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# PERFECT CODES ON SOME ORDERED SETS

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ABSTRACT. Using the concept of codes on ordered sets introduced by Brualdi, Graves and Lawrence, we consider perfect codes on the ordinal sum of two ordered sets, the standard ordered sets and the disjoint sum of two chains.

In the classical coding theory all perfect codes are completely described in terms of parameters (cf. [2, 3]). They are mathematically interesting structures. Brualdi et al. [1] recently introduced the notion of a code on an ordered set. In this paper we consider perfect codes on the ordinal sum of two ordered sets, the standard ordered sets and the disjoint sum of two chains.

A *chain* is an ordered set in which every two elements are comparable and an *antichain* in which no two elements are comparable. Then we denote by $\mathbf{n}$ and $\underline{n}$ denote the chain and the antichain, respectively, on the set $\{1, 2, \ldots, n\}$. Let $P$ and $Q$ be two disjoint ordered sets. The *disjoint sum* $P + Q$ of $P$ and $Q$ is the ordered set on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in $P$ or $x, y \in Q$ and $x < y$ in $Q$, and the *ordinal sum* $P \oplus Q$ of $P$ and $Q$ is obtained from $P + Q$ by adding the new relations $x < y$ for all $x \in P$ and $y \in Q$.

Let $\mathbb{F}_q$ be a finite field with $q = p^d$ ($p$ a prime, $d$ a positive integer). For $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$, the support of $u$ and the Hamming weight of $u$ are respectively given by

$$Supp(u) = \{i : 1 \leq i \leq n, u_i \neq 0\},$$

$$w_H(u) = |Supp(u)|.$$

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Brualdi et al. [1] generalized the notion of Hamming weight to that of $P$-weight, where $P$ is an ordered set on the set of coordinate positions of vectors in $\mathbb{F}_q^n$. For such a $P$, the *P-weight* $w_P(u)$ of $u \in \mathbb{F}_q^n$ is defined to be

$$w_P(u) = |\langle Supp(u) \rangle|,$$

where $\langle Supp(u) \rangle$ denotes the smallest ideal containing $Supp(u)$. (Recall a subset $I$ of an ordered set is an *ideal* if $a \in I$ and $b < a \Rightarrow b \in I$.) Now one can show that *P-distance* $d_P(u,v) = w_P(u-v)$ is a metric on $\mathbb{F}_q^n$. If $P$ is an antichain, then $P$-weight and $P$-distance are, respectively, Hamming weight and Hamming distance of classical coding theory. If $\mathbb{F}_q^n$ is endowed with $P$-distance, then a subset $C$ of $\mathbb{F}_q^n$ is called a *code on P* over $\mathbb{F}_q$. Let $x$ be a vector in $\mathbb{F}_q^n$ and $r$ be a nonnegative integer. The *P-sphere* with *center* $x$ and *radius* $r$ is the set

$$S_P(x;r) = \{y \in \mathbb{F}_q^n : d_P(x,y) \leq r\}.$$

Then $C$ is called a *perfect code* on $P$ over $\mathbb{F}_q$ provided there exists an integer $r$ such that the $P$-spheres of radius $r$ with centers at the codewords of $C$ are pairwise disjoint and their union is $\mathbb{F}_q^n$.

LEMMA 1. *Let $C$ be a perfect code on an ordered set over $\mathbb{F}_q$. Then the number of codewords of $C$ is a power of $q$.*

*Proof.* Cf. Lemma 34 [2]. □

First we characterize perfect codes on the ordinal sum of two ordered sets. From now on, we use $x_1x_2\cdots x_n$ instead of the usual vector notation $(x_1, x_2, \ldots, x_n)$. So, for $x = x_1x_2\cdots x_m \in \mathbb{F}_q^m$ and $y = y_1y_2\cdots y_n \in \mathbb{F}_q^n$, we write $xy = x_1x_2\cdots x_my_1y_2\cdots y_n \in \mathbb{F}_q^{m+n}$. In the following theorem, for $x \in \mathbb{F}_q^m$ and $y \in \mathbb{F}_q^n$ , $xy$ means that $x$ has its coordinate positions on $P$ and $y$ has its coordinate positions on $P'$.

THEOREM 1. *Let $P$ and $P'$ be ordered sets with $|P| = m$ and $|P'| = n$, respectively. Then every perfect code $C$ on $P \oplus P'$ over $\mathbb{F}_q$ satisfies one of the following:*

(i) $|C| \geq q^n$ *and, for each $y \in \mathbb{F}_q^n, \{x \in \mathbb{F}_q^m : xy \in C\}$ is a perfect code on $P$ of the same size.*

(ii) $|C| < q^n$ *and $C = \{x_yy : y \in C_0\}$, where $C_0$ is a perfect code on $P'$ and $y \mapsto x_y$ is a map from $C_0$ into $\mathbb{F}_q^m$.*

*Proof.* Let $C$ be a perfect code on $Q = P \oplus P'$ over $\mathbb{F}_q$. Then $|C| = q^k$ for some $k$ by Lemma 1. Suppose that the $Q$-spheres of radius $r$ with centers at the codewords of $C$ are pairwise disjoint and their union is $\mathbb{F}_q^{m+n}$. Now we have two cases to consider.

(i) If $k \geq n$, then $|S_Q(x;r)| \leq q^m$ and so $r \leq m$. For any $y \in \mathbb{F}_q^n$, there exits $x \in \mathbb{F}_q^m$ such that $xy \in C$. For, if there is $y \in \mathbb{F}_q^n$ such that $xy \notin C$ for any $x \in \mathbb{F}_q^m$, then $d_Q(xy, c) > m \geq r$ for any $c \in C$, which is a contradiction. Now, for each $y \in \mathbb{F}_q^n, \{x \in \mathbb{F}_q^m : xy \in C\}$ is a perfect code on $P$ of the same size.

(ii) If $k < n$, then $|S_Q(x;r)| > q^m$ and so $r > m$. If $x \neq x'$ and $xy, x'y \in C$ for $x, x' \in \mathbb{F}_q^m$ $y \in \mathbb{F}_q^n$, then $d_Q(xy, x'y) \leq m$ and so $r \leq m$, which is a contradiction. □

COROLLARY 1. [1] *Let $P = \mathbf{n}$. Then $C$ is a perfect code on $P$ over $\mathbb{F}_q$ if and only if there exists an integer $k$ with $0 \geq k \geq n$ such that $|C| = q^k$ and the set of all vectors $x_{n-k+1} \cdots x_n$ such that $x_1 \cdots x_n \in C$ for some $x_1 \cdots x_{n-k} \in \mathbb{F}_q^{n-k}$ equals to $\mathbb{F}_q^k$.*

COROLLARY 2. [1] *Let $P_n = \underline{1} \oplus \underline{n}$. Then, for each positive integer $m$, the extended binary Hamming code with parameters $[n = 2^m, 2^m - m - 1, 4]$ is a perfect code on $P_{n-1}$. In addition, the extended binary Golay code with parameters $[24, 12, 8]$ is a perfect code on $P_{23}$, and the extended ternary Golay code with parameters $[12, 6, 6]$ is a perfect code on $P_{11}$.*

Let $\mathcal{P}(X)$ be the power set of a set $X$ with $|X| = n$. For a natural number $n \geq 3$, consider the ordered set $S_n = (\{A \in \mathcal{P}(X) : |A| \in \{1, n-1\}\}, \subseteq)$, which is usually called the *n-dimensional standard ordered set.* Now we have a similar result to Theorem 1 for these important ordered sets. Here, for $x, y \in \mathbb{F}_q^n$ , $xy$ means that $x$ has its coordinate positions on the minimal elements of $S_n$ and $y$ has its coordinate positions on the maximal elements of $S_n$.

THEOREM 2. *For an integer $n \geq 3$, there is no perfect code $C$ on $S_n$ with $|C| = q^n$. Otherwise, every perfect code $C$ on $S_n$ satisfies one of the following:*

*(i) $|C| > q^n$ and, for each $y \in \mathbb{F}_q^n, \{x \in \mathbb{F}_q^n : xy \in C\}$ is a perfect code on the set of all minimal elements of $S_n$ of the same size.*

*(ii) $|C| < q^n$ and $C = \{x_y y : y \in C_0\}$, where $C_0$ is a perfect code on the set of all maximal elements of $S_n$ and $y \mapsto x_y$ is a map from $C_0$ into $\mathbb{F}_q^n$.*

*Proof.* Since $|S_{S_n}(x;r)| < q^n$ for $r < n$ and $|S_{S_n}(x;n)| = q^n + n(q-1)q^{n-1}$, there is no $S_n$-sphere of size $q^n$, whence there is no perfect code $C$ on $S_n$ with $|C| = q^n$. Now the rest of proof is similar to that of Theorem 1. □

Brualdi et al. [1] showed that there is no nontrivial perfect code on a disjoint sum of two chains of the same size. This is still true for a disjoint sum of two chains of different sizes.

THEOREM 3. *For any positive integers $m$ and $n$, there is no nontrivial perfect code on $\mathbf{m} + \mathbf{n}$.*

*Proof.* Let $P = \mathbf{m} + \mathbf{n}$ with $m \geq n \geq 1$, where $\mathbf{m} = \{1 < 2 < \cdots < m\}$ and $\mathbf{n} = \{1' < 2' < \cdots < n'\}$. Suppose that $C$ a nontrivial perfect code on $P$ and that the $P$-spheres of radius $r$ with centers at the codewords of $C$ are pairwise disjoint and their union is $\mathbb{F}_q^{m+n}$. Then we have two cases to consider.

Case 1. $r \geq m$.

Let $x = x_1 \cdots x_m x_{1'} \cdots x_{n'}$ and $y = y_1 \cdots y_m y_{1'} \cdots y_{n'}$ be any two codewords of $C$. Then the vector $x_1 \cdots x_m y_{1'} \cdots y_{n'}$ is contained in $S_P(x;r) \cap S_P(y;r)$, contradicting the assumption that $C$ is perfect.

Case 2. $r < m$.

Set $l = \min\{r, n\}$. For $x = x_1 \cdots x_m x_{1'} \cdots x_{n'} \in C$, let $B_0 = \{y_1 \cdots y_m\, y_{1'} \cdots y_{n'} \in S_P(x;r) \,:\, y_{i'} = x_{i'}, 1 \leq i \leq n\}$ and $B_i = \{y_1 \cdots y_m y_{1'} \cdots y_{n'} \in S_P(x;r) \,:\, y_{i'} \neq x_{i'}, y_{j'} = x_{j'}, i < j \leq l\}$ for $1 \leq i \leq l$. Clearly, $|B_0| = q^r$ and, for $1 \leq i \leq l$, $|B_i| = (q-1)q^{r-1}$. Now,

$$\begin{aligned} |S_P(x;r)| = \sum_{i=0}^{l} |B_i| &= l(q-1)q^{r-1} + q^r < q^{r-1}(l+1)q \\ &\leq q^{r-1}q^{l+1} = q^{r+l}. \end{aligned}$$

Hence, $|C| > q^{m-r}$ when $n \leq r$, i.e., $l = n$. By the pigeon-hole principle, there exist two distinct codewords $x = x_1 \cdots x_m x_{1'} \cdots x_{n'}$ and $y = y_1 \cdots y_m y_{1'} \cdots y_{n'}$ such that $x_i = y_i$ for $m \geq i > r$. Similarly, $|C| > q^{m+n-2r}$ when $r < n$, i.e., $l = n$. Again, there exist two distinct codewords $x = x_1 \cdots x_m x_{1'} \cdots x_{n'}$ and $y = y_1 \cdots y_m y_{1'} \cdots y_{n'}$ such that $x_i = y_i$ for $m \geq i > r$ and $x_{i'} = y_{i'}$ for $n \geq i > r$. In both cases, the vector $x_1 \cdots x_m y_{1'} \cdots y_{n'}$ is contained in $S_P(x;r) \cap S_P(y;r)$, which also contradicts the assumption that $C$ is perfect. □

## References

[1] R. A. Brualdi, J. Graves and K. M. Lawrence, *Codes with a poset metric*, Discrete Math. **147** (1995), no. 1-2, 57–72.

[2] F. J. MacWilliams and N. J. Sloane, *The Theory of Error-Correcting Codes* I, II, North-Holland, Amsterdam, 1977.

[3] V. Pless, *Introduction to the Theory of Error-Correcting Codes*, Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1998.

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