shape parameters. Derivatives of stiffness and mass matrices are directly calculated by derivatives of element matrices. The first and the second order derivatives of the eigenvalues are then mathematically derived from a dynamic equation of motion of FEM model. The calculation of the second order eigenvalue derivative requires the sensitivity of its corresponding eigenvector, which are developed by Nelson's direct approach. The modified eigenvalue of the structure is then evaluated by the Taylor series expansion with the first and the second derivatives of eigenvalue. Numerical examples for simple beam and plate are presented. First, eigenvalues of the structural system are numerically calculated. Second, the sensitivities of eigenvalues are then evaluated with respect to the element intrinsic parameters. The most effective parameter is determined by comparing sensitivities. Finally, we predict the modified eigenvalue by Taylor series expansion with the derivatives of eigenvalue for single parameter or multi parameters. The examples illustrate the effectiveness of the eigenvalue sensitivity analysis for the optimization of the structures.

Key Words : Finite Element Method(유한요소법), Eigenvalue(고유값), Eigenvector(고유벡터), Sensitivity(민감도), Design Modification(설계변경), Optimization(최적화)

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1. Introduction

The modification of physical characteristics of a structure requires iterative structural analysis due to design changes. The direct approach includes numerical methods such as the finite difference and the finite element method. Though this approach is suitable for relatively small proportion of design changes, it requires a lot of iterative calculation as well as much computational effort to reach constraint limits for a large system such as stress, displacement, frequency and weight limits. In addition, iterative methods, which are most useful for relatively small changes of the structure, have convergence problem, such that the convergence to the true solution is not always ensured or might be too slow. Thus, approximate reanalysis methods have become important tools for the analysis of a structure. This is intimately connected with the sensitivity.

When dynamic behavior such as natural frequency or mode shape is to be determined, the eigensystem should be solved. In order to predict the modified eigenvalue due to design changes, the derivatives of the eigenvalues and their corresponding eigenvectors should be evaluated. Two methods for calculating the eigensensitivity are the adjoint solution approach and the direct solution approach. In the adjoint method, Lancaster[1] derived expressions for the first and the second derivatives of eigenvalues with respect to a single parameter. Morgan developed the same idea, but eigenvectors are not explicitly obtained[2]. Garg and Rudisill proposed different direct methods to obtain the eigensensitivity for symmetric system, which do not require any left eigenvectors[3~5]. Nelson developed Rudisill and Chu's method for an arbitrary n^{th} symmetric or non-symmetric system[6]. This is a very useful method to calculate derivative of a eigenvector when the matrix of the system is sparse. Murthy and Haftka established the general guidelines for the selection of the most efficient method[7]. They found the efficiency of these methods depends on the number of design variables as well as the number of eigenvalues. The finite element method has been recently used for the analysis of structures extensively. Element parameters are taken as design variables in calculating the eigensensitivity. Kirch presented some approximate reanalysis methods based on series expansion and modified non-polynomial series[8]. Kirsch and Toledano suggested several effective approximation techniques and applied them to static problems[9]. Rizai and Bernard used the first and the second derivatives of eigenvalues for predicting the modified eigenvalue due to design changes. They determined the derivatives of stiffness and mass matrices by the finite difference method. Vanderplaats took element parameters as variables[10]. First derivatives of eigenvalues were applied to approximate optimization problems. They considered move limits of 80% changes relative to the predefined design, and showed that this approach allows the iteration number as well as computational time to be reduced.

In this paper, sensitivity analysis and modification by Taylor series with the first and the second derivatives of eigenvalue are combined together to effectively change the eigenvalue of a structure. For design changes, the most sensitive element parameter in the structure is first identified by eigensensitivity. This element is then modified and the new eigenvalue can be approximately predicted. Eigen-sensitivities with respect to element intrinsic parameters are derived. We then applied the Nelson's direct method to the eigenvalue problem of dynamic system in order to calulate the sensitivity of the eigenvector. The approximate eigenvalues can be predicted for design changes. The approximate method is based on the Taylor series expansion for a single or multiple variables in this work to find the modified frequency due to design changes. The modified frequencies are not only approximated by using a Taylor series expansion but also compared to direct solutions, which are evaluated from a finite element procedure. Intrinsic parameters such as a thickness of a plate element or the diameter of the circular beam are considered. A program has been developed based on the above procedure.

2. Derivatives of Eigenvalue and Eigenvector

2.1 Equation of Motion

Consider a continuum structure such as beam or plate with mass density per unit area $\rho(x_m)$, applied load $f_i(x_m,t)$. $\rho(x_m)$ and $f_i(x_m,t)$ are differentiable in the closure of domain D. The equation of motion of the structure for the forced vibration without damping is given by

$$-L[w(x_i,t)]+f(x_i,t)=M[w(x_i,t)] \quad x_i \in D$$

with boundary condition and initial condition as

$$B_m[(x_i,t)]=C_m[(x_i,t)] \quad x_i \in \Gamma$$
$$I_n[(x_i,t)]=g_n(x_i,t) \quad x_i \in D$$

L and M are self-adjoint linear operators on D, and are also self adjoint linear operators for the boundary condition with initial conditions. I_n is also self adjoint linear operator for the initial condition on domain D. Γ is the boundary. $f(x_i,t)$ is deleted for free vibration. The equation of motion with variable separation yields eigenvalue problem as

$$L[W(x_i)]=\lambda M[W(x_i)] \quad x_i \in D$$
$$B_m[(x_i)]=C_m[(x_i)] \quad x_i \in \Gamma$$

$\lambda=\omega^2$ and ω is natural frequency. The eigenvalue equation is then written in the form

$$([K]-\lambda_i[M])X_i=[0] \tag{1}$$

λ_i is the i^{th} eigenvalue, X_i is the i^{th} eigenvector. [K] and [M] are stiffness matrix and mass matrix respectively, which are symmetrical matrices Eigenvectors are taken to be M-orthogonal, and then equation (1) is pre-multiplicated by X_i^T. i^{th} The ith eigenvalue can be written by

$$\lambda_i=X_i^T[K]X_i \tag{2}$$

2.2 Derivative of Eigenvalue

The differentiation of equation (1) with respect to the design parameter, p, yields

$$\frac{\partial}{\partial p}[([K]-\lambda_i[M])X_i]=[0] \tag{3}$$

or

$$([K]-\lambda_i[M])\frac{\partial X_i}{\partial p}=-\left(\frac{\partial[K]}{\partial p}-\lambda_i\frac{\partial[M]}{\partial p}-\frac{\partial\lambda_i}{\partial p}[M]\right)X_i \tag{4}$$

Since $X_i^T([K]-\lambda_i[M])=[0]$ and $X_i^T[M]X_i=[I]$, pre-multiplication of equation (4) by X_i^T yields

$$\frac{\partial\lambda_i}{\partial p}=X_i^T\frac{\partial[G_i]}{\partial p}X_i \tag{5}$$

and

$$\frac{\partial[G_i]}{\partial p}=\frac{\partial[K]}{\partial p}-\lambda_i\frac{\partial[M]}{\partial p} \tag{6}$$

The eigenvalue sensitivities calculated by equation (5) tell us which parameter are the most sensitive variables and hence reduces the computational effort to predict approximate eigenvalues due to the design change. However, if the second order derivatives of eigenvalues are included in the approximation procedure, approximate eigenvalues may be much closer to exact solutions in the neighborhood of the baseline design parameter. The second derivative of the eigenvalues is drived by taking derivative of equation (5) in the expression.

$$\begin{aligned}\frac{\partial^2\lambda_i}{\partial p^2}=&X_i^T\left(\frac{\partial^2[K]}{\partial p^2}-\lambda_i\frac{\partial^2[M]}{\partial p^2}\right)X_i\\&-\left(X_i^T\frac{\partial[M]}{\partial p}X_i\right)\left(X_i^T\frac{\partial[G_i]}{\partial p}X_i\right)\\&-2X_i^T\frac{\partial[G_i]}{\partial p}\frac{\partial X_i}{\partial p}\end{aligned} \tag{7}$$

Equation (7) requires calculation of the first order eigenvalue sensitivity and its corresponding eigenvector sensitivity.

2.3 Derivative of Eigenvector

The sensitivity of eigenvector is required for the calculation of the second order derivative of eigenvector.

The substitution of equation (2) and (5) into equation (4) yields

$$([K]-\lambda_i[M])\frac{\partial X_i}{\partial p}=F_i \tag{8}$$

and

$$F_i=-\frac{\partial[G_i]}{\partial p}X_i+\left(X_i^T\frac{\partial[G_i]}{\partial p}X_i\right)[M]X_i \tag{9}$$

Since $X_i^TF_i=0$, the matrix on the left-hand side of equation (7) is of rank (n-1), but the n-equations are consistent. Although these equations can't be uniquely solved for the eigenvector derivative, any n-component vector can be represented as a linear combination of the remaining (n-1) eigenvectors. The derivative of eigenvector can be then expressed as

$$\frac{\partial X_i}{\partial p}=a_kX_k=XA \tag{10}$$

A is a column vector with element a_k. Substitution of equation (10) into equation (8) and pre-multiplication by X^T gives

$$X^T([K]-\lambda_i[M])XA=X^TF_i \tag{11}$$

or

$$([\Lambda]-\lambda_i[I]A=X^TF_i \tag{12}$$

$([\Lambda]-\lambda_i[I])$ denotes a diagonal matrix with elements $(\lambda_k-\lambda_i)$ at row and column k. The element of matrix A, is expressed as follows

$$a_k=\frac{X^TF_i}{\lambda_k-\lambda_i}\qquad k\neq i \tag{13}$$

The eigenvector derivative can be uniquely expressed in terms of (n-1) of the system eigenvectors and the i^{th} as follows.

$$\frac{\partial X_i}{\partial p}=\sum_{k=1}^{n}a_kX_k+a_iX_i \tag{14}$$

or

$$\frac{\partial X_i}{\partial p}=V_i+a_iX_i \tag{15}$$

Substituting equation (15) into equation (8) and applying $X_i^T([K]-\lambda_i[M])=[0]$ and $X_i^T[M]X_i=[I]$, equation (8) can be written in the form

$$([K]-\lambda_i[M])\,V_i=F_i \tag{16}$$

In order to determine a_i in equation (15), the partial derivative of $X_i^T[M]X_i=[I]$ is used

$$X_i^T\frac{\partial[M]}{\partial p}X_i+2X_i^T[M]\frac{\partial X_i}{\partial p}=0 \tag{17}$$

or

$$a_i=-X_i^T[M]\,V_i-\frac{1}{2}X_i^T\frac{\partial[M]}{\partial p}X_i \tag{18}$$

The eigenvector derivatives are calculated by equation (15) where V_i and a_i can be calculated from linear equation (16) and equation (18). However, V_i is very hard to determine since the inverse of $([K]-\lambda_i[M])$ is singular matrix and equation (16) is of rank (n-1). In comparing equation (1) with equation (16), X_i is the homogeneous solution, which is known and V_i is a particular solution. By partitioning of the matrix in equation (16) can be written in the form

$$\begin{pmatrix}(K-\lambda_iM)_{11} & (K-\lambda_iM)_{1k} & (K-\lambda_iM)_{13}\\ (K-\lambda_iM)_{k1} & (K-\lambda_iM)_{kk} & (K-\lambda_iM)_{k3}\\ (K-\lambda_iM)_{31} & (K-\lambda_iM)_{3k} & (K-\lambda_iM)_{33}\end{pmatrix}\begin{pmatrix}V_1\\ \nu_k\\ V_3\end{pmatrix}=\begin{pmatrix}F_1\\ f_k\\ F_3\end{pmatrix} \tag{19}$$

Elimination of k^{th} equation in the simultaneous linear equation (18) transforms the equation (19) as follows

$$\begin{pmatrix}(K-\lambda_iM)_{11} & (K-\lambda_iM)_{13}\\ (K-\lambda_iM)_{31} & (K-\lambda_iM)_{33}\end{pmatrix}\begin{pmatrix}V_1\\ V_3\end{pmatrix}=\nu_k\begin{pmatrix}(K-\lambda_iM)_{1k}\\ (K-\lambda_iM)_{3k}\end{pmatrix}+\begin{pmatrix}F_1\\ F_3\end{pmatrix} \tag{20}$$

The complete solution for V_1 and V_3 is obtained by solving equation (20) with setting $\nu_k=0$. Therefore, the k^{th} - k^{th} row and column element in equation (19) are zero except for k^{th} - k^{th} element, which can be written in the form.

$$\begin{pmatrix} (K-\lambda_i M)_{11} & 0 & (K-\lambda_i M)_{13} \\ 0 & 1 & 0 \\ (K-\lambda_i M)_{31} & 0 & (K-\lambda_i M)_{33} \end{pmatrix} \begin{pmatrix} V_1 \\ \nu_k \\ V_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ 0 \\ F_3 \end{pmatrix} \tag{21}$$

The sensitivities of eigenvectors are directly calculated by substituting V into equation (13), which can be written in the form

$$\frac{\partial X_i}{\partial p} = \begin{pmatrix} V_1 \\ \nu_k \\ V_3 \end{pmatrix} + a_i \begin{pmatrix} X_1 \\ x_k \\ X_3 \end{pmatrix} \tag{22}$$

The matrix on the left-hand side of equation (22) is of rank n and has the same bandedness as the characteristic equation (1). The important step in the above calculation is to decide the k^{th} element of the eigenvector X_i such that the absolute of x_k is the largest number in the elements of eigenvector $X_i^T = \{x_1, x_2, \cdots x_n\}$. If the absolute value of x_k is very small compared to the largest component in the X_i, the numerical solution of the equation (22) yields inaccurate results or will blow up. $[K]$, $[M]$, $\partial[K]/\partial p$, $\partial[M]/\partial p$, $\partial^2[K]/\partial p^2$ and $\partial^2[M]/\partial p^2$ are given in appendix

2.4 Approximate eigenvalue for design modification

Modified eigenvalues due to change of a single design parameter are approximately determined by Taylor series expansion with neglecting terms of higher than 3rd derivative

$$\lambda_i(p_{k0} + dp_k) = \lambda_i(p_{k0}) + \frac{\partial \lambda_i}{\partial p_k} dp_k + \frac{1}{2} \frac{\partial^2 \lambda_i}{\partial p_k} (dp_k)^2 \tag{23}$$

p_{k0} and dp_k are the predefined value of design parameter, p_k, and its increment. Now, consider a case that multiple parameters are simultaneously modified. The amount of change to each parameter is first to be determined. Design change to each parameter is determined based on the relative sensitivity, which is defined by

$$\xi_j = \frac{\lambda_{i,j}}{|\lambda_{i,m}|} \tag{24}$$

$\lambda_{i,m}$ is the sensitivity of i^{th} eigenvalue with respect to parameter, p_m. $|\lambda_{i,m}|$ is the maximum value in absolute eigenvalue derivatives. Design changes are then determined by

$$p_j = p_{jo} + dp_j \tag{25}$$

and

$$dp_j = \gamma \xi_j \tag{26}$$

γ is the weighting function, which depends on the type of parameter or structural model. The Taylor series expansion then becomes,

$$\lambda_i(\gamma) = \lambda_i(p_{10}, p_{20}, \cdots, p_{n0}) + \sum_{j=1}^{n} \frac{\partial \lambda_i}{\partial p_j}(\gamma \xi_j) \tag{27}$$

The modified eigenvalues are completely approximated by equation (27), although the determination of the weighting function depends on the experience of design engineering.

3. Numerical Results and Discussions

3.1 Circular cantilever beam

Fig. 1 shows a circular cantilever beam. It is composed of five elements with ten degrees of freedom. Its elastic modulus is 207GPa, mass density 7.754×10^{-9}kg/mm^3, initial diameter 6.35mm, length of the beam 127mm, Poisson's ratio 0.3. The element diameter is considered as a parameter to evaluate the sensitivity for the lowest eigenvalue, in which d_q means the diameter of element q. The lowest frequency is 283.03Hz. Mass matrix, stiffness matrices, their derivatives and eigenvalue derivatives of the system are directly calculated by self-developed program. Table 1 presents derivatives on the lowest

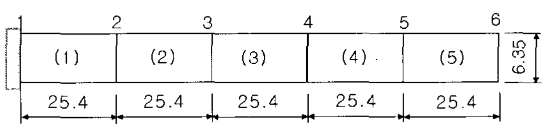

Fig. 1 FEM model of the circular cantilever beam

natural frequency and the absolute value of $\lambda_{1,1}$ is the largest as well as that of $\lambda_{1,3}$ is the smallest. This means that d_1 is the most sensitive parameter and d_3 is the least sensitive parameter for the modification of the fundamental natural frequency. The lowest natural frequency is then effectively modified as modifying d_1. Frequencies due to design change are approximately calculated based on a Taylor series expansion, in which first and second order terms are included.

Fig. 2~6 illustrates a comparison of FEM solutions, where mod. freq. 1 means approximate frequencies based on the first order sensitivity and mod. Freq. 2 means approximate frequencies based on the first and the second order derivatives. This shows that approximate frequency obtained by including the second order term is better than

Table 1 Derivatives of eigenvalue with respect to element diameter: f_1=283.03Hz

Diameter	d_1	d_2	d_3	d_4	d_5
$\partial f_1/\partial d_i$	53.47	24.60	4.30	-11.15	-26.73
$\partial^2 f_1/\partial d_i^2$	-0.5	-0.86	-0.60	-0.46	-0.17

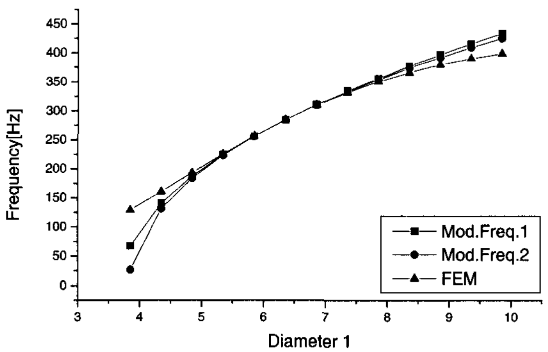

Fig. 2 Approximate frequency due to d_1 change

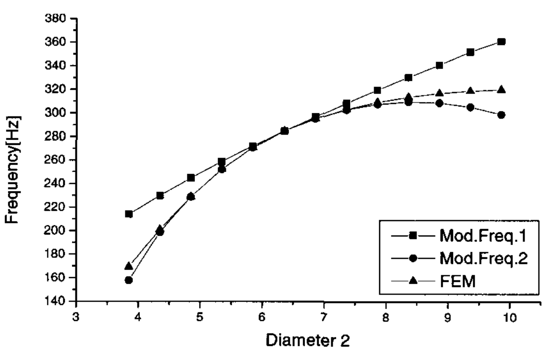

Fig. 3 Approximate frequency due to d_2 change

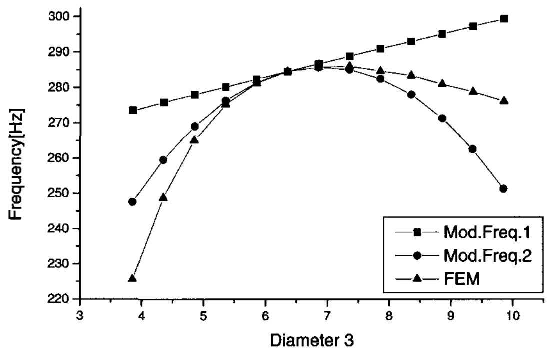

Fig. 4 Approximate frequency due to d_3 change

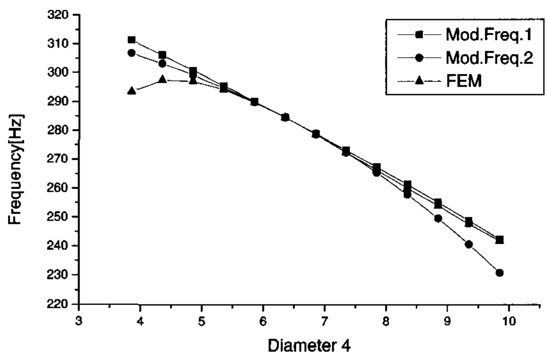

Fig. 5 Approximate frequency due to d_4 change

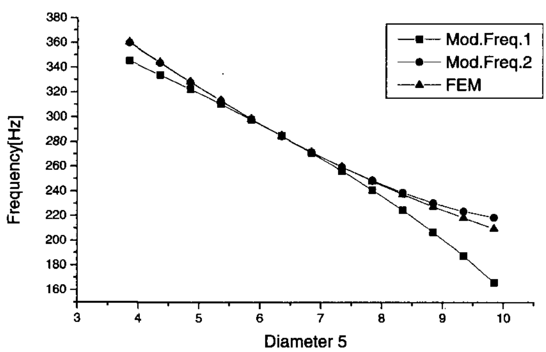

Fig. 6 Approximate frequency due to d_5 change

those obtained only by including the first order term except for $d_1 < 5.5$mm. Since the second order derivative turns rapidly from a positive value to a negative value in the neighborhood of the baseline d_{10}. It seems clear that it is necessary to re-evaluate the system, though frequencies for the deign change are effectively approximated by using second order sensitivity. Fig. 4 shows a result for the least sensitive design parameter d_3. Fig. 7 shows frequency changes with respect to weighting value γ. Fig. 8 shows that we optimize the circular beam from 283.03Hz to 400Hz with three times re-evaluations by simultaneous multivariable changes. Fig. 9 presents the

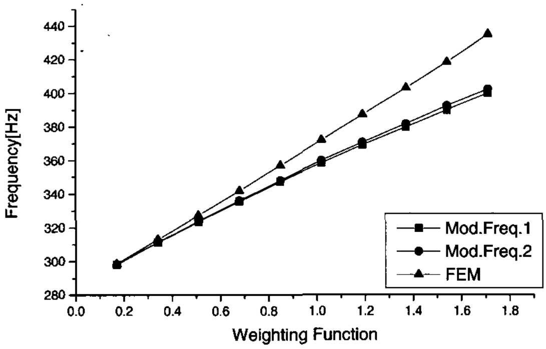

Fig. 7 Approximate frequency to weighting function

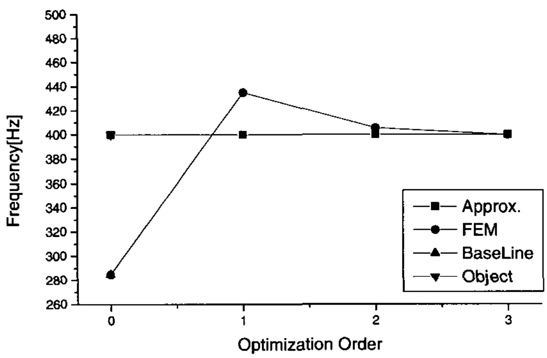

Fig. 8 Optimization order for the natural freq. of 400Hz

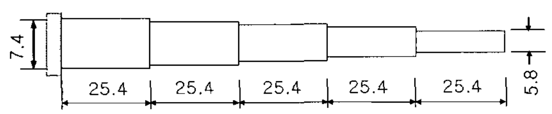

Fig. 9 Optimized beam with the natural freq. of 400Hz

beam geometry with 400Hz. Whenever the difference between the modified frequencies and the FEM solutions is greater than 5%, the cantilever beam is reevaluated by FEM. The third approximated frequencies are greatly close to the FEM results.

3.2 Fixed-fixed rectangular plate

The clamped rectangular plate is shown in Fig. 10. The plate has 64 nodes, 49 elements and 108 degrees of freedom. Its elastic modulus is 207GPa, mass density 7.754×10^{-9}kg/mm^3, thickness 25.4mm, the side length of the plate 2540mm, Poisson's ratio 0.3. Its fundamental frequency is 11.34Hz. We modified the plate with 20 Hz of the fundamental frequency by applying the same procedure for the previous beam design. Thickness of element was taken as design parameter. The most sensitive elements for the fundamental frequency were t_4, t_{22}, t_{28}, and t_{46}, whose 1st and 2nd derivative were 0.0263, 0.0012, respectively. The approximate frequencies calculated by using the 1st and 2nd order sensitivities are compared to FEM results in the Fig. 11. This shows that approximate frequency obtained by the second order sensitivity in the Taylor series expansion is closer to the FEM results. Fig. 12 shows the optimized cross section of the plate whose fundamental frequency is 20Hz.

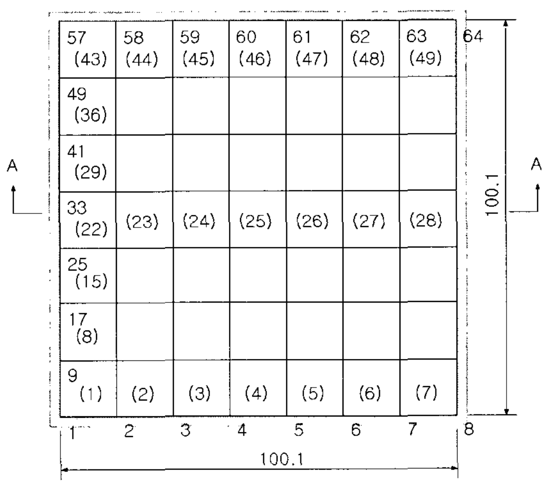

Fig. 10 FEM model of clamped plate

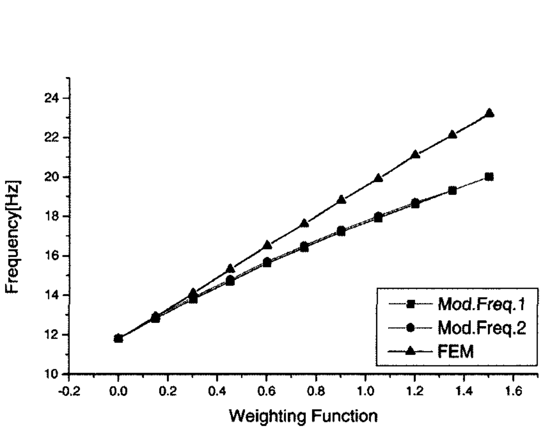

Fig. 11 Approximate frequency to weighting function

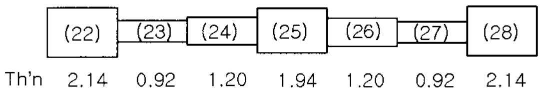

Fig. 12 Optimized plate (A-A cross section in Fig. 10)

4. Conclusions

A finite element program is developed for the calculation of sensitivity and modification of one and two dimensional structures. The program includes IMSL subroutines. The first and the second order sensitivity based on element design parameter are calculated for the eigenvalue. The first order sensitivity is used for locating the most sensitive element. The first and the second order sensitivity are used for design modification by Taylor series expansion. The approximated frequency by the first and the second derivatives is usually better than that by the first. In addition, the eigenvalue should be often reevaluated by FEM code, as the inflection point is close to the baseline and the change amount of parameter is large. Many design engineering problems allow for weighting factor to be generally greater than -1.0 and less than 1.0 for the design optimization. However, a convenient computational method for determining the move limit would be useful in a practical application. The numerical results presented show that the new natural frequencies are approximated from a Taylor series expansion. The sensitivity approach makes greatly reduce the routine efforts for the finite element analysis in order to optimize.

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Appendix

Stiffness matrix of element and mass matrix are formulated by

$$[K]_e = \int_v [B]^T [D][B]|J|\,dv \tag{A1}$$

$$[M]_e = \int_v [N]^T [\beta][N]|J|\,dv \tag{A2}$$

$[B]$ is the strain displacement, $[D]$ is the elasticity matrix, $[N]$ is the shape function matrix, $|J|$ is the Jacobian determinant, $[\beta]$ is the matrix of the element density and geometric variable. The first order derivatives of stiffness matrix and mass matrix are written in the form

$$\frac{\partial [K]_e}{\partial p} = \int_v [B]^T \frac{\partial [D]}{\partial p} [B]|J|\,dv \tag{A3}$$

$$\frac{\partial [M]_e}{\partial p} = \int_v [N]^T \frac{\partial [\beta]}{\partial p} [N]|J|\,dv \tag{A4}$$

The second order derivatives of stiffness matrix and mass matrix are written in the form,

$$\frac{\partial^2 [K]_e}{\partial p^2} = \int_v [B]^T \frac{\partial^2 [D]}{\partial p^2} [B]|J|\,dv \tag{A5}$$

$$\frac{\partial^2 [M]_e}{\partial p^2} = \int_v [N]^T \frac{\partial^2 [\beta]}{\partial p^2} [N]|J|\,dv. \tag{A6}$$