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# ON GENERALIZATION OF COVARIANCE AND VARIANCE

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*Dedicated to Professor Ka-Ying Lim on his retirement*

ABSTRACT. We introduce the notion of the generalized covariance and variance for bounded linear operators on a Hilbert space, and prove that the generalized covariance-variance inequality holds. It turns out that the inequality is a useful formula in the study of inequality involving linear operators in Hilbert spaces.

## 1. DEFINITION AND INTRODUCTION

Let $H$ be a Hilbert space over the field $C$ of complex numbers. Let $B(H)$ be the algebra of all bounded linear operators on $H$ into itself; $I$ denotes the identity operator, $O$ the zero operator, and $T^*$ the adjoint of $T \in B(H)$. The next definition was partially mentioned in our paper [8] without proof nor applications. Thus, the present paper is a continuation of [8].

**Definition 1.1.** For $S, T, R \in B(H)$ let $S, T$ and $R$ be acting on $x, y$ and $z$, respectively for every $x, y, z \in H$. The generalized covariance for $S$, $T$ and $R$ on $H$ is defined by

$$Ecov_R(S,T) = \| Rz \|^2 (Sx, Ty) - (Sx, Rz)(Rz, Ty),$$

where the symbol $(\cdot, \cdot)$ means the usual inner product in $H$. The generalized variance for $S$ and $R$ on $H$is a real number defined by

$$Evar_R(S) = Ecov_R(S, S) = \| Rz \|^2 \| Sx \|^2 - | (Sx, Rz) |^2 .$$

Recall in particular that the covariance for $S$ and $T$ on $H$, and the variance for

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$S$ on $H$, respectively, are defined in [7] as follows:

$$Cov_z(S,T) = \| z \|^2 (Sx, Tx) - (Sx, z)(z, Tx),$$

and

$$Var_z(S) = Cov_z(S,S) = \| z \|^2 \| Sx \|^2 - | (Sx, z) |^2 .$$

The covariance and variance for operators on $H$ were extensively studied in [7, 8] with applications in inequalities involving linear operators in $H$. In this paper we first prove that the generalized covariance-variance inequality hold. The inequality is used to create and to prove inequalities in $H$. Consequently, it turns out that many well-known inequalities in the literature follow easily as special cases; and related and improved inequalities are given. We show that $Ecov_R( , )$ is indeed a semi-inner product in $B(H)$; and relationships between this and the usual inner product $( , )$ in $H$ are explained in the last section.

## 2. Basic Results and Generalized Covariance-Variance Inequality

In this section we present basic results about the generalized covariance and variance; we prove that the generalized covariance-variance inequality holds true and the equality condition is given. Let $Re\ \alpha$ denote the real part of $\alpha \in C$.

**Lemma 2.1.** *For $S, T, R, Q \in B(H)$ let $S, T, R$ and $Q$ be acting on $x, y, z$ and $w$, respectively for every $x, y, z, w \in H$. Then the following relations hold.*

(2.1) $Ecov_R(S,S) = Evar_R(S) \geq 0$.

(2.2) $Ecov_R(S,S) = Evar_R(S) = 0$ *if and only if $Rz$ and $Sx$ are proportional.*

(2.3) $Ecov_R(Q \pm S, T) = Ecov_R(Q,T) \pm Ecov_R(S,T)$.

(2.4) $Ecov_R(\lambda S, T) = \lambda Ecov_R(S,T)$ *for any* $\lambda \in C$.

(2.5) $\overline{Ecov_R(T,S)} = Ecov_R(S,T)$.

(2.6) $Evar_R(Q \pm S) = Evar_R(Q) + Evar_R(S) \pm 2Re\, Ecov_R(Q,S)$.

(2.7) $| Ecov_R(S,T) |^2 \leq Evar_R(S) Evar_R(T)$.

*(We shall call (2.7) the generalized covariance-variance inequality, the g-c-v inequality in short).*

*Moreover, if $Evar_R(S) \neq 0 \neq Evar_R(T)$, then the g-c-v equality holds if and only if $Rz$ and $Sx - \lambda Ty$ are proportional, $\lambda \in C$.*

*Proof.* (2.1) and (2.2) are due to Definition 1 and the Cauchy-Schwarz inequality and equality; and (2.3), (2.4), (2.5) and (2.6) are also by Definition 1. We see that conditions (2.1), (2.3), (2.4) and (2.5) constitute a semi-inner product $\mathrm{Ecov}_R(\ ,\ )$ in $B(H)$ (it is not necessarily an inner product, since from (2.2) it does not have to follow that $S = O$). It follows that the Cauchy-Schwarz inequality holds in $B(H)$, which is precisely the g-c-v inequality (2.7). Neverthless, for the sake of completeness let us prove it directly as follows.

If $Evar_R(S) = 0$, then $Rz$ and $Sx$ are proportional by (2.2), and hence

$$Ecov_R(S,T) = 0.$$

Similarly for the case $Evar_R(T) = 0$. Assume $Evar_R(S) \neq 0 \neq Evar_R(T)$ and write $u = Evar_R(T) > 0$ and $v = Ecov_R(S,T)$, then, by (2.3), (2.4) and (2.5),

$$\begin{aligned} 0 &\leq \frac{1}{u} Ecov_R(uS - vT, uS - vT) \\ &= uEvar_R(S) - |v|^2 - |v|^2 + |v|^2 \\ &= u\,Evar_R(S) - |v|^2 \\ &= Evar_R(T)Evar_R(S) - |Ecov_R(S,T)|^2, \end{aligned}$$

and this proves the g-c-v inequality. It follows by above that the g-c-v equality holds if and only if $Ecov_R(uS - vT, uS - vT) = 0$, if and only if $Rz$ and $uSx - vTy$ are proportional by (2.2), and the proof is completed. □

Now, firstly we require a special operator: For $w \in H$ let $P_w \in B(H)$ be defined by $P_w(x) = (x,w)w$ for every $x \in H$. In particular, $P_e(e) = e$. Secondly observe that for every nonzero vector $x \in H$ there exists a unit vector orthogonal to $x$. For example, for any nonzero vectors $y, w \in H$ let $e = \frac{w}{\|w\|}$ with $w = y - \frac{(y,x)x}{\|x\|^2}$. The next result is an application of the g-c-v inequality, which will be used in section five.

**Corollary 2.2.** *For a unit vector $e \in H$ and $S, T, R, P_e \in B(H)$ let $S, T, R$ and $P_e$ be acting on $x, y, z$ and $e$, respectively for every $x, y, z \in H$. Then*

$$\begin{aligned} &|Ecov_R(S,T) - Ecov_R(S,P_e)Ecov_R(P_e,T)|^2 \\ &\leq [Evar_R(S) - |Ecov_R(S,P_e)|^2][Evar_R(T) - |Ecov_R(P_e,T)|^2] \end{aligned}$$

*if* $\mathrm{Evar}_R(P_e) = 1$. *The equality holds if and only if $Rz$ and $Sx + \alpha Ty - \beta e$ are proportional, $\alpha, \beta \in C$.*

*Proof.* Let $u = Ecov_R(S, P_e)$ and $v = Ecov_R(T, P_e)$, and note that

$$Evar_R(uP_e) = \mid u \mid^2 \text{ and } Evar_R(vP_e) = \mid v \mid^2 .$$

Then

$$\begin{aligned}
&\mid Ecov_R(S,T) - Ecov_R(S,P_e)Ecov_R(P_e,T) \mid^2 \\
&= \mid Ecov_R(S,T) - u\overline{v} \mid^2 = \mid Ecov_R(S - uP_e, vP_e - T) \mid^2 \\
&\leq Evar_R(S - uP_e)Evar_R(vP_e - T) \text{ by the g-c-v inequality} \\
&= [Evar_R(S) - \mid u \mid^2][Evar_R(T) - \mid v \mid^2] \text{ by (2.6) of Lemma 2.1,}
\end{aligned}$$

and the required inequality follows. The equality holds if and only if $Rz$ and $Sx - ue - \lambda(ve - Ty)$ are proportional by (2.2) in Lemma 2.1, $u, v, \lambda \in C$, which is the given condition.

## 3. Inequalities by Generalized Covariance and Variance

Before proceeding further about inequalities in $H$, we require some notations and their propperties. Let $A$, $B$, $T$, $U \in B(H)$. If $A$ is a positive operator, we write $A \geq O$. If $A$ and $B$ are selfadjoint, we write $A \geq B$ when $A - B \geq O$. Let $T = U \mid T \mid$ be the polar decomposition of $T$ with $U$ the partial isometry, and $\mid T \mid$ the positive square root of the positive operator $T^*T$. A basic well-known property about the polar decomposition of $T$ is that the equality $\mid T^* \mid^c = U \mid T \mid^c U^*$ holds for any $c > 0$, and $U^*U = I$ [4, p. 752]; this formula will also be used frequently in the next two sections. We recall that a complex number $\gamma \neq 0$ is a normal eigenvalue for $T$ if both relations $Tx = \gamma x$ and $T^*x = \overline{\gamma}x$ hold associated with the same eigenvector $x \neq 0$.

In the next result the proof of each inequality is nothing but expanding and simplifying a suitable g-c-v inequality, which is easy and a straightforward process. This shows the usefulness of the g-c-v inequality and simplicity in the proof of inequalities in $H$.

**Theorem 3.1.** *Let $S, T, R, U \in B(H)$. Then the following inequalities hold for every $x, y, z, e \in H$ with $\| e \| = 1$.*

$$\begin{aligned}
(3.1)\ &\mid \| Rz \|^2 (Sx, Ty) - (Sx, Rz)(Rz, Ty) \mid^2 \\
&\leq [\| Rz \|^2 \| Sx \|^2 - \mid (Sx, Rz) \mid^2][\| Rz \|^2 \| Ty \|^2 - \mid (Ty, Rz) \mid^2].
\end{aligned}$$

*The equality holds if and only if $Rz$ and $Sx - \lambda Ty$ are proportional, $\lambda \in C$.*

$$(3.2)\ |\|Rz\|^2((S-\gamma I)x,Ty)-((S-\gamma I)x,Rz)(Rz,Ty)|^2 \le [\|Rz\|^2\|(S-\gamma I)x\|^2-|((S-\gamma I)x,Rz)|^2][\|Rz\|^2\|Ty\|^2 - |(Ty,Rz)|^2]$$

*for* $\gamma \in C$ *and* $Sx \neq \gamma x$. *The equality holds if and only if* $Rz$ *and* $(S-\gamma I)x-\lambda Ty$ *are proportional.* $\lambda \in C$.

(3.3) *Let* $r, s \geq 0$, $\alpha, \beta \in (0,1]$ *with* $\alpha(1+2r)+\beta(1+2s) \geq 1$, *and* $T = U \mid T \mid$ *the polar decomposition. Then*

$$\begin{aligned} &|\||T|^{\alpha(1+2r)}z\|^2(T|T|^{\alpha(1+2r)+\beta(1+2s)-1}x,y) \\ &\quad -(|T|^{2\alpha(1+2r)}x,z)(z,|T|^{\alpha(1+2r)+\beta(1+2s)}U^*y)|^2 \\ &\le [\||T|^{\alpha(1+2r)}z\|^2\||T|^{\alpha(1+2r)}x\|^2-|(|T|^{2\alpha(1+2r)}x,z)|^2] \\ &\quad\cdot[\||T|^{\alpha(1+2r)}z\|^2\||T^*|^{\beta(1+2s)}y\|^2-|(z,|T|^{\alpha(1+2r)+\beta(1+2s)}U^*y)|^2]. \end{aligned}$$

*The equality holds if and only if* $U \mid T \mid^{\alpha(1+2r)} z$ *and* $U \mid T \mid^{\alpha(1+2r)} x - \lambda \mid T^* \mid^{\beta(1+2s)} y$ *are proportional,* $\lambda \in C$.

*Proof.* (3.1) Let $S$, $T$ and $R$ be acting on $x$, $y$ and $z$, respectively, and use the g-c-v inequality $| Ecov_R(S,T) |^2 \leq Evar_R(S)Evar_R(T)$ to expand.

(3.2) Let $S-\gamma I$, $T$ and $R$ be acting on $x$, $y$ and $z$, respectively, and use the g-c-v inequality $|Ecov_R(S-\gamma I,T)|^2 \leq Evar_R(S-\gamma I)Evar_R(T)$ to expand.

(3.3) Let $S = U \mid T \mid^{\alpha(1+2r)}$, $T = \mid T^* \mid^{\beta(1+2s)}$ and $R = U \mid T \mid^{\alpha(1+2r)}$ be acting on $x$, $y$ and $z$, respectively. Use the g-c-v inequality $\mid Ecov_R(S,T) \mid^2 \leq Evar_R(S)Evar_R(T)$, and notice that $\mid T^* \mid^c = U \mid T \mid^c U^*$, $c > 0$. So,

$$\begin{aligned} (Sx,Ty) &= (U|T|^{\alpha(1+2r)}x,|T^*|^{\beta(1+2s)}y) \\ &= (U|T|^{\alpha(1+2r)}x,U|T|^{\beta(1+2s)}U^*y) \\ &= (U|T|^{\alpha(1+2r)+\beta(1+2s)}x,y) \\ &= (T|T|^{\alpha(1+2r)+\beta(1+2s)-1}x,y), \end{aligned}$$

and similarly, $(Rz,Ty) = (z, \mid T \mid^{\alpha(1+2r)+\beta(1+2s)} U^*y)$. The required inequality thus follows now. We remark that the condition $\alpha(1+2r)+\beta(1+2s) \geq 1$ is unnecessary if $T$ is positive or $T$ is invertible as mentioned in [5].

## 4. Applications

The following corollaries about inequalities in $H$ are consequences of inequalities in Theorem 3.1. Some of them are generalizations and/or sharpenings of well-known inequalities in the literature. We shall waive the discussion about equality conditions and leave it to the reader.

**Corollary 4.1.** *Let* $S, T \in B(H)$, $e, x, y \in H$ *with* $\| e \| = 1$. *Then*

$$| (Sx, Ty) - (Sx, e)(e, Ty) |^2 \leq [\| Sx \|^2 - | (Sx, e) |^2][\| Ty \|^2 - | (Ty, e) |^2].$$

*Proof.* Let $R(z) = e$ in (3.1) of Theorem 3.1. □

We note that Corollary 4.1 appeared in [8, Theorem 1] with a complicated proof. It is the main formula used to sharpen and characterize inequalities in [8].

**Corollary 4.2.** *Let* $x, y, z \in H$. *Then*

$$\begin{aligned} &|\| z \|^2 (x, y) - (x, z)(z, y) |^2 \\ &\leq [\| z \|^2 \| x \|^2 - | (x, z) |^2][\| z \|^2 \| y \|^2 - | (y, z) |^2]. \end{aligned}$$

*Proof.* Let $S = T = R = I$ in (3.1) of Theorem 3.1. □

A particular case of Corollary 4.2 is the inequality

$$\| x \|^2 \leq \| z \|^2 [\| x \|^2 \| y \|^2 - | (x, y) |^2],$$

if $(x, z) = 0$ and $(y, z) = 1$, which is known as the extended Ostrowski inequality in vectors [2, Theorem 4.1].

**Corollary 4.3.** *Let* $S \in B(H)$ *and* $x, z, e \in H$. *If* $e$ *is a unit eigenvector corresponding to an eigenvalue* $\overline{\gamma}$ *of* $S^*$, *and* $Sx \neq \gamma x$. *Then*

$$| (e, z) |^2 \leq \frac{\| z \|^2 \| (S - \gamma I)x \|^2 - | ((S - \gamma I)x, z) |^2}{\| (S - \gamma I)x \|^2}.$$

*Proof.* By assumption, $((S - \gamma I)x, e) = (Sx, e) - (\gamma x, e) = (x, \overline{\gamma} e) - (x, \overline{\gamma} e) = 0$. Let $R = I$ and $Ty = e$ in (3.2) of Theorem 3.1. Then

$$\begin{aligned} &| ((S - \gamma I)x, z)(z, e) |^2 \\ &\leq [\| z \|^2 \| (S - \gamma I)x \|^2 - | ((S - \gamma I)x, z) |^2][\| z \|^2 - | (e, z) |^2]. \end{aligned}$$

The required inequality follows by simplifying above. □

Remark that in above if both operators $S - \gamma I$ and $S$ are acting on the vector $x$, then $\text{Evar}_x(S - \gamma I) = \text{Evar}_x(S)$ by Definition 1.1 and a straightforward simplification. This reminds us of the Bernstein's inequality [1] which says that if $e$ is a unit eigenvector corresponding to an eigenvalue $\gamma \neq 0$ of a selfadjoint operator $S$, then, for every $x \in H$ and $Sx \neq \gamma x$,

$$|(e,x)|^2 \leq \frac{\|x\|^2\|Sx\|^2 - |(Sx,x)|^2}{\|(S-\gamma I)x\|^2}.$$

Clearly, the inequality follows easily by letting $z = x$ in Corollary 4.3. We mention also that the inequality in Corollary 4.3 appeared in [7, Theorem 2] with a lengthy proof; and a different generalization of the Bernstein's inequality may be found in Corollary 4.6 below.

The next result generalizes both [3, Theorem 1] and [6, Theorem 1].

**Corollary 4.4.** *Let $T \in B(H)$ and $x, y, z \in H$. For $r, s \geq 0$, $\alpha, \beta \in (0,1]$ with $\alpha(1+2r)+\beta(1+2s) \geq 1$, and $T = U|T|$ the polar decomposition, if $z$ is orthogonal to $|T|^{\alpha(1+2r)+\beta(1+2s)}U^*y$ and $|T|^{\alpha} z \neq 0$, then*

$$|(T|T|^{\alpha(1+2r)+\beta(1+2s)-1}x,y)|^2 + \frac{(|T^*|^{2\beta(1+2s)}y,y)\,|(|T|^{2\alpha(1+2r)}x,z)|^2}{(|T|^{2\alpha(1+2r)}z,z)}$$
$$\leq (|T|^{2\alpha(1+2r)}x,x)(|T^*|^{2\beta(1+2s)}y,y).$$

*Proof.* This is a simple consequence of (3.3) in Theorem 3.1. □

**Corollary 4.5.** *Let $T \in B(H)$ and $x, y \in H$. For $r, s \geq 0$, $\alpha, \beta \in (0,1]$ with $\alpha(1+2r)+\beta(1+2s) \geq 1$, and $T = U|T|$ the polar decomposition, if a unit vector $e$ is orthogonal to $|T|^{\beta(1+2s)}U^*y$, then*

$$|(T|T|^{\alpha(1+2r)+\beta(1+2s)-1}x,y)|^2 + \||T^*|^{\beta(1+2s)}y\|^2|(|T|^{\alpha(1+2r)}x,e)|^2$$
$$\leq \||T|^{\alpha(1+2r)}x\|^2\||T^*|^{\beta(1+2s)}y\|^2.$$

*Proof.* We may take $\frac{|T|^{\alpha(1+2r)}z}{\||T|^{\alpha(1+2r)}z\|} = e$, so that

$$(z, |T|^{\alpha(1+2r)+\beta(1+2s)}U^*y) = (|T|^{\alpha(1+2r)}z, |T|^{\beta(1+2s)}U^*y) = 0,$$

i.e., $z$ is orthogonal to $|T|^{\alpha(1+2r)+\beta(1+2s)}U^*y$. The required inequality follows by Corollary 4.4, since

$$\frac{|(|T|^{2\alpha(1+2r)}x,z)|^2}{(|T|^{2\alpha(1+2r)}z,z)} = |(|T|^{\alpha(1+2r)}x, \frac{|T|^{\alpha(1+2r)}z}{\||T|^{\alpha(1+2r)}z\|})|^2 = |(|T|^{\alpha(1+2r)}x,e)|^2.$$

□

Remark that Corollary 4.5 is a generalization of [7, (1) in Theorem 4] and the present proof is direct and much shorter. In particular we have

$$| (T \mid T |^{\alpha(1+2r)+\beta(1+2s)-1} x, y) | \leq \| \mid T |^{\alpha(1+2r)} x \| \| \mid T^* |^{\beta(1+2s)} y \| .$$

Interestingly, the inequality above may be obtained directly from the Cauchy-Schwarz inequality $| (x, y) | \leq \| x \| \| y \|$; just replacing $x$ by $U \mid T |^{\alpha(1+2r)} x$, and $y$ by $| T^* |^{\beta(1+2s)} y$. According to [4] the relation $| (Tx, y) | \leq \| \mid T |^{\alpha} x \| \| \mid T^* |^{1-\alpha} y \|$, $\alpha \in (0, 1]$, is called the Heinz inequality. We see that Corollary 4.5 is obviously its generalization and sharpening.

The next result generalizes both [3, Theorem 4] and [6, Theorem 3].

**Corollary 4.6.** *Let $T \in B(H)$, $x, y \in H$, $s \geq 0$, $\beta \in (0, 1]$, and $T = U \mid T \mid$ the polar decomposition. If $T$ has a normal eigenvalue $\gamma \neq 0$ associated with a unit eigenvector $e$, then*

$$| \gamma |^2 | (x, e) |^2 \leq \frac{\| Tx \|^2 \| \mid T^* |^{\beta(1+2s)} y \|^2 - | (T \mid T |^{\beta(1+2s)} x, y) |^2}{\| \mid T^* |^{\beta(1+2s)} y \|^2},$$

*Proof.* Since, by assumption, $(| T |^2 e, e) = (Te, Te) = | \gamma |^2$, and $(| T |^2 x, e) = (Tx, \gamma e) = \overline{\gamma}(x, \overline{\gamma} e) = | \gamma |^2 (x, e)$. The required inequality is obtained by letting $\alpha = 1$, $r = 0$ and $z = e$ in Corollary 4.4. □

The next two lemmas are required for Corollary 4.9 below. Lemma 4.7 is an excellent generalization of the Löwner-Heinz inequality: $A^{\alpha} \geq B^{\alpha}$ if $A \geq B \geq O$ for $\alpha \in (0, 1]$. But the inequality does not hold in general if $\alpha > 1$.

**Lemma 4.7** (Furuta inequality [5]). *If $A \geq B \geq O$, then for eachr $\geq 0$,*

$$(B^r A^p B^r)^{\frac{\alpha(1+2r)}{p+2r}} \geq B^{\alpha(1+2r)} \text{ and } A^{\alpha(1+2r)} \geq (A^r B^p A^r)^{\frac{\alpha(1+2r)}{p+2r}}$$

*hold for any $p \geq 1$ and $\alpha \in (0, 1]$.*

**Lemma 4.8.** *Let $T$, $A$, $B \in B(H)$ satisfying conditions $\| Tx \| \leq \| Ax \|$ and $\| T^* y \| \leq \| By \|$ for all $x, y \in H$. Also let $p$, $q \geq 1$, $r$, $s \geq 0$, $\alpha, \beta \in (0, 1]$ with $\alpha(1 + 2r) + \beta(1 + 2s) \geq 1$, and $T = U \mid T \mid$ the polar decomposition. Then we have*

$$(| T |^{2\alpha(1+2r)} x, x) \leq ((| T |^{2r} A^{2p} | T |^{2r})^{\frac{\alpha(1+2r)}{p+2r}} x, x);$$

*and*

$$(| T^* |^{2\beta(1+2s)} y, y) \leq ((| T^* |^{2s} B^{2q} | T^* |^{2s})^{\frac{\beta(1+2s)}{q+2s}} y, y).$$

*Proof.* This is easy and was mentioned in [5, p. 80]. In fact, relations $\| Tx \| \leq \| Ax \|$ and $\| T^*y \| \leq \| By \|$ are equivalent to $| T |^2 \leq A^2$ and $| T^* |^2 \leq B^2$, respectively. Now, apply the first inequality in Lemma 4.7 to get both required inequalities. □

The next result without the first inequality appeared in [3, Theorem 3] with a different proof.

**Corollary 4.9.** *Let $T, A, B \in B(H)$ satisfying conditions $\| Tx \| \leq \| Ax \|$ and $\| T^*y \| \leq \| By \|$ for all $x, y \in H$. Also let $p, q \geq 1$, $r, s \geq 0$, $\alpha, \beta \in (0,1]$ with $\alpha(1+2r) + \beta(1+2s) \geq 1$, and $T = U \mid T \mid$ the polar decomposition such that $(T \mid T \mid^{\alpha(1+2r)+\beta(1+2s)-1} z, y) = 0$. Then*

$$\begin{aligned}
&| (T \mid T \mid^{\alpha(1+2r)+\beta(1+2s)-1} x, y) |^2 + \frac{(| T^* |^{2\beta(1+2s)} y, y) \mid (| T |^{2\alpha(1+2r)} x, z) |^2}{(| T |^{2\alpha(1+2r)} z, z)} \\
&\leq \| | T |^{\alpha(1+2r)} x \|^2 \| | T^* |^{\beta(1+2s)} y \|^2 \\
&\leq ((| T |^{2r} A^{2p} | T |^{2r})^{\frac{\alpha(1+2r)}{p+2r}} x, x)((| T^* |^{2s} B^{2q} | T^* |^{2s})^{\frac{\beta(1+2s)}{q+2s}} y, y).
\end{aligned}$$

*Proof.* Since $(z, | T |^{\alpha(1+2r)+\beta(1+2s)} U^*y) = (T \mid T \mid^{\alpha(1+2r)+\beta(1+2s)-1} z, y) = 0$, the inequality (3.3) in Theorem 3.1 becomes

$$\begin{aligned}
&| \| | T |^{\alpha(1+2r)} z \|^2 (T \mid T \mid^{\alpha(1+2r)+\beta(1+2s)-1} x, y) |^2 \\
&\leq [\| | T |^{\alpha(1+2r)} z \|^2 \| | T |^{\alpha(1+2r)} x \|^2 - | (| T |^{2\alpha(1+2r)} x, z) |^2] \\
&\quad \cdot \| | T |^{\alpha(1+2r)} z \|^2 \| | T^* |^{\beta(1+2s)} y \|^2 .
\end{aligned}$$

Rewrite it in the following form,

$$\begin{aligned}
&| \| | T |^{\alpha(1+2r)} z \|^2 (T \mid T \mid^{\alpha(1+2r)+\beta(1+2s)-1} x, y) |^2 \\
&\quad + \| | T |^{\alpha(1+2r)} z \|^2 \| | T^* |^{\beta(1+2s)} y \|^2 | (| T |^{2\alpha(1+2r)} x, z) |^2 \\
&\leq \| | T |^{\alpha(1+2r)} z \|^4 \| | T |^{\alpha(1+2r)} x \|^2 \| | T^* |^{\beta(1+2s)} y \|^2 .
\end{aligned}$$

The inequality above devided by $\| | T |^{\alpha(1+2r)} z \|^4$ on both sides, and applying Lemma 4.8 yield the desired inequality. □

At this stage we have to mention that part of the inequality in Corollary 4.9, i.e.,

$$\begin{aligned}
&| (T \mid T \mid^{\alpha(1+2r)+\beta(1+2s)-1} x, y) |^2 \\
&\leq ((| T |^{2r} A^{2p} | T |^{2r})^{\frac{\alpha(1+2r)}{p+2r}} x, x)((| T^* |^{2s} B^{2q} | T^* |^{2s})^{\frac{\beta(1+2s)}{q+2s}} y, y).
\end{aligned}$$

is equivalent to Lemma 4.7, cf. [5, Theorem 1 and p. 82]. We also mention that Corollary 4.9 (let $r = s = 0$ there) generalizes and sharpens the so called Heinz-Kato-Furuta inequality [3, p. 224], which says that for $A, B \geq O$ if $\| Tx \| \leq \| Ax \|$ and $\| T^*y \| \leq \| By \|$, then

$$| (T \mid T |^{\alpha+\beta-1} x, y) | \leq \| A^\alpha x \| \| B^\beta y \|$$

for every $x, y \in H$, $\alpha, \beta \in (0,1]$ with $\alpha + \beta \geq 1$. In particular, it is called the Heinz-Kato inequality if $\alpha + \beta = 1$.

**Corollary 4.10.** *Let $S, K, V \in B(H)$, $S \geq O$, $SK$ be selfadjoint, and let $SK = V \mid SK \mid$ be the polar decomposition. For $x, y \in H$, $r, s \geq 0$, $\alpha, \beta \in (0,1]$ with $\alpha(1+2r) + \beta(1+2s) \geq 1$ and $p, q \geq 1$, if there exists a unit vector $e$ orthogonal to $| SK |^{\beta(1+2s)} V^*y$, then*

$$\begin{aligned}
&| (SK \mid SK |^{\alpha(1+2r)+\beta(1+2s)-1} x, y) |^2 \\
&\quad + \| | SK |^{\beta(1+2s)} y \|^2 | (| SK |^{\alpha(1+2r)} x, e) |^2 \\
&\leq \| | SK |^{\alpha(1+2r)} x \|^2 \| | SK |^{\beta(1+2s)} y \|^2 \\
&\leq \| K \|^{\frac{2p(1+2r)\alpha}{p+2r} + \frac{2q(1+2s)\beta}{q+2s}} ((| SK |^{2r} S^{2p} | SK |^{2r})^{\frac{\alpha(1+2r)}{p+2r}} x, x) \\
&\quad \cdot ((| SK |^{2s} S^{2q} | SK |^{2s})^{\frac{\beta(1+2s)}{q+2s}} y, y).
\end{aligned}$$

*Proof.* The first inequality is obtained by replacing $T$ by $SK$ in Corollary 4.5, and the second inequality was proved in [9, Proof of Theorem 1, p. 857] using the Furuta inequality. □

Consequently, a particular case of Corollary 4.10 (let $r, s = 0$ there) is the inequality

$$| (SK \mid SK |^{\alpha+\beta-1} x, y) | \leq \| K \|^{\alpha+\beta} \| S^\alpha x \| \| S^\beta y \|,$$

which is equivalent to the Löwner-Heinz inequality [9, Corollary 1]. Notice also that Corollary 4.10 is a generalization and sharpening of the Reid's inequality:

$$| (SKx, x) | \leq \| K \| (Sx, x) \text{ [10].}$$

In fact, let $\alpha = \beta = \frac{1}{2}$, $r = s = 0$ and $p = q = 1$ in Corollary 4.10. Then

$$\begin{aligned}
&| (SKx, y) |^2 + (| SK | y, y) | (| SK |^{1/2} x, e) |^2 \\
&\leq (| SK | x, x)(| SK | y, y) \\
&\leq \| K \|^2 (Sx, x)(Sy, y).
\end{aligned}$$

## 5. Semi-inner Product $Ecov_R(\,,)$ in $B(H)$ and Inner Product $(\,,)$ in $H$

Let $S$, $T$, $R \in B(H)$ and let $S$, $T$ and $R$ be acting on $x$, $y$ and $z$, respectively. In this final section we would like to explain the relationships between the semi-inner product $Ecov_R(S,T)$ in $B(H)$ and the inner product $(x,y)$ in $H$. In fact, by section two it is understandable that $Ecov_R(S,T)$ corresponds to $(x,y)$, and that $Evar_R(S)$ $(\neq 0)$ to $(x,x) = \| x \|^2$ $(\neq 0)$. Moreover, a single vector $x$ corresponds to the operator $S$. In other words, to every inequality in $H$ there is an inequality expressed in terms of covariance and variance, and vice versa. For instance, to Corollary 2.2 we have

**Corollary 5.1.** *For $x$, $y$, $e \in H$ with $\| e \| = 1$, then*

$$| (x,y) - (x,e)(e,y) |^2 \leq [\| x \|^2 - | (x,e) |^2][\| y \|^2 - | (e,y) |^2].$$

*Proof.* The proof can be done similarly and correspondingly as in Corollary 2.2, i.e., let $u = (x,e)$ and $v = (y,e)$. Then

$$\begin{aligned} &| (x,y) - (x,e)(e,y) |^2 = | (x,y) - u\bar{v} |^2 = | (x - ue, ve - y) |^2 \\ &\leq \| x - ue \|^2 \| ve - y \|^2 \text{ by the Cauchy-Schwarz inequality} \\ &= [\| x \|^2 - | u |^2][\| y \|^2 - | v |^2]. \end{aligned}$$

□

Remark that Corollary 5.1 is also obtained from Corollary 4.1 if $S = T = I$.

On the other hand, notice that an extension of the Cauchy-Schwarz inequality in three vectors $x, y$ and $w$ is as follows:

$$| (w,x)(x,y) | \leq \frac{\| y \| \| w \| + | (w,y) |}{2} \| x \|^2 \quad [8, \text{p. } 248].$$

This may be obtained, among other proofs, by replacing $x$ by $2(w,x)x - \| x \|^2 w$ in the Cauchy-Schwarz inequality, and note that $\| 2(w,x)x - \| x \|^2 w \| = \| x \|^2 \| w \|$. Therefore,

$$\begin{aligned} &2 | (w,x)(x,y) | - \| x \|^2 | (w,y) | \\ &\leq | 2(w,x)(x,y) - \| x \|^2 (w,y) | \\ &= | (2(w,x)x - \| x \|^2 w, y) | \\ &\leq \| x \|^2 \| w \| \| y \|, \end{aligned}$$

and we have the Cauchy-Schwarz inequality in three vectors.

To the inequality above we have the next result which is an extension of the g-c-v inequality for four operators $S, T, Q, R \in B(H)$. The proof will be done similarly and correspondingly as above.

**Corollary 5.2.** *For $S$, $T$, $Q$, $R \in B(H)$ let $S, T, Q$ and $R$ be acting on $x$, $y$, $w$ and $z$, respectively for every $x$, $y$, $w$, $z \in H$. Then*

$$\begin{aligned}&| Ecov_R(Q,S)Ecov_R(S,T) | \\ &\leq \frac{[\mathrm{Evar}_R(T)]^{1/2}[\mathrm{Evar}_R(Q)]^{1/2}+ | Ecov_R(Q,T) |}{2} Evar_R(S).\end{aligned}$$

*Proof.* We shall use formulas in Lemma 2.1 to simplify relations. First, replace $S$ in the g-c-v inequality by the operator $2Ecov_R(Q,S)S - Evar_R(S)Q$, and notice that

$$\begin{aligned}&Evar_R(2Ecov_R(Q,S)S - Evar_R(S)Q) \\ &= Evar_R(2Ecov_R(Q,S)S) + Evar_R(Evar_R(S)Q) \\ &\quad - 2ReEcov_R(2Ecov_R(Q,S)S, Evar_R(S)Q) \text{ by (2.6) of Lemma 2.1} \\ &= 4 | Ecov_R(Q,S) |^2 Evar_R(S) + [Evar_R(S)]^2 Evar_R(Q) \\ &\quad - 4 | Ecov_R(Q,S) |^2 Evar_R(S) \\ &= [Evar_R(S)]^2 Evar_R(Q).\end{aligned}$$

Now,

$$\begin{aligned}&2 | Ecov_R(Q,S)Ecov_R(S,T) | - Evar_R(S) | Ecov_R(Q,T) | \\ &\leq | 2Ecov_R(Q,S)Ecov_R(S,T) - Evar_R(S)Ecov_R(Q,T) | \\ &= | Ecov_R(2Ecov_R(Q,S)S - Evar_R(S)Q, T) | \\ &\leq [Evar_R(2Ecov_R(Q,S)S - Evar_R(S)Q)]^{1/2}[Evar_R(T)]^{1/2} \text{ by the g-c-v inequality} \\ &= Evar_R(S)[Evar_R(Q)]^{1/2}[Evar_R(T)]^{1/2} \text{ by the notice above}\end{aligned}$$

and the proof is completed. □

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