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ON INJECTIVITY AND P-INJECTIVITY

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ABSTRACT. The following results are extended from P-injective rings to AP-injective rings: (1) R is left self-injective regular if and only if R is a right (resp. left) AP-injective ring such that for every finitely generated left R-module M, ${}_R(M/Z(M))$ is projective, where $Z(M)$ is the left singular submodule of ${}_RM$; (2) if R is a left nonsingular left AP-injective ring such that every maximal left ideal of R is either injective or a two-sided ideal of R, then R is either left self-injective regular or strongly regular. In addition, we answer a question of Roger Yue Chi Ming [13] in the positive. Let R be a ring whose every simple singular left R-module is YJ-injective. If R is a right MI-ring whose every essential right ideal is an essential left ideal, then R is a left and right self-injective regular, left and right V-ring of bounded index.

1. Introduction

Throughout this paper, a ring R denotes an associative ring with identity and all modules are unitary. A ring R is called (von Neumann) regular if for any $a \in R$, there exists $b \in R$ such that $a = aba$; R is called strongly regular if for any $a \in R$, there exists $b \in R$ such that $a = a^2b$. We write J and $Z(R)$ for the Jacobson radical of R and the left singular ideal of R respectively. $l(X), r(X)$ denote respectively the left and the right annihilator of X in R. If $X = \{a\}$, we will write it for $l(a), r(a)$. For a left R-module M, $Z(M) = \{z \in M \mid l(z)$ is an essential left ideal of $R\}$ is called the left singular submodule of M. M is called left nonsingular (resp. singular) if $Z(M) = 0$ (resp. $Z(M) = M$).

The concept of P-injective modules was introduced in 1974 to study von Neumann regular rings, V-rings, self-injective rings and their generalizations (see [6], [7]). This was generalized to YJ-injective modules and AP-injective modules. It is well-known that von Neumann regular

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rings are P-injective since all left (right) modules are P-injective (see [6]). We are thus motivated to study P-injective modules over rings which are not necessarily von Neumann regular.

Let R be a ring. A right R-module M is called P-injective [6] if every right R-homomorphism from any principal right ideal aR to M extends to one from R_R to M; R is called right P-injective if the right R-module R_R is P-injective. A right R-module M is called YJ-injective [11] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and every right R-homomorphism from a^nR to M extends to one from R_R to M; R is called right YJ-injective if the right R-module R_R is YJ-injective; Similarly, we may define left YJ-injective rings. A module M_R is said to be almost principally injective (or AP-injective for short) [3] if, for any $a \in R$, there exists an S-submodule X_a of M such that $l_M(r_R(a)) = Ma \oplus X_a$ as left S-modules, where $S = End(M)$. If R_R is an AP-injective module, then R is called a right AP-injective ring. Thus, R is right AP-injective if and only if, for any $a \in R$, there exists a left ideal L_a of R such that $l(r(a)) = Ra \oplus L_a$. It is well known that R is right P-injective if and only if $l(r(a)) = Ra$ for any $a \in R$ (see [2]), which implies that P-injective rings are AP-injective. But AP-injective rings need not be neither P-injective nor YJ-injective by [3, Example 1.5].

In this paper, we extend the following results from P-injective rings to AP-injective rings:

(1) R is left continuous regular if and only if R is a right (resp. left) AP-injective ring such that for every cyclic left R-module M, ${}_R(M/Z(M))$ is projective;

(2) R is left self-injective regular if and only if R is a right (resp. left) AP-injective ring such that for every finitely generated left R-module M, ${}_R(M/Z(M))$ is projective;

(3) If R is a left nonsingular left AP-injective ring such that every maximal left ideal of R is either injective or a two-sided ideal of R, then R is either left self-injective regular or strongly regular. We answer a question of Roger Yue Chi Ming [13] in the positive and prove the following facts:

(1) If R is a ring whose every simple singular left R-module is YJ-injective and every essential right ideal is an essential left ideal, then R is von Neumann regular;

(2) If R is right MI-ring whose every simple singular left R-module is YJ-injective and every essential right ideal is an essential left ideal,

then R is a left and right self-injective regular, left and right V-ring of bounded index.

2. Main results

It has been demonstrated in [13, Page 230] that if R is left nonsingular, then (1) $Z(M)$ is injective for every injective left R-module M and (2) for any complement left ideal C of R, $Z(R/C) = 0$. A ring R is said to be left continuous (Y. Utumi [4]) if (1) every left ideal of R isomorphic to a direct summand of ${}_RR$ is a direct summand of ${}_RR$ and (2) every complement left ideal of R is a direct summand of ${}_RR$. Thus, if R is left continuous, then $J = Z(R)$ and $R/Z(R)$ is von Neumann regular. By [5, Proposition 3.3], we have the following fact which will be needed.

LEMMA 2.1. *If R is left AP-injective, then any left ideal isomorphic to a direct summand of ${}_RR$ is a direct summand of ${}_RR$.*

PROPOSITION 2.2. *The following are equivalent for a ring R:*

(1) *R is left continuous regular;*

(2) *R is a right AP-injective ring such that for every cyclic left R-module M, ${}_R(M/Z(M))$ is projective;*

(3) *R is a left AP-injective ring such that for every cyclic left R-module M, ${}_R(M/Z(M))$ is projective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). By [13, Theorem 13] and all regular rings are both left and right AP-injective.

(2) $\Rightarrow$ (1). By assumption, $R/Z(R)$ is projective which implies $Z(R)$ is a direct summand of ${}_RR$, whence $Z(R) = 0$ (since $Z(R)$ contains no non-zero idempotent elements). Then for every complement left ideal K of R, $Z(R/K) = 0$. By assumption again, the left R-module R/K is projective which implies ${}_RK$ is a direct summand of ${}_RR$. Since R is right AP-injective, there exists a left ideal L of R such that $l(r(a)) = Ra \oplus L$. Note that $Z(R) = 0$, which implies that $l(r(a))$ is a complement left ideal of R and hence $l(r(a))$ is a direct summand of ${}_RR$. This shows that Ra is a direct summand of ${}_RR$. Therefore, R is left continuous regular.

(3) $\Rightarrow$ (1). By Lemma 2.1 and apply the proof in "(2) $\Rightarrow$ (1)". $\square$

The following result give a characterization of left self-injective regular rings and extends [13, Theorem 14].

THEOREM 2.3. *The following are equivalent for a ring R:*

(1) *R is left self-injective regular;*

(2) *R is a right AP-injective ring such that for every finitely generated left R-module M, ${}_R(M/Z(M))$ is projective;*

(3) *R is a left AP-injective ring such that for every finitely generated left R-module M, ${}_R(M/Z(M))$ is projective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). By [13, Theorem 14] and all regular rings are both left and right AP-injective.

(2) $\Rightarrow$ (1). By assumption and Proposition 2.2, R is left continuous regular. Denote ${}_RE$ as the injective hull of ${}_RR$. For any $u \in E$, $B = R + Ru$ is a finitely generated nonsingular left R-module. By our assumption, ${}_RB$ is projective. Since the left annihilator of any proper finitely generated right ideal of R is nonzero, by a well-known theorem of H. Bass, ${}_RR$ is a direct summand of ${}_RB$. But ${}_RR$ is essential in ${}_RB$ which implies $R = B$. This proves that $R = E$ is left self-injective regular.

(3) $\Rightarrow$ (1). As the proof in "(2) $\Rightarrow$ (1)" and by Proposition 2.2, we may complete our proof. □

Recall that a ring R is called reduced if it contains no non-zero nilpotent elements. It is well-known that a reduced left P-injective ring is strongly regular. By [10, Proposition 1(2)], if R is a reduced left YJ-injective ring, then R is strongly regular. Now, we have the same result for left AP-injective rings.

LEMMA 2.4. *If R is a reduced left AP-injective ring, then R is strongly regular.*

Proof. For any $0 \neq a \in R$, $l(a) = l(a^2)$ by assumption. Thus there exists a right ideal L of R such that

$$a \in r(l(a)) = r(l(a^2)) = a^2R \oplus L.$$

Then $a = a^2r + x$ for some $r \in R, x \in L$, which implies $a^2 - a^2ra = xa \in a^2R \cap L = 0$. Hence $a^2 = a^2ra$, so $a = a^2r$ since R is reduced. This proves that R is strongly regular. □

By [3, Corollary 2.3], the following result is immediate.

LEMMA 2.5. *If R is a left AP-injective ring, then $J = Z(R)$.*

A ring R is called a left (resp. right) MI-ring [12] if R contains an injective left (resp. right) ideal. In [12], R. Yue Chi Ming gave an example, in which R is an MI-ring and not left self-injective. The following result extends [13, Proposition 11].

PROPOSITION 2.6. *Let R be a left nonsingular left AP-injective ring such that every maximal left ideal is either injective or an ideal of R. Then R is either left self-injective or strongly regular.*

Proof. Since R is a left AP-injective ring, by Lemma 2.5, $J = Z(R) = 0$. First suppose that every maximal left ideal of R is an ideal of R. Since $J = 0$, by [13, Lemma 2], R is reduced. Thus R is strongly regular by Lemma 2.4. Now suppose there exists a maximal left ideal M of R which is not an ideal. By assumption, M is left injective and R is a left MI-ring. Note that R is semiprime, thus eR is a minimal right ideal of R if and only if Re is a minimal left ideal of R. By [13, Lemma 10], R is left self-injective. Thus R is regular since $J = 0$. □

In general, if R is a ring whose every simple singular left R-module is P-injective, then $J \cap Z(R) = 0$ (cf. [8, Proposition 3]). In [13], R. Yue Chi Ming raised a question: is it true that $J \cap Z(R) = 0$ if every simple singular left R-module is YJ-injective ? Now, we answer it in the positive. Let's start with the following lemma.

LEMMA 2.7. *If R is a ring whose every simple singular left R-module is YJ-injective, then $J \cap Z(R)$ contains no nonzero nilpotent elements.*

Proof. Take any $b \in J \cap Z(R)$ with $b^2 = 0$. If $b \neq 0$, then $l(b) + RbR$ is an essential left ideal of R. We will prove that $l(b) + RbR = R$. If not, there exists a maximal essential left ideal M of R containing $l(b) + RbR$. By assumption, the simple singular left R-module R/M is YJ-injective, thus there exists a positive n such that $b^n \neq 0$ and every left R-homomorphism from Rb^n to R/M extends to one from ${}_RR$ to R/M. Therefore $n = 1$ and every left R-homomorphism from Rb to R/M extends to one from ${}_RR$ to R/M. Since $l(b) \subseteq M$, the left R-homomorphism $f : Rb \to R/M$ by $f(rb) = r + M$ is well defined. Since R/M is left YJ-injective, there exists $c \in R$ such that $1 + M = bc + M$. Note that $bc \in RbR \subseteq M$, which implies $1 \in M$, it is a contradiction. This proves that $l(b)+RbR = R$ and hence $b = db$ for some $d \in RbR \subseteq J$. This implies $b = 0$, which is required contradiction. □

LEMMA 2.8. *If R is a ring whose every simple singular left R-module is YJ-injective, then for any $b \in J \cap Z(R)$, $l(b^n) = l(b)$ for any positive integer $n > 1$.*

Proof. Take any $x \in l(b^n)$. Then $xb^n = 0$, and thus $(b^{n-1}xb^{n-1})^2 = 0$. Note that $b^{n-1}xb^{n-1} \in J \cap Z(R)$, so $b^{n-1}xb^{n-1} = 0$ by Lemma 2.7. This implies $(xb^{n-1})^2 = 0$, by Lemma 2.7, and thus $xb^{n-1} = 0$ since

$xb^{n-1} \in J \cap Z(R)$. This proves that $x \in l(b^{n-1})$ and $l(b^n) = l(b^{n-1})$. By induction, $l(b^n) = l(b)$ for any positive integer $n > 1$. □

THEOREM 2.9. *If R is a ring whose every simple singular left R-module is YJ-injective, then $J \cap Z(R) = 0$.*

Proof. Take any $b \in J \cap Z(R)$. If $b \neq 0$, then $l(b) \neq R$ and $l(b) + RbR$ is an essential left ideal of R. We will prove $l(b) + RbR = R$. If not, as the proof in Lemma 2.7, there exist a maximal essential left ideal M of R containing $l(b) + RbR$ and a positive integer n such that $b^n \neq 0$ and every left R-homomorphism from Rb^n to R/M extends to one from ${}_RR$ to R/M. Since $l(b) \subseteq M$, so $l(b^n) \subseteq M$ by Lemma 2.8. Thus the left R-homomorphism $f : Rb^n \to R/M$ by $f(rb^n) = r + M$ is well defined. Since the simple singular left R-module R/M is YJ-injective, there exists $c \in R$ such that $1 + M = b^n c + M$. Note that $b^n c \in RbR \subseteq M$, which implies $1 \in M$, it is a contradiction. This proves that $l(b) + RbR = R$ and hence $b = db$ for some $d \in RbR \subseteq J$. This implies $b = 0$, which is required contradiction. □

If R is a ring whose every simple left R-module is YJ-injective, then $J = 0$. But the result need not be true for a ring whose every simple singular left R-module is P-injective.

The following example shows that there exists a ring whose every simple singular left R-module is P-injective with $J \neq 0$, and thus the ring R is a ring whose every simple singular left R-module is YJ-injective with $J \neq 0$.

EXAMPLE 2.10. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $J(R) \neq 0$. Note that $M = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is the unique maximal essential left ideal. Since M is an ideal and the right R-module R/M is flat, so the simple singular left R-module R/M is P-injective. Thus R is a ring whose every simple singular left R-module is P-injective with $J \neq 0$.

PROPOSITION 2.11. *If R is a ring whose every simple singular left R-module is YJ-injective, then J contains no nonzero nilpotent elements if and only if $J = 0$.*

Proof. Assume J contains no nonzero nilpotent elements. Write $L = Rb + l(b)$ for any $b \in J$. If $L = R$, then there exists $a \in R$, $c \in l(b)$ such that $1 = ab + c$. Hence $a = ab^2$ and $(b - bab)^2 = 0$, which implies $b = bab$ since $b - bab \in J$. Let $e = ab$, then $e^2 = e \in J$, which implies $e = 0$ and $b = 0$. If $L \neq R$, then there exists a left ideal K of R such

that $L \oplus K$ is an essential left ideal of R. We claim that $L \oplus K = R$. If not, there is a maximal essential left ideal M of R containing $L \oplus K$. By assumption, the simple singular left R-module R/M is YJ-injective. Since J contains no nonzero nilpotent elements and $b \in J$, a left R-homomorphism $f : Rb^n \to R/M$ by $f(rb^n) = r + M$ is well defined. Thus there exists $c \in R$ such that $1 - b^n c \in M$. Note that $b^n c \in J \subseteq M$, which implies that $1 \in M$, contradicting with M is maximal. This shows that $L \oplus K = R$. Then $Rb + l(b) = Re$ with $e^2 = e \in R$, so $b^2 = beb = bab^2$ for some $a \in R$. But $b \in J$, thus $b = 0$ by the preceding proof. This gives that $J = 0$. The converse is obvious. □

It is well-known that if R is semiprime, then every essential right ideal of R which is an ideal of R must be left essential. But the converse is obviously not true.

THEOREM 2.12. *If R is a ring whose every simple singular left R-module is YJ-injective and every essential right ideal of R is an essential left ideal, then R is von Neumann regular.*

Proof. We first prove that R is a semiprime ring. If not, then there exists $0 \neq b \in R$ such that $RbRb = 0$. Then $b \in J$. Let K be a complement right ideal of R such that $RbR \oplus K$ is an essential right ideal of R. Then $Kb \subseteq K \cap RbR = 0$ which implies $RbR \oplus K \subseteq l(b)$. By assumption, $RbR \oplus K$ is an essential left ideal of R, thus $b \in Z(R)$. This implies $b \in J \cap Z(R)$, so $b = 0$ by Theorem 2.9. It is a contradiction. This proves that R is a semiprime ring. By [8, Proposition 6], R is fully left idempotent. By [9, Proposition 9], R is von Neumann regular since R is a ring whose every essential right ideal is an ideal. □

LEMMA 2.13. *Let R be a ring whose every essential right ideal is an essential left ideal with $J \cap Z(R) = 0$. If $J = Z(R_R)$, where $Z(R_R)$ is the right singular ideal of R, then $J = 0$.*

Proof. Assume $J \neq 0$. Then there exists $0 \neq b \in J = Z(R_R)$, so $r(b)$ is an essential right ideal of R. By assumption, $r(b)$ is an essential left ideal. Thus $r(b) \cap Rb \neq 0$, and hence there exists $a \in R$ such that $0 \neq ab \in J$ and $bab = 0$. Note that $r(b)$ is an ideal, so $bRab = 0$. This implies $abRab = 0$. Let K be a complement right ideal of R such that $RabR \oplus K$ is an essential right ideal. As the proof in Theorem 2.12, we have $ab = 0$ since $J \cap Z(R) = 0$. It is a contradiction. Therefore $J = 0$. □

THEOREM 2.14. *Let R be a right MI-ring whose every essential right ideal is an essential left ideal. If R is a ring whose every simple singular*

left R-module is YJ-injective, then R is a left and right self-injective regular, left and right V-ring, and with bounded index.

Proof. By Theorem 2.9, $J \cap Z(R) = 0$. As the proof in Theorem 2.12, R is semiprime since R is a ring whose every essential right ideal is an essential left ideal. Thus eR is a minimal right ideal of R if and only if Re is a minimal left ideal of R. As the proof in [13, Lemma 10], R is right self-injective. This proves $J = Z(R_R)$ and R/J is regular. By Lemma 2.13, $J = 0$. Therefore R is right self-injective regular. Let M be a maximal ideal. Then R/M is simple, thus R/M is regular simple since R is regular. Moreover, R/M is a ring whose every maximal essential right ideal is an ideal and every maximal essential left ideal is an ideal. Write $B = R/M$. We will prove $Soc({}_BB) = B$, where $Soc({}_BB)$ denote the socle of ${}_RB$. If not, there exists an maximal left ideal L of B containing $Soc({}_BB)$ and L is an essential left ideal of B. Thus L is an ideal. But B is a simple ring, so $L = 0$ or $L = B$. It is a contradiction. This proves $Soc({}_BB) = B$ and hence R/M is Artinian semisimple. By [1, Page 79], R/M is a ring of bounded index. Thus R is left self-injective by [1, Theorem 6.20] and [1, Corollary 6.22]. □

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