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THE METHOD TO CONSTRUCT THE STRONG COMBINED-OPTIMAL DESIGN

PI-HSIANG HUANG[1] AND PEN-HWANG LIAU[2]

ABSTRACT

The technique of foldover is usually used by experimenters to de-alias the effects that are interesting in follow-up experiment. Employing a 2^{k-p} design with resolution III or higher, Li and Lin (2003) developed an algorithm and used computer programs to search its corresponding optimal foldover design for selected 16-run and 32-run experiments. Based on the minimum aberration criterion, the strong combined-optimal design, defined by Li and Lin, is the better choice of the initial design. In this article, we apply the technique of blocking to find the strong combined-optimal designs. Furthermore, we will tabulate all 16-run and 32-run strong combined-optimal designs and their corresponding core foldover plans for practical use. Some new designs that have not appeared in the other literature but constructed by the technique of blocking are also proposed in this article.

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1. INTRODUCTION

The 2^{k-p} fractional factorial designs, a subset of the full factorial, are widely used in industrial research or other fields to reduce the cost of the experiments. They are also utilized to identify practicable experiments in the early stages of the work. These designs have p independent defining words, where a "word" consists of letters which are the names of the factors denoted by 1, 2,. . . , k. The number

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[1]Corresponding author. Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, Taiwan, R.O.C. (e-mail: phhuang@nknucc.nknu.edu.tw)
[2]Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, Taiwan, R.O.C. (e-mail: phliau@nknucc.nknu.edu.tw)

of letters in a word is called the "wordlength". The group generated by these p defining words is called the defining contrast subgroup. For a 2^{k-p}, let A_i denote the number of words of length i in its defining contrast subgroup. The vector $W = (A_3, A_4, \ldots, A_k)$ is called wordlength pattern (WLP) of the design. The resolution of a 2^{k-p} design is defined to be the smallest r such that $A_r \geq 1$. For any two designs d_1 and d_2, let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$. Then d_1 is said to have less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If there is no design with less aberration than d_1, then d_1 has minimum aberration (MA).

After performing the initial experiment, should one need to run an additional follow-up experiment? What kind of criterion should be applied for selecting the second plan? Traditionally, an experimenter adds another 2^{k-p} fraction that is obtained by reversing each of the k factors of the initial design to run the follow-up experiment. Recently, Li and Mee (2002) focused on investigating an alternative foldover fraction of resolution III design that is not only increase the resolution to IV but also separate some of the aliased two-factor interactions. Li and Lin (2003) used the minimum aberration criterion to search the optimal foldover plan for giving k and p in the 2^{k-p} design. In their article, they tabulated the optimal foldover plans for selected 16-run and 32-run designs and found that all combined-optimal designs are the strong combined-optimal designs except the 2^{10-6} design in their tables. Liau (2006) showed that the existence of the strong combined-optimal design in general.

The lower bound of the runs for any resolution IV 2^{k-p} design is $2k$ runs. Resolution IV designs that attain this lower bound are called minimal designs. The regular 2^{k-p} design of resolution IV or higher is called maximal if its resolution reduces to three whenever an extra factor is added. According to this definition, all the minimal designs are the maximal designs.

Without loss of generality, we denote $1, \ldots, k-p$ as the basic factors and $k-p+1, \ldots, k$ as the generated factors in the 2^{k-p} fractional factorial design. For a regular 2^{k-p} design of resolution III or higher, $2^p - 1$ of the $2^k - 1$ factorial effects appear in the defining relation. The remaining $2^k - 2^p$ effects are partitioned into $g \equiv 2^{k-p} - 1$ alias sets of size 2^p and each alias set contains only one word that consists of basic factors. For a regular 2^{k-p} fractional factorial design d, let $\rho_i(d)$ be the length of the shortest word in the i^{th} alias set, $i = 1, \ldots, g$, then the estimation index of d is $\rho(d) = \max\{\rho_i(d) : i = 1, \ldots, g\}$ (Chen and Cheng, 2004). Chen and Cheng (2004) also showed that $\rho(d) = 2$ for any resolution III 2^{k-p} design when $2^{k-p-1} < k < 2^{k-p} - 1$. We may also obtain that the estimation index is 2 for maximal designs. We will give an explanation for this fact, using the idea of blocking, in the following section. Chen and Cheng (2003) showed that a design which is generated from a maximal design by the process of doubling is still a maximal design. By the process of doubling, it leads to a family of maximal resolution IV designs with $k = 5n/16$, for $n = 2^t$, $t \geq 5$. They also proved that the designs in this family have minimum aberration. This result originates from Butler (2003).

The major work of this article is that we use the method to construct the strong combined-optimal designs. Moreover, we tabulate all these kind of designs for 16 runs and 32 runs. Furthermore, we find some new designs that have not appeared in the other literature. The article is organized as follows. We will give a brief description of the relationship between foldover design and block design in section 2. In section 3, we discuss the strong combined-optimal foldover design for all 16-run and 32-run in detail. Characterization of even-word design and the blocking technique will also be discussed in section 3 and eventually followed by conclusion in section 4.

2. THE CORE FOLDOVER PLAN

In Li and Lin (2003), if a foldover plan consists of only folding on the generated factors, then it is called "a core foldover plan". They showed that every foldover plan is equivalent to a specific core foldover plan. Moreover, for each core foldover plan, there are 2^{k-p} plans that are equivalent to it. For easily explaining the construction of the core foldover plan, we denote the foldover plan as f. In addition, the notation f_{12}, for example, means that factors 1 and 2 are folded. For convenience, the original plan is written as f_0. The following example is given to illustrate the idea of the core foldover plan.

Example 2.1 Consider a MA 2^{5-2} design with two generators $I = 124$ and $I = 135$. The four core foldover plans can be expressed by f_0, f_4, f_5 and f_{45}. We may write down the combinations of basic factors in standard order:

$$0,\ 1,\ 2,\ 12,\ 3,\ 13,\ 23,\ 123,$$

where letter 0 is an identity element. In order to decide whether the letters 4 or 5 should enter the above order or not, we propose the rule that if the letter i ($i = 1, 2, \ldots, 5$) appears in the generator, then the sign of generator should be changed. For instance,

(a) letter 1, the second one in the standard order, appears in the $I = 124$, so letter 4 should enter to keep the sign of $I = 124$ positive,

(b) letter 2, the third one in the standard order, appears in the $I = 124$, so letter 4 should enter to keep the sign of $I = 124$ positive,

(c) letter 4 does not need to enter the next position because 12 changes the sign at the same time.

Following the same procedure, we obtain

$$0,\ 14,\ 24,\ 12,\ 3,\ 134,\ 234,\ 123.$$

If the same idea is applied to the letter 5 in the other generator $I = 135$, we finally obtain

$$0,\ 145,\ 24,\ 125,\ 35,\ 134,\ 2345,\ 123.$$

Consequently, we get the set that the original points are still presented after performing the process of foldover

$$\text{(i)} : \{f_0, f_{145}, f_{24}, f_{125}, f_{35}, f_{134}, f_{2345}, f_{123}\}.$$

Note that the set, {0, 145, 24, 125, 35, 134, 2345, 123}, is a subgroup. The cosets are obtained by the following algebraic operations

$$f_i \cdot f_i = f_{i^2} = f_0,\ f_i \cdot f_{ij} = f_{i^2j} = f_{0j} = f_j \ \text{ for } i, j \in \{1, 2, \ldots, k\}.$$

Three cosets are presented below:

$$\text{(ii)} : \{f_4, f_{15}, f_2, f_{1245}, f_{345}, f_{13}, f_{235}, f_{1234}\},$$
$$\text{(iii)} : \{f_5, f_{14}, f_{245}, f_{12}, f_3, f_{1345}, f_{234}, f_{1235}\},$$
$$\text{(iv)} : \{f_{45}, f_1, f_{25}, f_{124}, f_{34}, f_{135}, f_{23}, f_{12345}\}.$$

Because any foldover plan in (i) does not change the signs of two generators $I = 124$ and $I = 135$, we may use the notation (+, +) for this set. For similar reason, we use the notation (–, +) for the second set (ii), the notation (+, –) for the third set (iii) and the notation (–, –) for the last set (iv).

Using these kinds of notation, it is easy for us to recognize which elements are in the same set. For example, given the 2^{8-3} design with generators $I = 1236$, $I = 1247$ and $I = 23458$, the foldover plans f_{238} and f_{458} are in the same set with notation (+, –, –).

Consider the MA 2^{6-2} design with generators 1235 and 2346. The alias sets not containing main effects and two-factor interactions are

$$124 = 345 = 136 = 256, \; 134 = 245 = 126 = 356.$$

Hence, the estimation index of this design is three. If we choose the word 124 as the block effect, then the complete defining relation for this block is $I = 1235 = 2346 = 1456 = 124 = 345 = 136 = 256$. This is a 2^{6-3} design of resolution III. On the other hand, one new factor, say $7 = 124$, may be added to form the new design, then the resulting design is a 2^{7-3} design of resolution IV with defining relation $I = 1235 = 2346 = 1456 = 1247 = 3457 = 1367 = 2567$. There is an implication for this kind of construction. It indicates that a regular 2^{k-p} maximal design has the estimation index two. Otherwise, there exists at least one alias set not containing any main effect and two-factor interaction. One can use this alias set to define a new factor. Then the resolution of the new design is still higher than IV.

If the 2^{k-p} design can be blocked into two designs, $2^{k-(p+1)}$ runs with resolution III or higher for each, then the two designs fold over to each other. Next question is how to find the block effect and how to find the foldover factors between these two blocks. Obviously, we may block a design, whose estimation index is three or higher, into two designs with resolution III or higher. By above discussions, any maximal design can not be blocked into two designs with resolution at least III. As we mention in former section, each alias set contains only one word that consists of basic factors. Without loss of generality, we will use this word as the candidate of the block effect in the next section.

3. THE STRONG COMBINED-OPTIMAL DESIGN

The combined design consists of the initial design and its foldover design. For a 2^{k-p} design, there are 2^p ways to construct its foldover plan. The optimal foldover design is the one such that the combined design has the least aberration among all combined designs. In this situation, the combined design is called an optimal combined design. A 2^{k-p} design is called a combined-optimal design if the resulting optimal combined design has the minimum aberration among all combined optimal foldover designs. Furthermore, a 2^{k-p} design is called a strong combined-optimal design if the combined design is the minimum aberration $2^{k-(p-1)}$ design when we combine its optimal foldover. The readers may refer to Li and Lin's paper in detail.

Using the method described in section 2, if we start a MA design and apply the idea of blocking, then we may obtain the strong combined-optimal design. In Appendix, the complete collections of the strong combined-optimal designs with 16 runs and 32 runs are given for practical use. Some of them were not presented in the literature. For instance, the 2^{9-4} design in Li and Lin (2003) with wordlength pattern W=(3,3,4,4,1,0), defined by 6 = 12, 7 = 13, 8 = 14 and 9 = 2345, is marked by 9 − 4.9. The corresponding optimal folding factors are 6, 7, 8 and 9. This core foldover plan is denoted by f_{6789}. Another 2^{9-4} design with the same W=(3,3,4,4,1,0) is marked by 9 − 4.10 and it is defined by 6 = 12, 7 = 13, 8 = 24 and 9 = 345. Li and Lin (2003) claimed that both designs are not strong combined-optimal designs. But, if we select a MA 2^{9-3} design with generators 1237, 12458 and 13469, which is a design with the estimation index four, we may find the block effect 235, for example, such that it has the same wordlength pattern as designs 9 − 4.9 and 9 − 4.10. That is, the 2^{9-4} design with generators 1237, 12458, 13469 and 235 has W=(3,3,4,4,1,0). This is a strong combined-optimal design with its corresponding optimal foldover plan f_{58}. Applying the same method, other block effects 1345, 236 and 1246 can also form the 2^{9-4} design with the same W=(3,3,4,4,1,0). They all are strong combined-optimal designs. Another strong combined-optimal 2^{9-4} design defined by generators 2346, 1237, 12458 and 129(or 13469) has W=(2,4,6,2,0,1), which is neither presented in Li and Lin (2003) nor in Chen *et al.* (1993).

For the 32 and 64 runs MA designs, we tabulate their corresponding block effects and folding factors in Appendix. To save space, if they have the same wordlength pattern, just one of them will be tabulated in the table. For the same reason of saving space, at most six non-zero components of the wordlength patterns are given in the tables. Moreover, if a design is not presented in Li and Lin or Chen *et al.*, an asterisk is placed on the first column for indicating the new design.

Usage of tables is illustrated in the following. In Tables A.1 and A.2, the first nine factors are denoted by 1,..., 9, the $(10+i)^{th}$ factor is denoted by t_i, the $(20+i)^{th}$ factor by u_i and the $(30+i)^{th}$ factor by v_i, $i = 0, 1, \ldots, 9$. The first column lists the 2^{k-p} MA designs and the second column lists their generators. For the 32-run designs, the table only tabulates block effects and folding factors with $k = 6, \ldots, 15$, because the MA 2^{16-11} design is the maximal design. Furthermore, the estimation index is less than or equal to two for k greater than or equal to 17. For the 64-run design, the table tabulates block effects and folding factors with $k = 7, \ldots, 31$, and it should be noted that $k = 32$

MA design is a maximal design. Also, if k is 33 or higher, then the estimation index is less than or equal to two for all MA designs. We use the following example to explain the usage of the tables.

Example 3.1 Consider the MA 2^{8-3} design with generators 1236, 1247 and 13458. The alias sets not containing main effects and two-factor interactions are

$$234=146=137=267=1258=3568=4578=12345678,$$
$$125=356=1234567=2348=1468=1378=2678,$$
$$2345=1456=2567=128=368=478=1234678.$$

As a result, we may choose one of 234, 125 and 2345 to be the block effect. If we choose words 2345 or 125 to be the block effect, then we observe the strong combined-optimal 2^{8-4} design with the same $W = (3,7,4,0,1,0)$. We only list block effect, 2345, and its corresponding optimal foldover factors, 5 and 8, in the table and put the word 125 behind the word 2345. Moreover, if the word 234 is used to be the block effect, it produces the design with different $W = (4,6,4,0,0,1)$. Indeed, this word and its corresponding optimal foldover factors are listed in the next row in the table.

Here, we present an algorithm to find the optimal foldover factors of the 2^{k-p} design between two blocks. Suppose the word B is chosen to be the block effect, then the only difference in generators between two blocks is that one block is defined by B and the other one is defined by $-B$. We let word B consist of the basic factors only:

Step 1. We may randomly select a letter, say j, in the word B as a folding factor.

Step 2. We check whether the letter j is in each word of generators or not.

Step 3. If the letter j appears in the generator, then the corresponding generated factor is a foldover factor.

Step 4. Finally, we find the folding over factors between the two blocks.

Example 3.2 Suppose that the word 2345 is selected to be the block effect in example 3.1, the generators for the two blocks are 1236, 1247, 13458 and 2345, and 1236, 1247, 13458 and −2345, respectively. We may randomly choose a letter, for example letter 5, from the word 2345. Next, we check whether the letter 5 appears in each generators of 2^{8-3} design or not. For the first word, say 1236, since the letter 5 does not appear in the word 1236, the generated factor 6 can not to be a folding factor. Similarly, for the second word, say 1247, the generated factor 7 can not become a folding factor because the letter 5 does not appear in the word 1247. Finally, the generated factor 8 is a folding factor as a result of the simultaneous existence of the factors 5 and 8. So the folding factors between these two blocks are factors 5 and 8. Suppose we choose the letter 2 as the initial consideration, with the same method, the folding factors between two blocks are 2, 6 and 7. If the letter 3 is chosen, then the folding factors between two blocks are 3, 6 and 8. If the letter 4 is chosen, then the folding factors between two blocks are 4, 7 and 8. In fact, the foldover plans f_{267}, f_{368}, f_{478} and f_{58} are in the same equivalent set. In Tables A.1 and A.2, the last column marked by optimal foldover factors denotes the folding factors which form the core foldover plan.

4. CONCLUSION

Suppose that the experimenter wants the combined-design to reach the optimal condition when the follow-up plan is necessary to conduct. He may start to choose an optimal design with estimation index three or higher and try to split this design into two designs. After that, he may randomly select one of two designs as the initial design. Although the block effect is set to consist of basic factors only in the 2^{k-p} designs, it is till not easy to find all the solutions of block effect. Generally, in the 2^{k-p} design with resolution III or higher, there are $2^{k-p} - 1 - (k-p) - \{(k-p)(k-p-1)/2\}$ ways to select block effect. As the number of $k-p$ becomes larger, it will be more complicated to search for block effect. In this article, we apply the technique of blocking and the concept of the estimation index to split the 2^{k-p} design into two $2^{k-(p+1)}$ designs and we tabulate all 16-run and 32-run strong combined-optimal designs for practical use. At the same time, we also list their corresponding optimal foldover factors by the idea of core foldover plan.

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APPENDIX

TABLE A.1 *The blocking scheme of 2^{k-p} MA design with 32 runs*

$k-p$	*Generators*	*block effect*	*WLP of* $2^{k-(p+1)}$	*optimal foldover factors*
6-1	123456	345 (123; 124; 134; 234; 125; 135; 235; 145; 245)	(2,0,0,1)	5, 6
7-2	1236, 12457	2345 (134; 234; 135; 235; 1345)	(2,3,2,0,0)	5, 7
8-3	1236, 1247, 12458	2345 (125)	(3,7,4,0,1,0)	5, 8
	same as above	234	(4,6,4,0,0,1)	4, 7, 8
9-4	1236, 1247, 1258, 13459	2345	(4,14,8,0,4)	5, 8, 9
10-5	same as above + $2345t_0$	none		
11-6	1236, 1247, 1348, 1259, $135t_0$, $145t_1$	12345 (234; 235; 245; 345)	(13,25,25,27,23)	5, 9, t_0, t_1
12-7	1236, 1247, 1348, 2349, $125t_0$, $135t_1$, $145t_2$	12345 (234; 235; 245; 345)	(17,38,44,52,54)	5, t_0, t_1, t_2
*13-8	1236, 1247, 1348, 2349, $125t_0$, $135t_1$, $235t_2$, $145t_3$	12345 (245; 345)	(22,55,72,96,116)	same as above + t_3
*14-9	same as above + $245t_4$	12345 (345)	(28,28,77,112)	same as above + t_4
*15-10	same as above + $345t_5$	12345	(35,105,168,280)	same as above + t_5

TABLE A.2 *The blocking scheme of 2^{k-p} MA design with 64 runs*

$k-p$	*Generators*	*block effect*	*WLP of* $2^{k-(p+1)}$	*optimal foldover factors*
7-1	1234567	3456 (♯123; 124;...)	(1,1,0,0,1)	6, 7
8-2	12347, 12568	23456 (♯♯135; 235;...)	(1,2,3,1,0)	6, 8
	same as above	123456 (1235; 1245)	(2,1,2,2,0)	6, 8
9-3	1237, 12458, 13469	23456 (1256; 1356; 456)	(1,5,6,2,1)	6, 9
	same as above	123456 (156; 256; 356; 12356; 1456; 2456; 3456)	(2,3,6,4,0)	6, 9
*	same as above	2346 (135; 2345; 126; 2346)	(2,4,6,2,0)	6, 9
*	same as above	234	(2,5,5,2,0)	4, 8, 9
*	same as above	1246 (235; 1345; 236)	(3,3,4,4,1)	6, 9
10-4	1237, 12458, 12469, $1356t_0$	23456 (134; 2356; 456)	(2,8,12,4,2)	6, 9, t_0
*	same as above	123456 (235; 1345; 236; 1346; 1456; 2456; 3456)	(3,6,11,8,1)	6, 9, t_0

(*continued*)

$k-p$	*Generators*	*block effect*	*WLP of* $2^{k-(p+1)}$	*optimal foldover factors*
*	same as above	1256 (234)	(4,6,8,8,4)	6, 9, t_0
11-5	1237, 1248, 13459, $1346t_0$, $1256t_1$	23456	(2,14,22,8,6)	6, t_0, t_1
*	same as above	123456 (1356; 2356; 1456; 2456; 3456)	(4,10,20,16,4)	6, t_0, t_1
*12-6	1237, 1248, 13459, $1346t_0$, $1256t_1$, $23456t_2$	234	(8,15,24,32,24)	4, 8, 9, t_0, t_2
*13-7	1237, 1248, 1359, $145t_0$, $236t_1$, $2456t_2$, $3456t_3$	12345	(8,26,45,48,48)	5, 9, t_0, t_2, t_3
*	same as above	235(134; 234; 125; 235)	(9,24,42,54,52)	same as above
14-8	1237, 1248, 1259, $2345t_0$, $136t_1$, $146t_2$, $156t_3$, $3456t_4$	123456 (1345)	(8,43,64,80,112)	6, t_1, t_2, t_3, t_4
*	same as above	256 (126; 236; 246)	(11,36,64,88,110)	same as above
*15-9	same above + $123456t_5$	256 (1345; 126; 236; 246)	(13,51,96,144)	same as above + t_5
*16-10	1237, 1248, 1349, $125t_0$, $135t_1$, $126t_2$, $136t_3$, $1456t_4$, $2456t_5$, $3456t_6$	123456 (234; 235; 236)	(16,70,135,231)	6, t_2, t_3, t_4, t_5, t_6
*17-11	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $126t_3$, $136t_4$, $1456t_5$, $2456t_6$, $3456t_7$	123456 (235; 236)	(19,95,186,354)	6, t_3, t_4, t_5, t_6, t_7
18-12	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $126t_4$, $136t_5$, $1456t_6$, $2456t_7$, $3456t_8$	123456 (236)	(22,126,252,532)	6, t_4, t_5, t_6, t_7, t_8
19-13	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $126t_4$, $136t_5$, $236t_6$, $1456t_7$, $2456t_8$, $3456t_9$	123456	(25,164,336,784)	same as above + t_9
20-14	same above + $123456u_0$	none		
*21-15	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $145t_4$, $126t_5$, $146t_6$, $246t_7$, $156t_8$, $356t_9$, $456u_0$, $23456u_1$	12346 (136; 236; 346)	(46,204,624,1680)	6, t_5, t_6, t_7, t_8, t_9, u_0, u_1
*	same as above	13456 (345; 12345; 256; 12356; 12456)	(47,204,616,1680)	same as above
*	same as above	245	(48,204,609,1680)	5, t_1, t_2, t_3, t_4, t_8, t_9, u_0, u_1

(*continued*)

$k-p$	*Generators*	*block effect*	*WLP of* $2^{k-(p+1)}$	*optimal foldover factors*
*22-16	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $145t_4$, $126t_5$, $136t_6$, $146t_7$, $246t_8$, $156t_9$, $356u_0$, $456u_1$, $23456u_2$	13456 (245; 345; 12345; 236; 346; 12346; 256; 12356; 12456)	(54,250,801,2304)	same as above + u_2
*23-17	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $145t_4$, $126t_5$, $146t_6$, $245t_7$, $156t_8$, $356t_9$, $456u_0$, $23456u_1$, $136u_2$, $346u_3$	12456 (236; 246; 12346; 256; 12356; 12456)	(61,304,1033)	6, t_5, t_6, t_8, t_9, u_0, u_1, u_2, u_3
*	same as above	13456 (345; 12345)	(63,304,1015)	same as above
*24-18	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $145t_4$, $126t_5$, $146t_6$, $245t_7$, $156t_8$, $356t_9$, $456u_0$, $23456u_1$, $136u_2$, $245u_3$, $236u_4$	12346 (345; 12345; 346)	(70,365,1302)	6, t_5, t_6, t_8, t_9, u_0, u_1, u_2, u_4
*	same as above	13456 (256; 12356; 12456)	(71,365,1292)	same as above
*25-19	same as above + $345u_5$	13456 (12345; 346; 12346; 256; 12356; 12456)	(80,435,1623)	6, t_5, t_6, t_8, t_9, u_0, u_1, u_2, u_4
26-20	1237, 1248, 1349, $234t_0$, $125t_1$, $135t_2$, $235t_3$, $145t_4$, $126t_5$, $146t_6$, $245t_7$, $156t_8$, $356t_9$, $456u_0$, $136u_1$, $245u_2$, $236u_3$, $345u_4$, $346u_5$, $256u_6$	23456 (12345; 12346; 12356; 12456; 13456)	(90,515,2013)	6, t_5, t_6, t_8, t_9, u_0, u_1, u_3, u_5, u_6
27-21	same as above + $12345u_7$	23456 (12346; 12356; 12456; 13456)	(101,605,2473)	same as above
28-22	same as above + $12346u_8$	23456 (12356; 12456; 13456)	(113,706,3012)	same as above + u_8
*29-23	same as above + $12356u_9$	23456 (13456)	(126,819,3640)	same as above + u_9
*30-24	same as above + $12356v_0$	same as above	(140,945,4368)	same as above + v_0
*31-25	same as above + $13456v_1$	same as above	(155,1085,5208)	same as above + v_1

NOTE: ♯ *There are 35 block effects that have the same wordlength pattern.*
♯♯ *There are 18 block effects that have the same wordlength pattern.*

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