HYPERBOLIC CURVATURE AND $k$-CONVEX FUNCTIONS

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ABSTRACT. Let $\gamma$ be a $C^2$ curve in the open unit disk $D$. Flinn and Osgood proved that $K_D(z, \gamma) \geq 1$ for all $z \in \gamma$ if and only if the curve $f \circ \gamma$ is convex for every convex conformal mapping $f$ of $D$, where $K_D(z, \gamma)$ denotes the hyperbolic curvature of $\gamma$ at the point $z$. In this paper we establish a generalization of the Flinn-Osgood characterization for a curve with the hyperbolic curvature at least 1.

1. INTRODUCTION

We begin with a brief introduction to hyperbolic regions in the complex plane $\mathbb{C}$. A general discussion of hyperbolic regions can be found in [1] and [6]. A region $\Omega$ in $\mathbb{C}$ is called hyperbolic if the complement of $\Omega$ with respect to $\mathbb{C}$ contains at least two points. Let $D = \{z : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. The hyperbolic metric on $D$ is defined by

$$\lambda_D(z) |dz| = \frac{2|dz|}{1 - |z|^2}.$$

If a region $\Omega$ is hyperbolic, then, by the uniformization theorem [2, p.39], there is a holomorphic universal covering projection $\varphi$ of $D$ onto $\Omega$. The density $\lambda_\Omega(z)$ of the hyperbolic metric $\lambda_\Omega(z) |dz|$ on a hyperbolic region $\Omega$ is obtained from

$$\lambda_\Omega(\varphi(z)) |\varphi'(z)| = \lambda_D(z),$$

where $\varphi$ is any holomorphic universal covering projection of $D$ onto $\Omega$. The hyperbolic metric is invariant under holomorphic covering projections. In particular, the hyperbolic metric is a conformal invariant.

A hyperbolic simply connected region $\Omega$ is said to be $k$-convex ($k > 0$) if $|a - b| < 2/k$ for any pair of distinct points $a, b \in \Omega$ and the intersection $E_k[a, b]$ of two closed disks of radii $1/k$ that have both $a$ and $b$ on their boundaries lies in $\Omega$. A hyperbolic
simply connected region $\Omega$ is said to be 0-convex if $E_0[a,b]$ is in $\Omega$ for any pair of distinct points $a$ and $b$ in $\Omega$, where $E_0[a,b]$ is the closed line segment joining $a$ and $b$. We will always use convex instead of 0-convex. Mejia and Minda [4] proved that if $\Omega$ is a hyperbolic simply connected region bounded by a simple closed curve $\partial \Omega$ of class $C^2$ and if $K_e(z, \partial \Omega) \geq k$ for all $z \in \partial \Omega$, then $\Omega$ is $k$-convex. Here $K_e(z, \partial \Omega)$ denotes the euclidean curvature of $\partial \Omega$ at the point $z$.

Let us recall the definition of the hyperbolic curvature. For more details, see [3] and [5]. If $\gamma$ is a $C^2$ curve in a hyperbolic region $\Omega$ with parametrization $z = z(t)$, then the hyperbolic curvature of $\gamma$ at the point $z = z(t)$ is given by

$$K_{\Omega}(z, \gamma) = \frac{1}{\lambda_{\Omega}(z)} \left[ K_e(z, \gamma) + 2\text{Im} \left\{ \frac{\partial \log \lambda_{\Omega}(z)}{\partial z} \left( \frac{z'(t)}{|z'(t)|} \right) \right\} \right],$$

where

$$K_e(z, \gamma) = \frac{1}{|z'(t)|} \text{Im} \left[ \frac{z''(t)}{z'(t)} \right]$$

denotes the euclidean curvature of $\gamma$ at $z = z(t)$. Since $\lambda_D(z) = 2 \left( 1 - |z|^2 \right)$, we have

$$K_D(z, \gamma) = \frac{1}{2} \left( 1 - |z|^2 \right) K_e(z, \gamma) + \text{Im} \left[ \frac{\overline{z(t)} z'(t)}{|z'(t)|} \right]$$

for a $C^2$ curve $\gamma : z = z(t)$ in $\mathbb{D}$. Because the hyperbolic metric is invariant under holomorphic covering projections, so is the hyperbolic curvature. In particular, the hyperbolic curvature is conformally invariant.

Let $\gamma$ be a positively oriented circle in $\mathbb{D}$ with center 0 and radius $r \in (0, 1)$. A parametrization of $\gamma$ is $z = z(t) = re^{it}$, $0 \leq t \leq 2\pi$. Then

$$K_e(z, \gamma) = \frac{1}{|z'(t)|} \text{Im} \left\{ \frac{z''(t)}{z'(t)} \right\} = \frac{1}{r}.$$

As $\overline{z(t)} z'(t) / |z'(t)| = i \tau$ so that

$$K_D(z, \gamma) = \frac{1 - r^2}{2} \frac{1}{r} + r = \frac{1}{2} \left( r + \frac{1}{r} \right).$$

Note that $r + \frac{1}{r} > 2$. Since the hyperbolic curvature is a conformal invariant, it follows that any circle in $\mathbb{D}$ has the hyperbolic curvature strictly larger than 1.

Let $\gamma$ be the positively oriented circle $|z - a| = 1 - a$ where $0 < a < 1$. This circle is internally tangent to the unit circle at the point 1. A parametrization of $\gamma$ is $z = z(t) = a + (1 - a)e^{it}$, $0 < t < 2\pi$. Note that $z(0) = z(2\pi) = 1 \notin \mathbb{D}$. 

Then we obtain \( K_\varepsilon(z, \gamma) = 1/(1 - a) \). Since \( 1 - |z|^2 = 2(1 - a)(a - a \cos t) \) and \( \frac{z(t)z'(t)}{|z'(t)|} = i(a \cos t + 1 - a) \), we obtain

\[
K_\mathbb{D}(z, \gamma) = \frac{2(1 - a)(a - a \cos t)}{2} \frac{1}{1 - a} + a \cos t + 1 - a = 1.
\]

Since the hyperbolic curvature is a conformal invariant, it follows that any oricycle in \( \mathbb{D} \), that is, a circle internally tangent to the unit circle, has the hyperbolic curvature 1.

A conformal mapping \( f \) of the unit disk \( \mathbb{D} \) is called \( k \)-convex provided \( f(\mathbb{D}) \) is a \( k \)-convex region. A \( C^2 \) curve \( \gamma \) is said to be \( k \)-convex provided \( K_\varepsilon(z, \gamma) \geq k \) for all \( z \in \gamma \). Let \( \gamma \) be a \( C^2 \) curve in \( \mathbb{D} \). Flinn and Osgood [3] proved that \( K_\mathbb{D}(z, \gamma) \geq 1 \) for all \( z \in \gamma \) if and only if the curve \( f \circ \gamma \) is convex for every convex conformal mapping \( f \) of \( \mathbb{D} \). In this paper we establish a generalization of the Flinn-Osgood characterization for a curve with the hyperbolic curvature at least 1. More precisely, we prove that \( K_\mathbb{D}(z, \gamma) \geq 1 \) for all \( z \in \gamma \) if and only if the curve \( f \circ \gamma \) is \( k \)-convex for every \( k \)-convex conformal mapping \( f \) of \( \mathbb{D} \).

2. **Main Results**

Let \( \Omega \) be a hyperbolic region in \( \mathbb{C} \). Fix \( a \in \Omega \) and let \( w = \varphi(z) \) be a holomorphic universal covering projection \((\mathbb{D}, 0)\) onto \((\Omega, a)\). From the identity

\[
\lambda_\Omega(\varphi(z)) |\varphi'(z)| = \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D},
\]

we obtain

\[
\log \lambda_\Omega(\varphi(z)) + \frac{1}{2} \log \varphi'(z) + \frac{1}{2} \log \overline{\varphi'(z)} = \log 2 - \log(1 - z\overline{z}).
\]

We apply the operator \( \partial/\partial z \) to both sides of this identity and obtain

\[
\frac{\partial}{\partial w} \log \lambda_\Omega(\varphi(z)) \cdot \varphi'(z) + \frac{1}{2} \frac{\varphi''(z)}{\varphi'(z)} = \frac{\overline{z}}{1 - z\overline{z}}.
\]

For \( z = 0 \), this identity yields

\[
(1) \quad \frac{\partial}{\partial w} \log \lambda_\Omega(\varphi(a)) = \frac{1}{2} \frac{\varphi''(0)}{\varphi'(0)^2}.
\]

Mejia and Minda [4] proved that if \( \Omega \) is a \( k \)-convex region, then for \( z \in \Omega \)

\[
(2) \quad \left| \frac{\partial}{\partial z} \log \lambda_\Omega(z) \right| \leq \frac{1}{2} \lambda_\Omega(z) \sqrt{1 - \frac{2k}{\lambda_\Omega(z)}}
\]
with equality if and only if \( \Omega \) is a disk of radius \( 1/k \). We establish a sufficient condition for a curve in a \( k \)-convex region to be \( k \)-convex.

**Theorem 1.** Let \( \gamma \) be a \( C^2 \) curve in a \( k \)-convex region \( \Omega \) with nonvanishing tangent and \( z \in \gamma \). Then \( K_{\Omega}(z, \gamma) \geq 1 \) implies \( K_e(z, \gamma) > k \).

**Proof.** From the definition of the hyperbolic curvature, we obtain

\[
K_e(z, \gamma) = K_{\Omega}(z, \gamma) \lambda_{\Omega}(z) - 2 \text{Im} \left\{ \frac{\partial \log \lambda_{\Omega}(z)}{\partial z} \frac{z'(t)}{|z'(t)|} \right\}
\]

\[
\geq K_{\Omega}(z, \gamma) \lambda_{\Omega}(z) - 2 \left| \frac{\partial \log \lambda_{\Omega}(z)}{\partial z} \right|.
\]

Since \( K_{\Omega}(z, \gamma) \geq 1 \), it follows from (2) and (3) that

\[
K_e(z, \gamma) \geq \lambda_{\Omega}(z) \left( 1 - \sqrt{1 - \frac{2k}{\lambda_{\Omega}(z)}} \right)
\]

\[
= \frac{2k}{1 + \sqrt{1 - \frac{2k}{\lambda_{\Omega}(z)}}} > k.
\]

We now establish a condition for a curve in \( \mathbb{D} \) to have the hyperbolic curvature at least 1.

**Theorem 2.** Let \( \gamma \) be a \( C^2 \) curve in the open unit disk \( \mathbb{D} \) with nonvanishing tangent. Then \( K_{\mathbb{D}}(z, \gamma) \geq 1 \) for all \( z \in \gamma \) if and only if the curve \( f \circ \gamma \) is \( k \)-convex for every \( k \)-convex conformal mapping \( f \) of \( \mathbb{D} \).

**Proof.** Suppose \( K_{\mathbb{D}}(z, \gamma) \geq 1 \) for all \( z \in \gamma \). Let \( f \) be a \( k \)-convex conformal mapping of \( \mathbb{D} \) onto a \( k \)-convex region \( \Omega \). Since the hyperbolic curvature is a conformal invariant,

\[
K_{\Omega}(f(z), f \circ \gamma) = K_{\mathbb{D}}(z, \gamma) \geq 1.
\]

Because \( \Omega \) is \( k \)-convex, Theorem 1 yields \( K_e(f(z), f \circ \gamma) \geq k \).

Conversely, suppose the curve \( f \circ \gamma \) is \( k \)-convex for every \( k \)-convex conformal mapping \( f \) of \( \mathbb{D} \). We note that for \( \alpha > 0 \), the function

\[
w = f(z) = \frac{\alpha z}{1 - \sqrt{1 + \alpha k z^2}} = \alpha z + \alpha \sqrt{1 + \alpha k z^2} + \cdots
\]

is a \( k \)-convex conformal mapping of \( \mathbb{D} \). The region \( \Omega = f(\mathbb{D}) \) is the disk with center \( -\sqrt{1 + \alpha k} / k \) and radius \( 1/k \). Since the hyperbolic curvature is invariant under conformal mappings, we may assume that \( z = 0 \) without loss of generality. Furthermore, we may also assume that \(-i\) is the unit tangent to \( \gamma \) at the origin,
that is, $-i = z'(t_0)/|z'(t_0)|$, where $z(t_0) = 0$. Since $f'(0) = \alpha > 0$, it follows that $-i$ is also the unit tangent to $f \circ \gamma$ at the origin. From (1), we obtain

\begin{equation}
\text{Im} \left\{ \frac{\partial \log \lambda_\Omega(0)}{\partial w} \left| \frac{w'(t_0)}{w'(t_0)} \right| \right\} = \frac{1}{2} \frac{f''(0)}{f'(0)^2}.
\end{equation}

Since $\lambda_\Omega(0) = 2/f'(0)$ and $K_\Omega(0, f \circ \gamma) \geq k$, it follows from the definition of the hyperbolic curvature and (4) that

\begin{align*}
K_\Omega(0, f \circ \gamma) & \geq \frac{f'(0)}{2} \left[ k + \frac{f''(0)}{f'(0)^2} \right] \\
& = \frac{\alpha k}{2} + \sqrt{1 + \alpha k} > 1.
\end{align*}

**Remark.** If we put $k = 0$ in Theorem 2, we recover the corresponding result for convex regions which was established by Flinn and Osgood [3].

Let $\Delta$ be a disk in $\mathbb{D}$, and let $\gamma$ denote the positively oriented boundary of $\Delta$. Then $\gamma$ is either a circle in $\mathbb{D}$ with $K_\mathbb{D}(z, \gamma) > 1$ or an oricycle in $\mathbb{D}$ with $K_\mathbb{D}(z, \gamma) = 1$. Thus, $K_\mathbb{D}(z, \gamma) \geq 1$ in all cases. Hence Theorem 2 yields the following.

**Corollary 3.** Let $\Delta$ be a disk in the open unit disk $\mathbb{D}$. If $f$ is a $k$-convex conformal mapping of $\mathbb{D}$, then $f(\Delta)$ is $k$-convex.

**References**


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