

ENERGY DECAY ESTIMATES FOR A KIRCHHOFF MODEL WITH VISCOSITY

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ABSTRACT. In this paper we study the uniform decay estimates of the energy for the nonlinear wave equation of Kirchhoff type

$$y''(t) - M(|\nabla y(t)|^2)\Delta y(t) + \delta y'(t) = f(t)$$

with the damping constant $\delta > 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$.

1. Introduction

Let Ω be a bounded in $\mathbb{R}^n (n \geq 1)$, having a boundary $\Gamma := \partial\Omega$ of class C^2 such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. We denote by ν (respectively, $\partial/\partial\nu$) the unit normal of Γ pointing into the exterior of Ω (respectively, the normal derivative). Let x^0 be an arbitrary but fixed point in \mathbb{R}^n and set $\ell(x) = x - x^0, x \in \mathbb{R}^n$,

$$(1.1) \quad R = \sup\{|\ell(x)| : x \in \Omega\},$$

$$(1.2) \quad \begin{aligned} \Gamma_0 &= \{x \in \Gamma | \ell(x) \cdot \nu(x) \leq 0\} \quad \text{and} \\ \Gamma_1 &= \{x \in \Gamma | \ell(x) \cdot \nu(x) > 0\} (\neq \emptyset), \end{aligned}$$

where $u \cdot v$ will denote the usual inner product for any $u, v \in \mathbb{R}^n$.

Let $M(\cdot) \in C^1([0, \infty), \mathbb{R})$ be a function such that

$$(1.3) \quad M(t) \geq m_0 > 0 \quad \text{for all } t \geq 0,$$

where m_0 is a constant.

Now we will consider the following nonlinear damped wave equation of Kirchhoff type:

$$(1.4) \quad y''(t) - M(|\nabla y(t)|^2)\Delta y(t) + \delta y'(t) = f(t) \text{ in } \Omega \times (0, \infty)$$

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with the undamped mixed boundary conditions;

$$(1.5) \quad y(t) = 0 \quad \text{on } \Gamma_0 \times (0, \infty)$$

$$(1.6) \quad \frac{\partial y(t)}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \times (0, \infty)$$

$$(1.7) \quad y(0) = y_0, \quad y'(0) = y_1 \quad \text{in } \Omega,$$

where $\delta > 0$ is a constant.

The motivation which the problem (1.4)-(1.7) has attracted the attention of several researchers (see [1], [3], [5]-[10], [12]-[16] and references therein) is of its intimate connection with a mathematical model for the transverse deflection of an elastic string of length $L > 0$ whose ends are held a fixed distance apart are written in the form of the hyperbolic equation :

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \left(\alpha + \beta \int_0^L \left| \frac{\partial u(x, t)}{\partial x} \right|^2 dx \right) \frac{\partial^2 u(x, t)}{\partial x^2} = 0,$$

which was proposed by Kirchhoff [8], where $u(x, t)$ is the deflection of the point x of the string at the time t and $\alpha > 0$, β are constants. We introduce the energy

$$(1.8) \quad E(t) = \frac{1}{2} [|y'(t)|^2 + \overline{M}(|\nabla y(t)|^2)],$$

where $\overline{M}(s) = \int_0^s M(t)dt$.

Note that from assumption on $M(\cdot)$ the energy $E(t)$ satisfies

$$(1.9) \quad \frac{1}{2} [|y'(t)|^2 + m_0 |\nabla y(t)|^2] \leq E(t).$$

The main purpose of this paper is to give the uniform decay estimates for the problem (1.4)-(1.7) under certain appropriate hypotheses.

Problems with the undamping, that is, $\delta = 0$ or with the damping occurring in the boundary and $M(\cdot) \equiv 1$ were studied by many authors; Quinn and Russel [14], Chen [2], Lions [11], Lasiecka and Tataru [10], Komornik and Zuazua [9]. They had obtained a uniform decay result of the form :

$$E(t) \leq C e^{-\alpha t} E(0), \quad t \geq 0,$$

where $C \geq 1$ and $\alpha > 0$ being some constants. On the other hand, Gorain [6] has considered the stability of the solution of strong damped wave equation with $M(\cdot) \equiv 1$ and $f(t) \equiv 0$ in (1.4), $-\delta \Delta y'(t)$ instead of $\delta y'(t)$, and homogeneous boundary conditions. Later, when $f(t) \not\equiv 0$ in

(1.4), Cavalcanti et al. [1] treated stability problems with nonhomogeneous conditions, that is,

$$M(|\nabla y(t)|^2) \frac{\partial y(t)}{\partial \nu} + \frac{\partial}{\partial t} \left(\frac{\partial y(t)}{\partial \nu} \right) = g(t) (\neq 0)$$

instead of (1.6) and strong damping $-\Delta y'(t)$. They have obtained the exponential decay using the perturbed energy method under some appropriate assumption on f and g . Also, Jung and Lee [7] obtained uniform decay estimates for the strong damped wave equations with homogeneous boundary conditions (1.5)-(1.7). They used a direct method as the method of proof, which is based on some integral inequalities; see for example, Komornik and Zuazua [9]. Since we do not have any information about the influence of the inner products $(f(t), \cdot)_{L_2(\Omega)}$ and about the sign of the derivative of $E(t)$, it is very difficult to treat stability problems for nonhomogeneous cases.

Note that the problem considered here is a generalization of the abstract dynamical system with the internal damping term which is always present in actual systems (see [4]). Furthermore our study can be applied to the linear damped extensible beam equation.

2. Preliminaries and main result

We first introduce some notations which will be used throughout this paper. Let $L_2(\Omega)$ be the space of square integrable with inner product (\cdot, \cdot) and norm $|\cdot|$. For any non-negative integer m , $H^m(\Omega)$ denotes the usual Sobolev space of order m and $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.

From now on, we will denote

$$(y(t), z(t)) := \int_{\Omega} y(t)z(t)dx \text{ and } (y(t), z(t))_{\tilde{\Gamma}} := \int_{\tilde{\Gamma}} y(t)z(t)d\Gamma.$$

Before stating our main results, let us recall the following result, which says existence of the regular solution for the problem (1.4)-(1.7) using the Yosida approximation method and its proof can be found on Theorem 5.1 in [13] (see also [12]).

We put

$$D(\gamma) := \{(y_0, y_1, f) : |\Delta y_0| \leq \gamma, |\nabla y_1| \leq \gamma, \int_0^\infty |\nabla f| dt \leq \gamma\},$$

where $(y_0, y_1, f) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (L^1([0, \infty); H_0^1(\Omega)) \cap L^\infty([0, \infty); L_2(\Omega)))$.

Then we have :

THEOREM A. *Let $f(t) \in C([0, \infty); L_2(\Omega))$. Then there exists a $\gamma_0(> 0)$ satisfying the following: For any $(y_0, y_1, f) \in D(\gamma_0)$, there exists a unique solution $y(t)$ on $[0, \infty)$ to the problem (1.4)-(1.7) and the solution $y(t)$ satisfies*

$$y(t) \in C([0, \infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L_2(\Omega)).$$

Furthermore, there exists a positive constant K such that

$$(2.1) \quad |\Delta y(t)| \leq K \text{ and } |\nabla y'(t)| \leq K \text{ for all } t \geq 0.$$

We now state the main result.

THEOREM B. *Let $y(t)$ be a regular solution to the problem (1.4)-(1.7). If we assume that*

$$(2.2) \quad \int_0^t e^{C(\epsilon)s} |f(s)|^2 ds = O(t^\alpha), \quad t \rightarrow \infty$$

hold for some positive constants $C(\epsilon)$ depending on $\epsilon > 0$ (see (3.12)) and $\alpha > 0$, then there exist a positive constant ϵ_0 such that, for all $\epsilon \in [0, \epsilon_0)$

$$(2.3) \quad E(t) = O\left(e^{-C(\epsilon)t}\right) \text{ as } t \rightarrow \infty.$$

In particular, if $f(t) \equiv 0$, then we have

COROLLARY. *Let $y(t)$ be a regular solution to the problem (1.4)-(1.7) with $f(t) \equiv 0$. Then the energy $E(t) \leq De^{-C(\epsilon)t}E(0)$ for all $t \geq 0$ and some constant $D > 0$, independent of δ .*

3. Proof of main theorem

In this section we shall prove the main theorem by using the multipliers technique and some integral inequalities.

Before proving the main result, we will give the following lemmas.

LEMMA 3.1. *Let $E(t)$ be the energy given by (1.8). Then the derivative $E'(t)$ of $E(t)$ satisfies the following equality;*

$$(3.1) \quad E'(t) = -\delta |y'(t)|^2 + (f(t), y'(t)).$$

Proof. From (1.8) and using Green's formula, we have

$$\begin{aligned} E'(t) &= (y''(t), y'(t)) + M(|\nabla y(t)|^2)(\nabla y'(t), \nabla y(t)) \\ &= M(|\nabla y(t)|^2)(y'(t), \frac{\partial y(t)}{\partial \nu})_{\Gamma} - \delta |y'(t)|^2 + (f(t), y'(t)). \end{aligned}$$

Considering boundary conditions (1.5) and (1.6), we obtain our result. \square

Let us define the positive constant λ_1 by

$$(3.2) \quad |y(t)|^2 \leq \lambda_1 |\nabla y(t)|^2 \quad (\lambda_1 > 1),$$

which arise due to Poincaré.

LEMMA 3.2. *We have the following inequalities;*

$$(3.3) \quad |\psi(t)| \leq \frac{\sqrt{\lambda_1}}{\sqrt{m_0}} E(t),$$

$$(3.4) \quad \psi'(t) \leq -L_0 E(t) - \delta(y'(t), y(t)) + (f(t), y(t)) + 2|y'(t)|^2,$$

where the function $\psi(t)$ is given by

$$(3.5) \quad \psi(t) = (y'(t), y(t))$$

and L_0 is some positive constant.

Proof. Simple calculations using the Schwarz inequality, (3.2), and $|\nabla y(t)|^2 \leq \overline{M}(|\nabla y(t)|^2)/m_0$, show that

$$|\psi(t)| \leq \frac{\sqrt{\lambda_1}}{\sqrt{m_0}} |y'(t)| [\overline{M}(|\nabla y(t)|^2)]^{1/2}$$

and hence by the Young inequality we obtain (3.3).

In order to prove (3.4), we first note that by (1.4) the derivative of $\psi(t)$ is given by

$$\begin{aligned} (3.6) \quad \psi'(t) &= (y''(t), y(t)) + |y'(t)|^2 \\ &= M(|\nabla y(t)|^2)(\Delta y(t), y(t)) - \delta(y'(t), y(t)) \\ &\quad + (f(t), y(t)) + |y'(t)|^2. \end{aligned}$$

Using Green's formula and the inequality,

$$(m_0/m_1)\overline{M}(|\nabla y(t)|^2) \leq M(|\nabla y(t)|^2)|\nabla y(t)|^2,$$

we obtain by (3.6),

$$\begin{aligned}
 \psi'(t) &= -M(|\nabla y(t)|^2)|\nabla y(t)|^2 - \delta(y'(t), y(t)) + (f(t), y(t)) + |y'(t)|^2 \\
 &\leq -\left(\frac{m_0}{m_1}\overline{M}(|\nabla y(t)|^2) + |y'(t)|^2\right) - \delta(y'(t), y(t)) \\
 &\quad + (f(t), y(t)) + 2|y'(t)|^2 \\
 &\leq -\frac{2m_0}{m_1}E(t) - \delta(y'(t), y(t)) + (f(t), y(t)) + 2|y'(t)|^2 \\
 &= -L_0E(t) - \delta(y'(t), y(t)) + (f(t), y(t)) + 2|y'(t)|^2,
 \end{aligned}$$

where $L_0 = 2m_0/m_1(> 0)$. The proof is completed. \square

We are now in a position to prove the main result.

Proof of Theorem B. Let ϵ be any nonnegative number.

We define the perturbed function $H_\epsilon(t)$ by

$$(3.7) \quad H_\epsilon(t) = \lambda E(t) + \epsilon \psi(t),$$

where λ is fixed and satisfies

$$(3.8) \quad \lambda > \frac{\sqrt{\lambda_1}}{\sqrt{m_0}}\epsilon.$$

From (3.3), (3.7) and (3.8), for some λ (we may choose λ as $\lambda = 2\epsilon\sqrt{\lambda_1}/\sqrt{m_0}$), we have

$$(3.9) \quad \frac{\lambda}{2}E(t) \leq H_\epsilon(t) \leq 2\lambda E(t)$$

for all $t \geq 0$.

Differentiating (3.7) with respect to t , applying Lemmas 3.1 and 3.2 and using the Young inequality, for any $\mu > 0$, we obtain

$$\begin{aligned}
 H'_\epsilon(t) &= \lambda E'(t) + \epsilon \psi'(t) \\
 &\leq -\lambda \delta |y'(t)|^2 + \frac{\lambda}{2\epsilon} |f(t)|^2 + \frac{\lambda \epsilon}{2} |y'(t)|^2 \\
 &\quad - \epsilon L_0 E(t) + \epsilon(\delta + 1)\lambda_1 \mu |\nabla y(t)|^2 + \frac{\delta \epsilon}{4\mu} |y'(t)|^2 \\
 &\quad + 2\epsilon |y'(t)|^2 + \frac{\epsilon}{4\mu} |f(t)|^2 \\
 &= -\lambda \delta |y'(t)|^2 + \epsilon \left(\frac{\lambda}{2} + \frac{\delta}{4\mu} + 2 \right) |y'(t)|^2 \\
 &\quad - \epsilon L_0 E(t) + \epsilon(\delta + 1)\lambda_1 \mu |\nabla y(t)|^2 + \left(\frac{\lambda}{2\epsilon} + \frac{\epsilon}{4\mu} \right) |f(t)|^2 \\
 &\equiv (I_0 + I_1) + (I_2 + I_3) + I_4.
 \end{aligned}$$

Choosing $\mu = m_0 L_0 / (2(\delta + 1)\lambda_1)$ and simple calculations using (1.9) yield

$$I_2 + I_3 \leq -\epsilon(L_0 - \epsilon(\delta + 1)\lambda_1 m_0^{-1}\mu)E(t) = -\frac{L_0}{2}\epsilon E(t)$$

and

$$I_0 + I_1 \leq \left(-\lambda\delta + \epsilon\left(\frac{\lambda}{2} + \frac{\delta(\delta + 1)\lambda_1}{2m_0 L_0} + 2\right)\right)|y'(t)|^2 \leq 0,$$

where $\epsilon \leq \epsilon_1 := \lambda\delta/N$ ($N = \lambda/2 + \delta(\delta + 1)\lambda_1/(2m_0 L_0) + 2$).

Define $\epsilon_0 > 0$ by

$$\epsilon_0 = \min \left\{ \epsilon_1, \frac{\lambda\sqrt{m_0}}{\sqrt{\lambda_1}} \right\}.$$

Then we obtain for $\epsilon \in [0, \epsilon_0)$,

$$(3.10) \quad H'_\epsilon(t) \leq -\frac{L_0}{2}\epsilon E(t) + C(\lambda, \delta, \epsilon)|f(t)|^2,$$

where $C(\lambda, \delta, \epsilon) = \lambda/2\epsilon + \epsilon(\delta + 1)\lambda_1/(2m_0 L_0)$.

From (3.9) and (3.10), we also have

$$(3.11) \quad H'_\epsilon(t) + \frac{L_0}{4\lambda}\epsilon H_\epsilon(t) \leq C(\lambda, \delta, \epsilon)|f(t)|^2.$$

Consequently, multiplying (3.11) by $e^{(L_0/4\lambda)\epsilon t}$ and integrating over $(0, t)$, we get

$$H_\epsilon(t) \leq \left(H_\epsilon(0) + C(\lambda, \delta, \epsilon) \int_0^t e^{\frac{L_0}{4\lambda}\epsilon s} |f(s)|^2 ds\right) e^{-\frac{L_0}{4\lambda}\epsilon t}.$$

Taking (3.9) into consideration, we can see that

$$(3.12) \quad \frac{\lambda}{2}E(t) \leq \left(2\lambda E(0) + C(\lambda, \delta, \epsilon) \int_0^t e^{\frac{L_0}{4\lambda}\epsilon s} |f(s)|^2 ds\right) e^{-\frac{L_0}{4\lambda}\epsilon t}.$$

Thus our proof is completed from the above inequality and assumption (2.2). \square

FURTHER REMARK. Adding the perturbed function $H_\epsilon(t)$ to $\rho(t) = (y'(t), \ell \cdot \nabla y(t))$ in (3.7), we may determine an upper bound of the value of δ consistent with stability(cf. [6]).

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