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A HYBRID VOLTERRA-TYPE EQUATION WITH TWO TYPES OF IMPULSES

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Abstract. We formulate and analyze a hybrid system model that involves Volterra integral operators with multiple integrals and two types of impulsive terms. We give a constructive proof, via an iteration method, of existence and uniqueness of solutions.

1. Introduction

The class of Volterra integral equations has been traditionally used to model the behavior of systems with memory. The question of existence and uniqueness of solutions of Volterra integral equations is carried out by applying appropriate fixed point theorems; in addition to the classical works [4, 8], we mention the papers [2, 9, 10] that contain techniques related to the material of the present paper.

In this paper, we not only formulate and analyze a novel class of equations that contain some of the characteristics of Volterra integral equations, but also fall into the general category of hybrid systems theory.

The discipline of hybrid systems aims to formulate and analyze models of systems that combine continuous and discrete features, as well as to solve associated problems of control theory and game theory. In the area of ordinary differential equations, the standard model of hybrid systems is the class of impulsive differential equations [1]. The topic of Volterra integral equations with impulses has received considerable attention in the recent research literature [5, 6, 7, 11]. To the best of our knowledge, the paper [3] is the first to contain a general Volterra type term describing the impulses; previous papers contained particular types of impulses.

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In this paper, we set out to explore the inclusion of multiple-integral Volterra terms and multiple types of discrete impulsive terms, with two qualitatively different types of impulses; the various ingredients can be combined in several different ways. It is expected that the present work will form the basis for further developments in the area of systems described by Volterra equations with multiple integral terms and also multiple integral-sum terms. In order to make these concepts clear at this stage, we shall describe the various types of integral equations that can arise as extensions and generalizations of the classical Volterra model.

The standard (nonlinear) Volterra equation of the second kind for a scalar-valued unknown function $y(t)$ is

$$y(t) = y_0(t) + \int_0^t f(t, s, y(s))ds \tag{1.1}$$

It is well known that general smooth phenomena with memory are not always modelled by single integral operators of the type $\int_0^t f(t, s, y(s))ds$, but rather they involve series of multiple integrals. The multiple integral series analogue of (1.1) is

$$y(t) = y_0(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t \cdots \int_0^t \int_0^t f_n(t, s_1, s_2, \cdots\cdots, s_n, y(s_1), y(s_2), \cdots, y(s_n))ds_1 ds_2 \cdots ds_n \tag{1.2}$$

and a continuous solution is sought over $0 \le t \le T$.

If the functions f_n are continuous in all their arguments, satisfy bounds

$$|f(t, s_1, \cdots, s_n, x_1, x_2, \cdots, x_n)| \le C_n$$

and Lipschitz conditions

$$|f_n(t, s_1, \cdots, s_n, x_1, x_2, \cdots, x_n) - f_n(t, s_1, \cdots, s_n, \xi_1, \xi_2, \cdots, \xi_n)| \le L_n \sum_{i=1}^{n} |x_i - \xi_i|$$

with the series $\sum_{i=1}^{n} M_n \frac{t^n}{n!}$, $\sum_{n=0}^{\infty} L_{n+1} \frac{t^n}{n!}$ having radius of convergence $R > T$, then the existence and uniqueness of a solution of (1.2) can be shown, in a constructive manner, by showing that the operator S, defined by

$$(Sx)(t) : = y_0(t) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t \cdots \int_0^t \int_0^t f_n(t, s_1, s_2, \cdots\cdots, s_n, x(s_1), x(s_2), \cdots, x(s_n))ds_1 ds_2 \cdots ds_n \tag{1.3}$$

is a contraction on $C(0, T; R)$(the space of continuous real-valued functions on $[0, T]$) in the norm

$$\|x\| := \max_{0<t<T} e^{-\mu t}|x(t)|. \tag{1.4}$$

If μ is sufficiently large, then the proof of the contraction property relies on the estimate

$$\begin{aligned}
&e^{-\mu t}|x(t) - \xi(t)| \\
&\quad -f_n(t, s_1, s_2, \cdots, s_n, \xi(s_1), \xi(s_2), \cdots, \xi(s_n))|ds_1 ds_2 \cdots ds_n \\
&\leq e^{-\mu t} \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t \cdots \int_0^t \int_0^t \sum_{j=1}^{n} L_n |x(s_j) - \xi(s_j)| \\
&\leq e^{-\mu t} \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t \cdots \int_0^t \int_0^t \sum_{j=1}^{n} e^{\mu s_j} L_n \|x - \xi\|_\mu \\
&= \frac{1 - e^{-\mu t}}{\mu} \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} n L_n \|x - \xi\|_\mu \\
&= \frac{1 - e^{-\mu t}}{\mu} \sum_{m=0}^{\infty} \frac{t^m}{m!} L_{m+1} \|x - \xi\|_\mu \\
&\leq \frac{1 - e^{-\mu T}}{\mu} \sum_{m=0}^{\infty} \frac{T^m}{m!} L_{m+1} \|x - \xi\|_\mu
\end{aligned} \tag{1.5}$$

and the contraction property follows from the convergence of $\sum_{m=0}^{\infty} \frac{T^m}{m!} L_{m+1}$ and the fact that $\lim_{\mu \longrightarrow \infty} \frac{1-e^{-\mu T}}{\mu} = 0$.

It is plain that we may assume, without loss of generality, that each function f_n is symmetric with respect to permutations of the symbols (s_i, x_i), $i = 1, 2, \cdots, n$. i.e. $f_n(t, s_{\sigma(1)}, \cdots, s_{\sigma(n)}, x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = f_n(t, s_1, \cdots, s_n, x_1, \cdots, x_n)$ for every permutation σ of $(1, 2, ..., n)$, otherwise we could replace each f_n by its symmetrization

$$\begin{aligned}
&\widetilde{f_n}(t, s_1, \cdots, s_n, x_1, \cdots, x_n) \\
&:= \frac{1}{n!} \sum_{\sigma \in \prod_n} f_n(t, s_{\sigma(1)}, \cdots, s_{\sigma(n)}, x_{\sigma(1)}, \cdots, x_{\sigma(n)}),
\end{aligned} \tag{1.6}$$

where $\prod_n$ is the set of all permutations of $(1, 2, ..., n)$, and we would have

$$\begin{aligned}
&\int_0^t \cdots \int_0^t \int_0^t f_n(t, s_1, s_2, \cdots, s_n, x(s_1), x(s_2), \cdots, x(s_n)) ds_1 ds_2 \cdots ds_n \\
&= \int_0^t \cdots \int_0^t \int_0^t \widetilde{f_n}(t, s_1, s_2, \cdots, s_n, x(s_1), x(s_2), \cdots, x(s_n)) ds_1 ds_2 \cdots ds_n.
\end{aligned} \tag{1.7}$$

Under the condition of symmetry of each f_n, (1.2) can be written as

$$\begin{aligned}
y(t) &= y_0(t) + \sum_{n=1}^{\infty} \int_{s_n=0}^{t} \int_{s_{n-1}=0}^{s_n} \cdots \int_{s_1=0}^{s_2} f_n(t, s_1, s_2, \cdots, s_n, \\
&\qquad y(s_1), y(s_2), \cdots, y(s_n)) ds_1 \cdots ds_{n-1} ds_n.
\end{aligned} \tag{1.8}$$

Another extension of the standard Volterra equation (1.1) has been introduced in [3]. This extension involves impulsive terms, and the impulsive Volterra equation becomes

$$y(t) = y_0(t) + \int_0^t f(t,s,y(s))ds + \sum_{i:\tau_i<t} g(t,\tau_i,y(\tau_i)). \tag{1.9}$$

(The problems in [3] included controlled equations of the type (1.9).)

The time-instants τ_i in (1.9) are the impulsive times. The unknown function y will have jump discontinuities at these instants; however, the effect of the jumps in y are more general than in the case of impulsive ordinary differential equations. A crucial feature of an impulsive Volterra equation of the type (1.9) is that a constructive proof of existence and uniqueness of solutions requires the use of a two-dimensional vector-valued metric and a concept of contractions with a 2×2 matrix, instead of a scalar, contraction "coefficient". We refer to [3] for the details.

In this paper, we further explore the interrelationships between continuous and discrete components of systems with memory. To this effect, we consider Volterra equations with integral terms up to order two, discrete terms up to order two, and mixed discrete and continuous terms up to order two, with two kinds of impulsive effects, impulses of order 1, at impulsive times denoted by τ_i, and impulses of order 2, at variable times denoted by $\sigma_i(t)$. The detailed formulation of this model, together with explanations of the terminology of higher-order impulses, is presented in the next section.

The constructive proof of existence and uniqueness of solutions for our second-order full impulsive model requires the use of a three-dimensional vector-valued metric, and proof of contraction property with the role of contraction coefficient played by a matrix.

2. PROPERTIES AND HYPOTHESES

In this section, we give some properties and hypotheses for the functions.

Now we consider the following basic Volterra integral equation

$$\begin{aligned} x(t) &= x_0(t) + \int_0^t f_1(t,s,x(s))ds + \int_0^t \int_0^s f_2(t,s,s_1,x(s),x(s_1))ds_1 ds \\ &+ \sum_{i:\tau_i<t} G_1(t,\tau_i,x(\tau_i^-)) + \sum_{i:\tau_i<t} \sum_{j=1}^{i-1} G_2(t,\tau_i,\tau_j,x(\tau_i^-),x(\tau_j^-)) \\ &+ \int_0^t \sum_{i:\sigma_i(s)<t} \sum_{j:\tau_j<t} g(t,s,\sigma_i(s),\tau_j,x(s),x(\sigma_i(s)^-),x(\tau_j^-))ds \end{aligned} \tag{2.1}$$

$$+\sum_{i:\sigma_i(t)<t}\sum_{j:\tau_j<t} G_3(t,\sigma_i(t),\tau_j,x(\sigma_i(t)^-),x(\tau_j^-)).$$

Let ρ_{ij}, $i=1,2,\cdots,N_\sigma$, $j=1,2,\cdots,N_{\sigma,i}$ be the solutions of equation

$$t=\sigma_i(t) \tag{2.2}$$

Define $\sigma=\{\sigma_i(\cdot):i=1,2,\cdots,N_\sigma\}$, and we assume that, for every $i=1,2,\cdots,N_\sigma$. equation (2.2) has a finite number of solutions. Let

$$\rho:=\{\rho_{ij}:1\le j\le N_{\sigma,i},1\le i\le N_\sigma\}. \tag{2.3}$$

We set

$$\tau:=\{\tau_0,\tau_1,\cdots,\tau_{N_\tau}\} \tag{2.4}$$

and

$$\mathbf{I}:=\tau\cup\rho \tag{2.5}$$

The space $C(0,T;R;\mathbf{I})$ is defined as the space of real-valued functions $x(\cdot)$ which are bounded and continuous on every open interval (α,β), with endpoints α and β in $\mathbf{I}$, and has limits $x(\alpha^+):=\lim_{t\to\alpha^+}x(t)$, $x(\alpha^-):=\lim_{t\to\alpha^-}x(t)$ at every point $\alpha\in\mathbf{I}\cap(0,T)$, as well as limits $x(0^+):=\lim_{t\to0^+}x(t)$, $x(T^-):=\lim_{t\to T^-}x(t)$.

Let $x(t)=\xi(t)$, $x(\tau_i^-)=\eta_i$ and $x(\sigma_i(t)^-)=\beta_i(t)$. Assume that the functions f_1,f_2,G_1,G_2,G_3 and g satisfy the following conditions:

(H1) $f_1(t,s,x(s))$ is continuous for $0\le s\le t$, $x\in C(0,T;R;\mathbf{I})$, there exist $L_1>0$ such that

$$|f_1(t,s,x_1)-f_1(t,s,x_2)|\le L_1|x_1-x_2|.$$

(H2) $f_2(t,s,s_1,x,y)$ is continuous for $0\le s_1\le s\le t$, $x,y\in C(0,T;R;\mathbf{I})$, there exist $L_{21},L_{22}>0$ such that

$$|f_2(t,s,s_1,x_1,y_1)-f_2(t,s,s_1,x_2,y_2)|\le L_{21}|x_1-x_2|+L_{22}|y_1-y_2|.$$

(H3) $G_1(t,\tau_i,x(\tau_i^-))$ is continuous impulse function for $0<\tau_i<t<T$, $x\in R^n$, there exist $L_{G_1}>0$ such that

$$|G_1(t,\tau_i,\eta_i^1)-G_1(t,\tau_i,\eta_i^2)|\le L_{G_1}|\eta_i^1-\eta_i^2|.$$

(H4) $G_2(t,\tau_i,\tau_j,x(\tau_i^-),x(\tau_j^-))$ is continuous impulse function for $0<\tau_j<\tau_i<t<T$, $x\in R^n$, there exist $L_{G_{21}},L_{G_{22}}>0$ such that

$$|G_2(t,\tau_i,\tau_j,\eta_i^1,\eta_j^1)-G_2(t,\tau_i,\tau_j,\eta_i^2,\eta_j^2)|\le L_{G_{21}}|\eta_i^1-\eta_i^2|+L_{G_{22}}|\eta_j^1-\eta_j^2|.$$

(H5) $G_3(t, \sigma_i(t), \tau_j, x(\sigma_i(t)^-), x(\tau_j^-))$ is continuous impulse function for $0 < \tau_j < \sigma_i(t) < t < T$, $x \in R^n$, there exist $L_{G_{31}}, L_{G_{32}} > 0$ such that

$$\begin{aligned}&|G_3(t, \sigma_i(t), \tau_j, \beta_i^1(t), \eta_j^1) - G_3(t, \sigma_i(t), \tau_j, \beta_i^2(t), \eta_j^2)| \\ &\leq L_{G_{31}}|\eta_j^1 - \eta_j^2| + L_{G_{32}}|\beta_i^1(t) - \beta_i^2(t)|.\end{aligned}$$

(H6) $g(t, s, \sigma_i(s), \tau_j, x(s), x(\sigma_i(s)^-), x(\tau_j^-))$ is continuous function for $0 < \tau_j < \sigma_i(s) < s < t < T$, $x \in C([0, T] : \tau)$, there exist $L_{g_1}, L_{g_2}, L_{g_3} > 0$ such that

$$\begin{aligned}&|g(t, s, \sigma_i(s), \tau_j, \xi^1(t), \beta_i^1(t), \eta_j^1) - |g(t, s, \sigma_i(s), \tau_j, \xi^2(t), \beta_i^2(t), \eta_j^2)| \\ &\leq L_{g_1}|\xi^1(t) - \xi^2(t)| + L_{g_2}|\eta_j^1 - \eta_j^2| + L_{g_3}|\beta_i^1(t) - \beta_i^2(t)|.\end{aligned}$$

(H7) There is a positive number h such that $\tau_i - \tau_{i-1} \geq h$ for all $i = 1, 2, \cdots, N_\tau$; $\sigma_j(t) - \sigma_{j-1}(t) \geq h$ for all $j = 1, 2, \cdots, N_\sigma$, and for all $t \in [0, T]$; whenever $\sigma_j(s) < \tau_i$, for some $j \in \{1, 2, \cdots, N_\sigma\}$ and $i \in \{1, 2, \cdots, N_\tau\}$, then $\tau_i - \sigma_j(s) \geq h$.

3. EXISTENCE AND UNIQUENESS OF SOLUTION OF THE STATE EQUATION

In this section, we will show the existence and uniqueness of solution for the nonlinear Volterra integral equation.

For each collection of impulse times, we seek a solution (2.1) in the space $C(0, T; R; \mathbf{I})$ of real valued function $x(t)$ that are bounded and continuous on every open interval (α, β), with endpoints α and β in $\mathbf{I}$ and have limits $x(\alpha^+) := \lim_{t \to \alpha^+} x(t)$, $x(\alpha^-) := \lim_{t \to \alpha^-} x(t)$ at every point $\alpha \in \mathbf{I} \cap (0, T)$, as well as limits $x(0^+) := \lim_{t \to 0^+} x(t)$, $x(T^-) := \lim_{t \to T^-} x(t)$.

Theorem 3.1. *Suppose that hypotheses (H1) $\sim$ (H7) are satisfied. Then for every τ and every σ, (2.1) has a unique solution $x(\cdot)$ in the space $C(0, T; R; \mathbf{I})$.*

Proof. We observe that the hybrid Volterra equation (2.1) implies the following impulsive conditions at the times τ_k and ρ_{kl}:

$$\begin{aligned}x(\tau_k^+) &= x(\tau_k^-) + G_1(\tau_k, \tau_k, x(\tau_k^-)) + \sum_{j=1}^{k-1} G_2(\tau_k, \tau_k, \tau_j, x(\tau_k^-), x(\tau_j^-)) \\ &\quad + \int_0^{\tau_k} \sum_{i:\sigma_i(s)<\tau_k} g(\tau_k, s, \sigma_i(s), \tau_k, x(s), x(\sigma_i(s)^-), x(\tau_k^-))ds \\ &\quad + \sum_{i:\sigma_i(\tau_k)<\tau_k} G_3(\tau_k, \sigma_i(\tau_k), \tau_k, x(\sigma_i(\tau_k)^-), x(\tau_k^-))\end{aligned} \tag{3.1}$$

$$x(\rho_{kl}^+) = x(\rho_{kl}^-) + \sum_{j:\tau_j<\rho_{kl}} G_3(\rho_{kl}, \rho_{kl}, \tau_j, x(\rho_{kl}^-), x(\tau_j^-)) \tag{3.2}$$

The interval $[0,T]$ can be expressed as $[0,T] = \cup_{0\leq l\leq M}[\alpha_l, \alpha_{l+1}]$, for some positive integer M, with each $\alpha_l \in \mathbf{I}$, and all the corresponding open intervals satisfying $(\alpha_l, \alpha_{l+1}) \cap (\alpha_k, \alpha_{k+1}) = \emptyset$ for $k \neq l$, $(\alpha_l, \alpha_{l+1}) \cap \mathbf{I} = \emptyset$.

The solution of (2.1) can be obtained inductively as follows: equation (2.1) can be solved on $[0, \alpha_1)$ as a Volterra integral equation(including double integral terms, for which existence and uniqueness is provided by the argument in the introduction), and therefore $x(\alpha_1^-)$ can be determined. Then $x(\alpha_1^+)$ can be found by the impulsive conditions (3.1), (3.2) above. The equation (2.1) becomes a Volterra integral equation(with multiple integral terms, but no impulses) for the restriction of $x(t)$ to the interval (α_1, α_2), and consequently $x(t)$ can be uniquely determined over the interval $[0, \alpha_2)$. Inductively, if $x(t)$ has been determined over the interval $[0, \alpha_l)$, then $x(\alpha_l^-)$ can be determined, and $x(\alpha_l^+)$ can be found from the impulsive conditions (3.1), (3.2), so that (2.1) becomes a Volterra integral equation(with multiple integral terms, but no impulses) for the restriction of $x(t)$ to the interval (α_l, α_{l+1}), which is uniquely solvable on (α_l, α_{l+1}), therefore $x(t)$ will be known over the interval $[0, \alpha_{l+1})$. This completes the induction. □

We consider the space $V = C(0,T;R;\mathbf{I}) \times R^{N_\tau} \times C(0,T;R;\mathbf{I})$ with a vector valued norm defined for each (ξ, η_i, β_i) in V denoted by $\xi(t) = x(t)$, $\eta_i = x(\tau_i^-)$, $\beta_i(t) = x(\sigma_i(t)^-)$ by

$$
\begin{aligned}
&\|\xi\|_\mu = \sup_{0\leq t\leq T} e^{-\mu t}|\xi(t)|, \\
&\|\eta\|_\mu = \max_{1\leq i\leq N} e^{-\mu\tau_i}|\eta_i|, \qquad\qquad \eta = [\eta_i : 0 \leq i \leq N_\tau], \\
&\|\beta\|_\mu = \max_{1\leq i\leq N}\sup_{0\leq t\leq T} e^{-\mu\sigma_i(t)}|\beta_i(t)|, \qquad \beta = [\beta_j : 0 \leq j \leq N_\sigma].
\end{aligned}
\tag{3.3}
$$

We define an operator S on V by

$$
S(\xi,\eta,\beta) = \begin{pmatrix} S_c(\xi,\eta,\beta) \\ [S_d(\xi,\eta,\beta)]_l \\ [S_m(\xi,\eta,\beta)]_p \end{pmatrix},
\tag{3.4}
$$

where $S_c : V \longrightarrow C(0,T;R;\mathbf{I})$, $S_d : V \longrightarrow R^{N_\tau}$ and $S_m : V \longrightarrow C(0,T;R;\mathbf{I})$;

$$
\begin{aligned}
&S_c(\xi,\eta,\beta)(t) \\
&= x_0(t) + \int_0^t f_1(t,s,\xi(s))ds + \int_0^t\int_0^s f_2(t,s,s_1,\xi(s),\xi(s_1))ds_1 ds \\
&\quad + \sum_{i:\tau_i<t} G_1(t,\tau_i,\eta_i) + \sum_{i:\tau_i<t}\sum_{j=1}^{i-1} G_2(t,\tau_i,\tau_j,\eta_i,\eta_j)
\end{aligned}
\tag{3.5}
$$

$$+\int_0^t \sum_{i:\sigma_i(s)<t}\sum_{j:\tau_j<t} g(t,s,\sigma_i(s),\tau_j,\xi(s),\beta_i(s),\eta_j)ds$$
$$+\sum_{i:\sigma_i(t)<t}\sum_{j:\tau_j<t} G_3(t,\sigma_i(t),\tau_j,\beta_i(t),\eta_j),$$

$$\begin{aligned}
&[S_d(\xi,\eta,\beta)]_l\\
&= x_0(\tau_l)+\int_0^{\tau_l} f_1(\tau_l,s,\xi(s))ds+\int_0^{\tau_l}\int_0^s f_2(\tau_l,s,s_1,\xi(s),\xi(s_1))ds_1ds\\
&\quad+\sum_{i<l} G_1(\tau_l,\tau_i,\eta_i)+\sum_{i<l}\sum_{j=1}^{i-1} G_2(\tau_l,\tau_i,\tau_j,\eta_i,\eta_j)\\
&\quad+\int_0^{\tau_l}\sum_{i:\sigma_i(s)<\tau_l}\sum_{j=1}^{l-1} g(\tau_l,s,\sigma_i(s),\tau_j,\xi(s),\beta_i(s),\eta_j)ds\\
&\quad+\sum_{i:\sigma_i(\tau_l)<\tau_l}\sum_{j=1}^{l-1} G_3(\tau_l,\sigma_i(\tau_l),\tau_j,\beta_i(\tau_l),\eta_j),
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&[S_m(\xi,\eta,\beta)]_p(t)\\
&= x_0(\sigma_p(t))+\int_0^{\sigma_p(t)} f_1(\sigma_p(t),s,\xi(s))ds\\
&\quad+\int_0^{\sigma_p(t)}\int_0^s f_2(\sigma_p(t),s,s_1,\xi(s),\xi(s_1))ds_1ds\\
&\quad+\sum_{i:\tau_i<\sigma_p(t)} G_1(\sigma_p(t),\tau_i,\eta_i)+\sum_{i:\tau_i<\sigma_p(t)}\sum_{j=1}^{i-1} G_2(\sigma_p(t),\tau_i,\tau_j,\eta_i,\eta_j)\\
&\quad+\int_0^{\sigma_p(t)}\sum_{i:\sigma_i(s)<\sigma_p(t)}\sum_{j:\tau_j<\sigma_p(t)} g(\sigma_p(t),s,\sigma_i(s),\tau_j,\xi(s),\beta_i(s),\eta_j)ds\\
&\quad+\sum_{i:\sigma_i(\sigma_p(t))<\sigma_p(t)}\sum_{j:\tau_j<\sigma_p(t)} G_3(\sigma_p(t),\sigma_i(\sigma_p(t)),\tau_j,\beta_i(\sigma_p(t)),\eta_j).
\end{aligned} \tag{3.7}$$

We shall call that S_c is continuous component of S, S_d is discrete component of S and S_m is mixed component of S.

Lemma 3.2. *The solution of equation* (2.1) *is equivalent to the problem of finding a fixed point of the operator S defined in* (3.5) $\sim$ (3.7) *above.*

Proof. It is clear that, if $x(t)$ is a solution of (2.1) in the space $C(0,T;R;\mathbf{I})$, then it is plain that the triple (ξ^*,η^*,β^*), defined by

$$\xi^*(t):=x(t),\quad \eta_i^*:=x(\tau_i^-),\quad \beta_j^*(t):=x(\sigma_j(t)^-)$$

is a fixed point of the operator S in the space V.

Conversely, suppose that $(\xi^*, \eta^*, \beta^*) \in V$ is a fixed point of S. Let the intervals $[\alpha_l, \alpha_{l+1}]$ be as in the proof of Theorem 3.1. It follows from (3.5) that $\xi^*(t)$ is a solution of (2.1) over the time-interval $[0, \alpha_1)$. If $\alpha_1 \in \tau$, it follows from (3.6) that $\eta_1^* = \xi^*(\alpha_1^-)$, and then it follows from (3.5) that $\xi^*(\alpha_1^+)$ satisfies the impulsive condition (3.1) at $t = \alpha_1$. If $\alpha_1 \in \rho$, it follows from (3.7) that $\beta_1^*(\alpha_1) = \xi^*(\sigma_1(\alpha_1)^-)$, and then it follows from (3.5) that $\xi^*(\sigma_1(\alpha_1)^+)$ satisfies the impulsive condition (3.2) at $t = \alpha_1$. Inductively, if $\xi^*(t)$ solves (2.1) over $[0, \alpha_l)$ and $\xi^*(\alpha_l^+)$ satisfies the appropriate impulsive condition, (3.1) or (3.2), at $t = \alpha_l$, then we shall show that $\xi^*(t)$ also solves (2.1) over $[0, \alpha_{l+1})$ and $\xi^*(\alpha_{l+1}^+)$ satisfies the appropriate impulsive condition, (3.1) or (3.2), at $t = \alpha_{l+1}$. Suppose that the points $\alpha_0, \alpha_1, \cdots, \alpha_l$ correspond to $\{\tau_\lambda : 0 \le \lambda \le \lambda_0\} \cup \{\sigma_\mu : 1 \le \sigma_\mu \le \mu_0\}$. Then it follows from (3.5) that $\xi^*(t)$ solves (2.1) over $[0, \alpha_{l+1})$, since the equation obtained from (2.1) over the interval $[\alpha_l, \alpha_{l+1})$ as in the proof of Theorem 3.1, contains no impulses in (α_l, α_{l+1}). If $\alpha_{l+1} \in \tau$, then it follows from (3.6) that $\xi^*(\alpha_{l+1}^-) = \eta_{\lambda_0+1}^*$, and then it follows from (3.5) that, in case $\alpha_{l+1} < T$, $\xi^*(\alpha_{l+1}^+)$ satisfies the impulsive condition (3.1) at $t = \alpha_{l+1} = \tau_{\lambda_0+1}$. If $\alpha_{l+1} \in \rho$, then it follows from (3.6) that $\xi^*(\alpha_{l+1}^-) = \beta_j^*(\sigma_j(\alpha_{l+1})^-)$ for some $j \ge \mu_0$, and then it follows from (3.5) that, in case $\alpha_{l+1} < T$, $\xi^*(\alpha_{l+1}^+)$ satisfies the impulsive condition (3.2) at $t = \rho_{jk} = \alpha_{l+1}$ for some k. The induction is complete. □

Lemma 3.3. If $\mu > 0$ is sufficiently large, then for any $(\xi^1, \eta^1, \beta^1), (\xi^2, \eta^2, \beta^2)$ of V, we have

$$\begin{aligned}
(i)\ & \|S_c(\xi^1, \eta^1, \beta^1) - S_c(\xi^2, \eta^2, \beta^2)\|_\mu \\
& \le a_{11}\|\xi^1 - \xi^2\|_\mu + a_{12}\|\eta^1 - \eta^2\|_\mu + a_{13}\|\beta^1 - \beta^2\|_\mu, \\
(ii)\ & \|S_d(\xi^1, \eta^1, \beta^1) - S_d(\xi^2, \eta^2, \beta^2)\|_\mu \\
& \le a_{21}\|\xi^1 - \xi^2\|_\mu + a_{22}\|\eta^1 - \eta^2\|_\mu + a_{23}\|\beta^1 - \beta^2\|_\mu, \\
(iii)\ & \|S_m(\xi^1, \eta^1, \beta^1) - S_m(\xi^2, \eta^2, \beta^2)\|_\mu \\
& \le a_{31}\|\xi^1 - \xi^2\|_\mu + a_{32}\|\eta^1 - \eta^2\|_\mu + a_{33}\|\beta^1 - \beta^2\|_\mu,
\end{aligned}$$

where all constants a_{ij} are nonnegative elements of the matrix $A := (a_{ij})_{1 \le i,j \le 3}$.
where all constants a_{ij} are nonnegative elements of the matrix $A := (a_{ij})_{1 \le i,j \le 3}$.

Proof. We will show that the operator S satisfies the contractive condition. It is enough to have $\lim_{\mu \to \infty} D = 0$, $\lim_{\mu \to \infty} T = 0$ and $\lim_{\mu \to \infty} S = 0$ for $D = det(A)$, $T = tr(A)$ and $S = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}$ (see Appendix I).

Let $L_2 = \max\{L_{21}, L_{22}\}$, $L_{G_2} = \max\{L_{G_{21}}, L_{G_{22}}\}$, $L_g = \max\{L_{g_1}, L_{g_2}, L_{g_3}\}$ and $L_{G_3} = \max\{L_{G_{31}}, L_{G_{32}}\}$. Then we have

$$
\begin{aligned}
(i)\ & e^{-\mu t}|S_c(\xi^1,\eta^1,\beta^1)(t) - S_c(\xi^2,\eta^2,\beta^2)(t)| \\
&\le e^{-\mu t}\Big\{\int_0^t |f_1(t,s,\xi^1(s)) - f_1(t,s,\xi^2(s))|ds \\
&+ \sum_{i:\tau_i<t} |G_1(t,\tau_i,\eta_i^1) - G_1(t,\tau_i,\eta_i^2)| \\
&+ \sum_{i:\tau_i<t}\sum_{j=1}^{i-1} |G_2(t,\tau_i,\tau_j,\eta_i^1,\eta_j^1) - G_2(t,\tau_i,\tau_j,\eta_i^2,\eta_j^2)| \\
&+ \int_0^t \sum_{i:\sigma_i(s)<t}\sum_{j:\tau_j<t} |g(t,s,\sigma_i(s),\tau_j,\xi^1(s),\beta_i^1(s),\eta_j^1) \\
&-g(t,s,\sigma_i(s),\tau_j,\xi^2(s),\beta_i^2(s),\eta_j^2)|ds \\
&+ \sum_{i:\sigma_i(t)<t}\sum_{j:\tau_j<t} |G_3(t,\sigma_i(t),\tau_j,\beta_i^1(t),\eta_j^1) - G_3(t,\sigma_i(t),\tau_j,\beta_i^2(t),\eta_j^2)|\Big\}.
\end{aligned}
$$

Hence, by the assumption $(H1) \sim (H7)$,

$$
\begin{aligned}
&\|S_c(\xi^1,\eta^1,\beta^1) - S_c(\xi^2,\eta^2,\beta^2)\|_\mu \\
&\le\Big\{L_1\frac{1-e^{-\mu t}}{\mu} + L_2\Big(\frac{t}{\mu} - \frac{te^{-\mu t}}{\mu}\Big) + L_g(N_t)^2\frac{1-e^{-\mu t}}{\mu}\Big\}\|\xi^1-\xi^2\|_\mu \\
&+\Big\{L_{G_1}\Big(1+\frac{1-e^{-\mu(N_t-1)h}}{e^{\mu h}-1}\Big) + L_{G_2}(N_t - 1 + (N_t-1)(N_t-2) \\
&+N_t(N_t-1))e^{-\mu h} \\
&+L_gN_t\Big(1+\frac{1-e^{-\mu(N_t-1)h}}{e^{\mu h}-1}T\Big)+L_{G_3}N_t\Big(1+\frac{1-e^{-\mu(N_t-1)h}}{e^{\mu h}-1}\Big)\Big\}\|\eta^1-\eta^2\|_\mu \\
&+\Big\{L_{G_3}N_t\Big(\frac{1-e^{-\mu N_t h}}{1-e^{-\mu h}}\Big) + L_gTN_t\Big(\frac{1-e^{-\mu N_t h}}{1-e^{-\mu h}}\Big)\Big\}\|\beta^1-\beta^2\|_\mu.
\end{aligned}
$$

$$
\begin{aligned}
(ii)\ & e^{-\mu\tau_l}|S_d(\xi^1,\eta^1,\beta^1)(\tau_l) - S_d(\xi^2,\eta^2,\beta^2)(\tau_l)| \\
&\le e^{-\mu\tau_l}\Big\{\int_0^{\tau_l} |f_1(\tau_l,s,\xi^1(s)) - f_1(\tau_l,s,\xi^2(s))|ds \\
&+\int_0^{\tau_l}\int_0^s |f_2(\tau_l,s,s_1,\xi^1(s),\xi^1(s_1)) - f_2(\tau_l,s,s_1,\xi^2(s),\xi^2(s_1))|ds_1ds \\
&+\sum_{i<l} |G_1(\tau_l,\tau_i,\eta_i^1) - G_1(\tau_l,\tau_i,\eta_i^2)|
\end{aligned}
$$

$$+\sum_{i<l}\sum_{j=1}^{i-1}|G_2(\tau_l,\tau_i,\tau_j,\eta_i^1,\eta_j^1)-G_2(\tau_l,\tau_i,\tau_j,\eta_i^2,\eta_j^2)|$$
$$+\int_0^{\tau_l}\sum_{i:\sigma_i(s)<\tau_l}+\sum_{j=1}^{l-1}|g(\tau_l,s,\sigma_i(s),\tau_j,\xi^1(s),\beta_i^1(s),\eta_j^1)$$
$$-g(\tau_l,s,\sigma_i(s),\tau_j,\xi^2(s),\beta_i^2(s),\eta_j^2)|ds$$
$$+\sum_{i:\sigma_i(\tau_l)<\tau_l}\sum_{j=1}^{l-1}|G_3(\tau_l,\sigma_i(\tau_l),\tau_j,\beta_i^1(\tau_l),\eta_j^1)$$
$$-G_3(\tau_l,\sigma_i(\tau_l),\tau_j,\beta_i^2(\tau_l),\eta_j^2)|\Big\}.$$

Hence, by the assumption $(H1)\sim(H7)$,

$$\|S_d(\xi^1,\eta^1,\beta^1)-S_d(\xi^2,\eta^2,\beta^2)\|_\mu$$
$$\leq\left\{L_1\frac{1-e^{-\mu\tau_l}}{\mu}+L_2\left(\frac{\tau_l}{\mu}-\frac{\tau_le^{-\mu\tau_l}}{\mu}\right)+L_g(l-1)N_{\tau_l}\frac{1-e^{-\mu\tau_l}}{\mu}\right\}\|\xi^1-\xi^2\|_\mu$$
$$+\left\{L_{G_1}\frac{1-e^{-\mu(l-1)h}}{e^{\mu h}-1}+L_{G_2}\frac{l(l-1)e^{-\mu h}}{2}+L_{G_2}\frac{1-e^{-\mu(l-2)h}}{(e^{\mu h}-1)^2}\right.$$
$$\left.+L_gN_{\tau_l}\tau_l\frac{1-e^{-\mu(l-1)h}}{e^{\mu h}-1}+L_{G_3}N_{\tau_l}\frac{1-e^{-\mu(l-1)h}}{e^{\mu h}-1}\right\}\|\eta^1-\eta^2\|_\mu$$
$$+\left\{L_g(l-1)\tau_le^{-\mu h}\frac{1-e^{-\mu(N_{\tau_l}-1)h}}{1-e^{-\mu h}}+L_{G_3}(l-1)e^{-\mu h}\frac{1-e^{-\mu(N_{\tau_l}-1)h}}{1-e^{-\mu h}})\right\}$$
$$\times\|\beta^1-\beta^2\|_\mu.$$

(iii) $e^{-\mu t}|[S_m(\xi^1,\eta^1,\beta^1)]_p(t)-[S_m(\xi^2,\eta^2,\beta^2)]_p(t)|$

$$\leq e^{-\mu t}\left\{\int_0^{\sigma_p(t)}|f_1(\sigma_p(t),s,\xi^1(s))-f_1(\sigma_p(t),s,\xi^2(s))|ds\right.$$
$$+\int_0^{\sigma_p(t)}\int_0^s|f_2(\sigma_p(t),s,s_1,\xi^1(s),\xi^1(s_1))$$
$$-f_2(\sigma_p(t),s,s_1,\xi^2(s),\xi^2(s_1))|ds_1ds$$
$$+\sum_{i:\tau_i<\sigma_p(t)}|G_1(\sigma_p(t),\tau_i,\eta_i^1)-G_1(\sigma_p(t),\tau_i,\eta_i^2)|$$
$$+\sum_{i:\tau_i<\sigma_p(t)}\sum_{j=1}^{i-1}|G_2(\sigma_p(t),\tau_i,\tau_j,\eta_i^1,\eta_j^1)-G_2(\sigma_p(t),\tau_i,\tau_j,\eta_i^2,\eta_j^2)|$$

$$+\int_0^{\sigma_p(t)} \sum_{i:\sigma_i(s)<\sigma_p(t)} \sum_{j:\tau_j<\sigma_p(t)} |g(\sigma_p(t), s, \sigma_i(s), \tau_j, \xi^1(s), \beta_i^1(s), \eta_j^1)$$

$$-g(\sigma_p(t), s, \sigma_i(s), \tau_j, \xi^2(s), \beta_i^2(s), \eta_j^2)|ds$$

$$+\sum_{i:\sigma_i(\sigma_p(t))<\sigma_p(t)} \sum_{j:\tau_j<\sigma_p(t)} |G_3(\sigma_p(t), \sigma_i(\sigma_p(t)), \tau_j, \beta_i^1(\sigma_p(t)), \eta_j^1)$$

$$-G_3(\sigma_p(t), \sigma_i(\sigma_p(t)), \tau_j, \beta_i^2(\sigma_p(t)), \eta_j^2)|\Big\}.$$

Assume that $\sigma_{i+1}(t) - \sigma_i(s) \geq h$ whenever $s \leq \sigma_i(t)$. Then, by the assumptions $(H1) \sim (H7)$,

$$\|S_m(\xi^1, \eta^1, \beta^1) - S_m(\xi^2, \eta^2, \beta^2)\|_\mu$$
$$\leq \left\{L_1 \frac{1-e^{-\mu\sigma_p(t)}}{\mu} + L_2\left(\frac{\sigma_p(t)}{\mu} - \frac{\sigma_p(t)e^{-\mu\sigma_p(t)}}{\mu}\right) + L_g(N_{\sigma_p(t)})^2 \frac{1-e^{-\mu\sigma_p(t)}}{\mu}\right\}$$
$$\times \|\xi^1 - \xi^2\|_\mu$$
$$+\left\{L_{G_1} \frac{1-e^{-\mu h N_{\sigma_p(t)}}}{1-e^{-\mu h}} + L_{G_2}\left(N_{\sigma_p(t)} + \frac{N_{\sigma_p(t)}(N_{\sigma_p(t)-1})e^{-\mu h}}{2} - \frac{1-e^{-\mu h N_{\sigma_p(t)}}}{1-e^{-\mu h}}\right)\right.$$
$$+L_{G_2}\left(N_{\sigma_p(t)}e^{-\mu h} + \frac{N_{\sigma_p(t)}(N_{\sigma_p(t)}-1)e^{-\mu h}}{2} - \frac{1-e^{-\mu N_{\sigma_p(t)}h}}{e^{\mu h}-1}\right)$$
$$\left.+L_g\sigma_p(t)N_{\sigma_p(t)} \frac{1-e^{-\mu h N_{\sigma_p(t)}}}{1-e^{\mu h}} + L_{G_3}N_{\sigma_p(t)} \frac{1-e^{-\mu h N_{\sigma_p(t)}}}{1-e^{-\mu h}}\right\} \|\eta^1 - \eta^2\|_\mu$$
$$+\left\{L_g\sigma_p(t)N_{\sigma_p(t)} \frac{1-e^{-\mu(N_{\sigma_p(t)}-1)h}}{e^{\mu h}-1} + L_{G_3}N_{\sigma_p(t)} \frac{1-e^{-\mu(N_{\sigma_p(t)}-1)h}}{e^{\mu h}-1})\right\}$$
$$\times \|\beta^1 - \beta^2\|_\mu.$$

If we take a_{ij}, $1 \leq i, j \leq 3$, in the above result inequalities as same as (i), (ii) and (iii) of this lemma, then $(a_{11}, a_{21}, a_{22}, a_{23}, a_{31}, a_{33}) \longrightarrow (0,0,0,0,0,0)$ as $\mu \longrightarrow \infty$, and a_{12}, a_{13}, a_{32} remains bounded as $\mu \longrightarrow \infty$; consequently $(D, T, S) \longrightarrow 0$ as $\mu \longrightarrow \infty$. That is, for μ sufficiently large, the operator S will be satisfied the contractive condition. □

We consider the following iterative scheme for the solution of (2.1): $(\xi(0), \eta(0), \beta(0))$ is an arbitrary element of V; for $k = 0, 1, 2, \cdots$, $(\xi(k+1), \eta(k+1), \beta(k+1))$ is defined by

(3.8)
$$\xi_{(k+1)}(t) = x_0(t) + \int_0^t f_1(t, s, \xi_{(k)}(s))ds + \int_0^t \int_0^s f_2(t, s, s_1, \xi_{(k)}(s), \xi_{(k)}(s_1))ds_1 ds$$

$$+\sum_{i:\tau_i<t} G_1(t,\tau_i,\eta_{(k),i}) + \sum_{i:\tau_i<t}\sum_{j=1}^{i-1} G_2(t,\tau_i,\tau_j,\eta_{(k),i},\eta_{(k),j})$$
$$+\int_0^t \sum_{i:\sigma_i(s)<t}\sum_{j:\tau_j<t} g(t,s,\sigma_i(s),\tau_j,\xi_{(k)}(s),\beta_{(k),i}(s),\eta_{(k),j})ds$$
$$+\sum_{i:\sigma_i(t)<t}\sum_{j:\tau_j<t} G_3(t,\sigma_i(t),\tau_j,\beta_{(k),i}(t),\eta_{(k),j}),$$

(3.9)
$$\eta_{(k+1),l}$$
$$= x_0(\tau_l) + \int_0^{\tau_l} f_1(\tau_l,s,\xi_{(k)}(s))ds + \int_0^{\tau_l}\int_0^s f_2(\tau_l,s,s_1,\xi_{(k)}(s),\xi_{(k)}(s_1))ds_1ds$$
$$+\sum_{i<l} G_1(\tau_l,\tau_i,\eta_{(k),i}) + \sum_{i<l}\sum_{j=1}^{i-1} G_2(\tau_l,\tau_i,\tau_j,\eta_{(k),i},\eta_{(k),j})$$
$$+\int_0^{\tau_l} \sum_{i:\sigma_i(s)<\tau_l}\sum_{j=1}^{l-1} g(\tau_l,s,\sigma_i(s),\tau_j,\xi_{(k)}(s),\beta_{(k),i}(s),\eta_{(k),j})ds$$
$$+\sum_{i:\sigma_i(\tau_l)<\tau_l}\sum_{j=1}^{l-1} G_3(\tau_l,\sigma_i(\tau_l),\tau_j,\beta_{(k),i}(\tau_l),\eta_{(k),j}),$$

(3.10)
$$\beta_{(k+1),p}(t)$$
$$= x_0(\sigma_p(t)) + \int_0^{\sigma_p(t)} f_1(\sigma_p(t),s,\xi_{(k)}(s))ds$$
$$+\int_0^{\sigma_p(t)}\int_0^s f_2(\sigma_p(t),s,s_1,\xi_{(k)}(s),\xi_{(k)}(s_1))ds_1ds$$
$$+\sum_{i:\tau_i<\sigma_p(t)} G_1(\sigma_p(t),\tau_i,\eta_{(k),i}) + \sum_{i:\tau_i<\sigma_p(t)}\sum_{j=1}^{i-1} G_2(\sigma_p(t),\tau_i,\tau_j,\eta_{(k),i},\eta_{(k),j})$$
$$+\int_0^{\sigma_p(t)} \sum_{i:\sigma_i(s)<\sigma_p(t)}\sum_{j:\tau_j<\sigma_p(t)} g(\sigma_p(t),s,\sigma_i(s),\tau_j,\xi_{(k)}(s),\beta_{(k),i}(s),\eta_{(k),j})ds$$
$$+\sum_{i:\sigma_i(\sigma_p(t))<\sigma_p(t)}\sum_{j:\tau_j<\sigma_p(t)} G_3(\sigma_p(t),\sigma_i(\sigma_p(t)),\tau_j,\beta_{(k),i}(\sigma_p(t)),\eta_{(k),j}).$$

Then we have:

Theorem 3.4. *Suppose that $(H1) \sim (H7)$ are satisfied. Then the iterative method defined by (3.8) $\sim$ (3.10) above converges to $(x(\cdot),\ (x(\tau_1^-), x(\tau_2^-), x(\tau_2^-), \cdots, x(\tau_{N_\tau}^-)),$ $x(\sigma_1(\cdot)^-), x(\sigma_2(\cdot)^-), \cdots, x(\sigma_{N_\sigma}(\cdot)^-))$, as $k \longrightarrow \infty$; the convergence of $\xi_{(k)}$ to $x(\cdot)$ is uniform on each closed interval $[\alpha,\beta]$ that does not contain points of **I** in the interior, in the sense that, if we define the restrictions $\xi_{(k)}^{\alpha\beta} = \xi_{(k)}$, $x^{\alpha\beta}(\cdot) = x$ to the* *interval* $[\alpha, \beta]$ *by* $\xi_{(k)}^{\alpha\beta}(t) = \xi_{(k)}(t)$ *for* $t \in [\alpha, \beta)$, $\xi_{(k)}^{\alpha\beta}(\beta) = \xi_{(k)}(\beta^-)$, $x^{\alpha\beta}(t) = x(t)$ *for* $t \in [\alpha, \beta)$, $x^{\alpha\beta}(\beta) = x(\beta^-)$, *then* $\xi_{(k)}^{\alpha\beta} \longrightarrow x^{\alpha\beta}$ *uniformly on* $[\alpha, \beta]$.

Proof. By Lemma 3.3, if μ is sufficiently large, the operator S is a contraction with respect to the vector-valued norm $\|\cdot\|_\mu$ on V. The contraction property with respect to the vector valued norm $\|\cdot\|_\mu$ means that, for all $x, y \in V$, we have $\|Sx - Sy\|_\mu \le A\|x - y\|_\mu$, where the 3×3 real matrix A has $\lim_{\mu\to\infty} D = 0$, $\lim_{\mu\to\infty} T = 0$, $\lim_{\mu\to\infty} S = 0$. Consequently, the iterates of S, with arbitrary initial data, converges to the unique fixed point of S in the topology induced on V by the vector valued norm $\|\cdot\|_\mu$; this is a well known extension of the standard Banach fixed point theorem to the case of a vector valued metric, and the proof proceeds as in the standard case. Convergence with respect to uniform convergence on each $[\tau_{i-1}, \tau_i]$. The fixed point of S gives the solution of (2.1) by Lemma 3.2. □

Remark. It follows from (3.5) $\sim$ (3.7) that $\eta_{(k+1),l} = \xi_{(k+1)}(\tau_l^-)$, $\beta_{(k+1),l} = \xi_{(k)}(\sigma_l(t)^-)$ for all $k \ge 0$, so that, for $k \ge 1$, (3.5) $\sim$ (3.7) can also be written in the form

$$\begin{aligned}\xi_{(k+1)}(t) &= x_0(t) + \int_0^t f_1(t, s, \xi_{(k)}(s))ds + \int_0^t \int_0^s f_2(t, s, s_1, \xi_{(k)}(s), \xi_{(k)}(s_1))ds_1 ds \\ &+ \sum_{i:\tau_i<t} G_1(t, \tau_i, \xi_{(k)}(\tau^-)) + \sum_{i:\tau_i<t} \sum_{j=1}^{i-1} G_2(t, \tau_i, \tau_j, \xi_{(k)}(\tau_i^-), \xi_{(k)}(\tau^-)) \\ &+ \int_0^t \sum_{i:\sigma_i(s)<t} \sum_{j:\tau_j<t} g(t, s, \sigma_i(s), \tau_j, \xi_{(k)}(s), \xi_{(k)}(s), \xi_{(k)}(\tau_j^-))ds \\ &+ \sum_{i:\sigma_i(t)<t} \sum_{j:\tau_j<t} G_3(t, \sigma_i(t), \tau_j, \xi_{(k)}(\sigma_i(t)^-), \xi_{(k)}(\tau_j^-)).\end{aligned}$$

Of course, if $\eta_{(0)}$ is chosen as $\eta_{(0),i} = \xi_{(0)}(\tau_i^-)$, $\beta_{(0)}$ is chosen as

$$\beta_{(0),i} = \xi_{(0)}(\sigma_i(t)^-),$$

then (2.1) holds for all $k = 0, 1, 2, \cdots\cdots$

Appendix I. Condition for a 3×3 matrix to be contractive. We shall derive the necessary and sufficient conditions for a 3×3 matrix $A = [a_{ij}]_{1 \le i,j \le 3}$ to be contractive, in the sense that all eigenvalues of A will have modulus strictly less than 1.

The characteristic polynomial of A is

$$(A.1) \qquad f(\lambda) = \lambda^3 - T\lambda^2 + S\lambda - D,$$

where $T = tr(A) = a_{11} + a_{22} + a_{33}$, the trace of A, $D = \det(A)$, the determinant of A, and

$$(A.2) \qquad \begin{aligned} S &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{13}a_{31} - a_{23}a_{32} - a_{12}a_{21}. \end{aligned}$$

We recall, from the theory of the Routh-Hurwitz stability criteria, that a cubic polynomial $\varphi(\omega) = p_0\omega^3 + p_1\omega^2 + p_2\omega + p_3$, with $p_0 > 0$, will have all its roots in the open left-half plane (on the plane of complex numbers) if and only if the following conditions are satisfied:

$$(A.3) \qquad p_1 > 0; \qquad p_3 > 0; \qquad p_1p_2 > p_0p_3.$$

The transformation $\lambda = \frac{\omega+1}{\omega-1}$ maps $Re(\omega) < 0$ onto $|\lambda| < 1$. By using this transformation into $(A.1)$, we find that A will have all its eigenvalues in the interior of the unit circle on the complex plane if and only if the polynomial

$$(A.4) \qquad \begin{aligned} g(\omega) &= [1 - T + S - D]\omega^3 + [3 - T - S + 3D]\omega^2 \\ &\quad + [3 + T - S - 3D]\omega + [1 - T + S + D] \end{aligned}$$

has all its roots in the half-plane $Re(\omega) < 0$. After some straightforward algebra, we find

$$(A.5) \qquad p_1p_2 - p_0p_3 = 8 - 4S - 8D^2 + 4DT.$$

Consequently, assuming $p_0 \equiv 1 - T + S - D > 0$, the matrix A will be contractive if and only if

$$(A.6) \qquad \begin{aligned} &3 - S + 3D - T > 0; \\ &1 + T + S + D > 0; \\ &8 - 4S - 8D^2 + 4DT > 0. \end{aligned}$$

Further, we note that a sufficient condition for A to be contractive is that the quantities S, T, D should be sufficiently small.

In case the elements a_{ij} of the matrix A depend on a parameter μ (as will be the case in the application to the mappings defined in Section 3 of this paper), a sufficient condition for A to be contractive is that $(S, D, T) \longrightarrow (0, 0, 0)$ as $\mu \longrightarrow \infty$.

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