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GLOBAL EXISTENCE OF SOLUTIONS TO THE PREY-PREDATOR SYSTEM WITH A SINGLE CROSS-DIFFUSION

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ABSTRACT. The prey-predator system with a single cross-diffusion pressure is known to possess a local solution with the maximal existence time $T \leq \infty$. By obtaining the bounds of W_2^1-norms of the local solution independent of T we establish the global existence of the solution. And the long-time behaviors of the global solution are analyzed when the diffusion rates d_1 and d_2 are sufficiently large.

1. Introduction

The following Lotka-Volterra Prey-Predator system models an ecological system in which a prey species and a predator species live correlated on the same habitat Ω :

$$(1.1)\qquad \begin{cases} u_t = u(a_1 - b_1 u - c_1 v) & \text{for } t \in (0,\infty), \\ v_t = v(a_2 + b_2 u - c_2 v) & \text{for } t \in (0,\infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, & \end{cases}$$

where $u(t)$ represents the population density of the prey species, and $v(t)$ represents the population density of the predator species at time t. The positive constant a_1 means that the prey is assumed to be sharing limited resource so that its population can increase a bit in the absence of predator. If $a_2 > 0$ the predator is assumed to have another source of food supply than the prey, sufficient to increase the predator population somewhat in the absence of prey. If $a_2 \leq 0$ the predator population will be decreasing in the absence of prey. The positive constants b_1 and c_2 account for the competitions within the prey species and predator

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species, respectively. $c_1 > 0$ represents the death rate of the prey due to the encounter with predator. And, $b_2 > 0$ is the growth rate of the predator due to their prey consumption. The properties of the solutions to (1.1) will be briefly discussed at the beginning of Section 4 of the present paper.

The semilinear Lotka-Volterra Prey-Predator system includes the diffusion factor for the prey and predator species :

$$(1.2) \qquad \begin{cases} u_t = d_1 \Delta u + u(a_1 - b_1 u - c_1 v) & \text{for } t \in (0,\infty), \\ v_t = d_2 \Delta v + v(a_2 + b_2 u - c_2 v) & \text{for } t \in (0,\infty), \\ u(0) = u_0 \geq 0, \quad v(0) = v_0 \geq 0, \end{cases}$$

where d_1 and d_2 are positive constants which represent the diffusion rates of the prey and predator species, respectively. The diffusion rates d_1 and d_2 can be interpreted to reflect the probabilistic random walk of the individuals of each species on the habitat Ω. Okubo's book [12] and review article [13] have more details on derivations of diffusions. For the qualitative properties of the solutions of the system (1.2) we refer the reader to Conway and Smoller [5], [6].

Further considerations on the directed movement of the individuals in each species resulted to introduce the cross-diffusion and self-diffusion terms in the prey-predator system as in the following :

$$(1.3) \qquad \begin{cases} u_t = \Delta[(d_1 + \alpha_{11} u + \alpha_{12} v)u] + u(a_1 - b_1 u - c_1 v) & \\ v_t = \Delta[(d_2 + \alpha_{21} u + \alpha_{22} v)v] + v(a_2 + b_2 u - c_2 v) & \\ & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. The coefficients α_{ij}'s are nonnegative constants for $i, j = 1, 2$. And d_i, b_i, c_i $(i = 1, 2)$, and a_1 are positive constants. Only a_2 may be nonpositive. System (1.3) is generally referred as the cross-diffusion systems with prey-predator. Throughout this paper we assume that the initial functions $u_0(x)$, $v_0(x)$ are not identically zero. By applying the strong maximum principle and the Hopf boundary lemma for parabolic equations (see [7], [15]) to the system (1.6), we have that

$$(1.4) \quad u(x,t) > 0, \qquad v(x,t) > 0 \qquad \text{for every } x \in [0,1] \text{ and } \quad t > 0.$$

The coefficients d_1 and d_2 are the diffusion rates of the two species, respectively. The positive cross-diffusion rates α_{12} and α_{21} mean that the prey tends to avoid higher density of the predator species and vice versa by diffusing away. The tendency to move in the direction of lower density of own species is represented by the self-diffusion rates α_{11} and α_{22} for the prey and predator, respectively. For details in the biological background, we refer the reader to the monograph of Okubo and Levin [14].

For the cross-diffusion systems with prey-predator type reaction functions, there are a few results mainly on the steady-state problems with the elliptic systems, see [1], [8], [9], [11], [16]. In [18] the author studied parabolic properties of the prey-predator Cross-Diffusion system (1.3) in

$$\begin{array}{ll} \text{Case(A)} & d_1 = d_2 \text{ and } \alpha_{11} = \alpha_{22} = 0, \\ \text{Case(B)} & 0 < \alpha_{21} < 8\alpha_{11} \text{ and } 0 < \alpha_{12} < 8\alpha_{22}. \end{array}$$

The local existence of solutions to (1.3) was established by Amann [2], [3], [4] which deal with more general form of equations :

$$(1.5)\quad \begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u\, f(x,u,v) & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v\, g(x,u,v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where f and g are functions in $C^\infty(\overline{\Omega} \times R^2, R)$. According to his results the system (1.5) has a unique nonnegative solution $u(\cdot,t)$, $v(\cdot,t)$ in $C([0,T), W_p^1(\Omega)) \cap C^\infty((0,T), C^\infty(\Omega))$, where $T \in (0,\infty]$ is the maximal existence time for the solution u, v. The following result is also due to Amann [3].

THEOREM 1. *Let u_0 and v_0 be in $W_p^1(\Omega)$. The system (1.5) possesses a unique nonnegative maximal smooth solution $u(x,t), v(x,t) \in C([0,T), W_p^1(\Omega)) \cap C^\infty(\overline{\Omega} \times (0,T))$ for $0 \leq t < T$, where $p > n$ and $0 < T \leq \infty$. If the solution satisfies the estimates $\sup_{0<t<T} \|u(\cdot,t)\|_{W_p^1(\Omega)} < \infty$, $\sup_{0<t<T} \|v(\cdot,t)\|_{W_p^1(\Omega)} < \infty$, then $T = +\infty$. If, in addition, u_0 and v_0 are in $W_p^2(\Omega)$ then $u(x,t), v(x,t) \in C([0,\infty), W_p^2(\Omega))$, and $\sup_{0\leq t<\infty} \|u(\cdot,t)\|_{W_p^2(\Omega)} < \infty$, $\sup_{0\leq t<\infty} \|v(\cdot,t)\|_{W_p^2(\Omega)} < \infty$.*

The system (1.5) is a special case of the concrete example (7), (8) in Introduction of [3], and the results stated in Theorem 1 is from the Theorem in Introduction of [3]. The results in Theorem 1 mean that once we establish the uniform W_p^1-bound, (with $p > n$), independent of the maximal existence time T for the solutions, the global existence of the solutions will follow. And also the uniform L_∞-bound of the solutions will be obtained from the Sobolev embedding theorems.

In this paper we consider the case when $\alpha_{11}, \alpha_{12}, \alpha_{22} > 0$ and $\alpha_{21} = 0$ for the system (1.3) in the spatial domain $\Omega = [0,1] \subset \mathbb{R}^1$. The system (1.3) is rewritten in this case as follows :

$$
(1.6) \quad \begin{cases} u_t = [(d_1 + \alpha_{11}\, u + \alpha_{12}\, v)u]_{xx} + u(a_1 - b_1 u - c_1 v) \\ v_t = [(d_2 \qquad\qquad + \alpha_{22}\, v)v]_{xx} + v(a_2 + b_2 u - c_2 v) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } [0,1] \times (0,\infty), \\ u_x(x,t) = v_x(x,t) = 0 \qquad\qquad\qquad \text{at } x = 0, 1, \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 \qquad \text{in } [0,1], \end{cases}
$$

where d_i, a, a_1, b_i, c_i are all positive constants for $i, j = 1, 2$, and a_2 is a real constant. Throughout this paper we assume that the initial functions $u_0(x)$, $v_0(x)$ are not identically zero and contained in the function space $W_2^1([0,1])$. This system does fall into neither Case(A) nor Case(B) that have been studied before by the author in [18]. One way to interpret the biological meanings of the cross- and self-diffusion terms in system (1.6) is that the prey species tends to avoid higher density of the predator species and its own species. And the predator species tends to avoid higher density of its own species.

The parabolic maximum principle does not apply to the prey-predator model. Thus, even though system (1.6) does not have the cross-diffusion term for the predator species v, it is impossible to obtain the uniform bound of $v(x,t)$ for $(x,t) \in [0,1] \times \Omega$. Hence it is impossible to use the same technique for system (1.6) as in author's paper [17] on the cross-diffusion system with competition type reaction functions and single non-zero cross-diffusion term.

We first obtain the uniform bound of the local solution of the systems (1.6) independent of T, the maximal existence time by applying the calculus inequalities of Gagliardo-Nirenberg type. From this we establish the existence of the global solution to system (1.6). In each step of estimates of the solution we look for the contribution of the diffusion coefficients d_1, d_2 and conclude that the uniform bound of the solution is independent of d_1, d_2 if $d, d_1, d_2 \geq \delta$ for any positive constant δ. Using this result we obtain results on the long-time behaviors of the solution to (1.6) for large d_1, d_2.

Here we state the main theorem of this paper on the global existence of the solution to (1.6).

THEOREM 2. *Suppose for system* (1.6) *that the initial functions* u_0, v_0 *are in* $W_2^2([0,1])$, *and let* $(u(x,t), v(x,t))$ *be the maximal solution* *obtained as in* Theorem 1. *Then there exist positive constants* t_0, $M' = M'(d_1, d_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$, *and* $M = M(d_1, d_2, \alpha_{11}, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ *such that*

$$\max\{\|u(\cdot,t)\|_{1,2}, \|v(\cdot,t)\|_{1,2} : t \in (t_0, T)\} \le M',$$
$$\max\{u(x,t),\ v(x,t) : (x,t) \in [0,1] \times (t_0, T)\} \le M,$$

and $T = +\infty$. *That is,* (u, v) *is the global solution to* (1.6). *In the case* $d_1, d_2 \ge \delta$ *for any positive constant* δ, *the constant* M *is independent of the diffusion coefficients* d_1 *and* d_2, *that is,* $M = M(\delta, \alpha_{11}, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$.

Following Section 1. Introduction, we list in Section 2 the Calculus inequalities that are necessary in the proof of Theorem 2. Section 3 is devoted to the proof of Theorem 2. In Section 4 the long-time behaviors of the global solution to (1.6) are investigated.

2. Calculus inequalities

THEOREM 3. *Let* $\Omega \in \mathbb{R}^n$ *be a bounded domain with* $\partial\Omega$ *in* C^m. *For every function* u *in* $W^{m,r}(\Omega)$, $1 \le q, r \le \infty$, *the derivative* $D^j u$, $0 \le j < m$, *satisfies the inequality*

$$|D^j u|_p \le C(|D^m u|_r^a |u|_q^{1-a} + |u|_q), \tag{2.1}$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$, *for all* a *in the interval* $\frac{j}{m} \le a < 1$, *provided one of the following three conditions :*

(i) $r \le q$,

(ii) $0 < \frac{n(r-q)}{mrq} < 1$, *or*

(iii) $\frac{n(r-q)}{mrq} = 1$ *and* $m - \frac{n}{q}$ *is not a nonnegative integer.*

(*The positive constant* C *depends only on* n, m, j, q, r, a.)

Proof. We refer the reader to A. Friedman [7] or L. Nirenberg [10] for the proof of this well-known calculus inequality. □

COROLLARY 4. *There exist positive constant* C *such that for every function* u *in* $W_2^1([0,1])$

$$|u|_2 \le C(|u_x|_2^{\frac{1}{3}} |u|_1^{\frac{2}{3}} + |u|_1). \tag{2.2}$$

Proof. $m = 1, r = 2, q = 1$ satisfy the condition (ii) in Theorem 3. □

LEMMA 5. *For every function* u *in* $W_2^2([0,1])$ *with* $u_x(0) = u_x(1) = 0$

$$|u_x|_2 \le |u_{xx}|_2^{\frac{1}{2}} |u|_2^{\frac{1}{2}}. \tag{2.3}$$

Proof. Using the given boundary conditions and Hölder's inequality

$$\int_0^1 u_x^2\,dx = -\int_0^1 u u_{xx}\,dx \le |u_{xx}|_2 |u|_2,$$

and thus the inequality (2.3) holds. □

LEMMA 6. *For every u in $W_2^3([0,1])$ with $u_x(0) = u_x(1) = 0$*

$$|u_{xx}|_2 \le |u_{xxx}|_2^{\frac{2}{3}} |u|_2^{\frac{1}{3}}, \tag{2.4}$$

Proof. Using the given boundary conditions, Hölder's inequality and the inequality (2.3) in the previous Lemma

$$\int_0^1 u_{xx}^2\,dx = -\int_0^1 u_x u_{xxx}\,dx \le |u_x|_2 |u_{xxx}|_2 \le |u_{xx}|_2^{\frac{1}{2}} |u|_2^{\frac{1}{2}} |u_{xxx}|_2.$$

Thus we have $|u_{xx}|_2^{\frac{3}{2}} \le |u_{xxx}|_2 |u|_2^{\frac{1}{2}}$, and thus the assertion is proved. □

LEMMA 7. *If a function f is in the space $W_2^1([0,1])$, then there exists a constant $C > 0$ such that*

$$|f^2|_\infty \le C((1 + \tfrac{1}{\epsilon})|f|_2^2 + \epsilon |f_x|_2^2), \tag{2.5}$$

for every $0 < \epsilon < 1$.

Proof. Suppose first $f \in C^1[0,1]$. By Lemma 5.2 of [7] there exists a function F in $C_0^1(\mathbb{R}^1)$ such that $F = f$ in the interval $[0,1]$ and $\|F\|_{W_2^j(\mathbb{R}^1)} \le C\|f\|_{j,2}$, $j = 0, 1$. For the function F we have the inequalities

$$\begin{aligned} |F^2|_{L_\infty(\mathbb{R}^1)} \le \int_{\mathbb{R}^1} |(F^2)_x|\,dx &= 2\int_{\mathbb{R}^1} |F F_x|\,dx \\ &\le \int_{\mathbb{R}^1} (\epsilon |F_x|^2 + \tfrac{1}{\epsilon}|F|^2)\,dx \\ &= \epsilon |F_x|^2_{L_2(\mathbb{R}^1)} + \tfrac{1}{\epsilon}|F|^2_{L_2(\mathbb{R}^1)}. \end{aligned}$$

Thus now for f we have

$$|f^2|_\infty \le |F^2|_{L_\infty(\mathbb{R}^1)} \le \epsilon |F_x|^2_{L_2(\mathbb{R}^1)} + \tfrac{1}{\epsilon}|F|^2_{L_2(\mathbb{R}^1)} \le \epsilon C\|f\|_{1,2}^2 + \tfrac{C}{\epsilon}|f|_2^2 \tag{2.6}$$

for every $\epsilon > 0$.

Suppose now that $f \in W_2^1([0,1])$. There exists a sequence $\{f_i\}$ in $C^1[0,1]$ such that $\|f_i - f\|_{1,2} \to 0$, $\|f_i - f\|_{0,2} \to 0$, $|f_i - f|_\infty \to 0$ as $i \to \infty$. Hence by passing limits in the inequality (2.6) for f_i we obtain the inequality (2.6) for $f \in W_2^1([0,1])$ and thus the inequality (2.5) for every $0 < \epsilon < 1$. □

3. W_1^2 estimates for the system (1.6)

Proof of Theorem 2. **Step 1**. Taking integration of the first equation in the system (1.6) over the domain $[0,1]$ we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 u(t)\,dx &= \int_0^1 (a_1 u - b_1 u^2 - c_1 uv)\,dx\\ &\le a_1\int_0^1 u\,dx - b_1\int_0^1 u^2\,dx\\ &\le a_1\int_0^1 u\,dx - b_1\left(\int_0^1 u\,dx\right)^2\\ &= b_1\left(\frac{a_1}{b_1} - \int_0^1 u\,dx\right)\int_0^1 u\,dx.\end{aligned}$$

In the case that $\int_0^1 u_0\,dx < \frac{a_1}{b_1}$ we have that $\int_0^1 u(x,t)\,dx \le \frac{a_1}{b_1}$ for all $t > 0$. In the case that $\int_0^1 u_0\,dx \ge \frac{a_1}{b_1}$ there exist positive constants ξ and τ_0 such that $\int_0^1 u(x,t)\,dx < \xi + \frac{a_1}{b_1}$ for all $t \in (\tau_0, \infty)$. Hence we obtain the L_1-bound of u for all time.

The L_1-bound of v does not come directly like the case of u above. We first have to observe the L_2-estimates for v. Multiplying the second equation in (1.6) by v, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 v^2\,dx\\ &= \int_0^1 v(d_2 v + \alpha_{22} v^2)_{xx}\,dx + \int_0^1 v^2(a_2 + b_2 u - c_2 v)\,dx\\ &= -d_2\int_0^1 v_x^2\,dx - 2\alpha_{22}\int_0^1 v v_x^2\,dx\\ &\quad + a_2\int_0^1 v^2\,dx + b_2\int_0^1 uv^2\,dx - c_2\int_0^1 v^3\,dx\\ &\le -d_2\int_0^1 v_x^2\,dx + a_2\int_0^1 v^2\,dx + b_2 M_0 |v^2|_\infty - c_2\left(\int_0^1 v^2\,dx\right)^{\frac{3}{2}}\\ &\le -d_2\int_0^1 v_x^2\,dx + a_2\int_0^1 v^2\,dx - c_2\left(\int_0^1 v^2\,dx\right)^{\frac{3}{2}}\\ &\quad + b_2 M_0 C\left(\left(1+\tfrac{1}{\epsilon}\right)\int_0^1 v^2\,dx + \epsilon\int_0^1 v_x^2\,dx\right),\end{aligned} \tag{3.1}$$

where we applied the calculus inequality (2.5) to the function v :

$$\begin{aligned}|v^2|_\infty &\le C\left(\left(1+\tfrac{1}{\epsilon}\right)|v|_2^2 + \epsilon|v_x|_2^2\right)\\ &= C\left(\left(1+\tfrac{1}{\epsilon}\right)\int_0^1 v^2\,dx + \epsilon\int_0^1 v_x^2\,dx\right),\end{aligned} \tag{3.2}$$

for every $0 < \epsilon < 1$.

$$\int_0^1 v^2\,dx \le \left(\int_0^1 (v^2)^{\frac{3}{2}}\,dx\right)^{\frac{2}{3}} = \left(\int_0^1 v^3\,dx\right)^{\frac{2}{3}},$$

thus

$$-\int_0^1 v^3\,dx \le -\left(\int_0^1 v^2\,dx\right)^{\frac{3}{2}}. \tag{3.3}$$

Now we take a small value for $\epsilon > 0$ so that we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v^2\,dx &\le -\hat{d}_2\int_0^1 v_x^2\,dx - c_2\left(\int_0^1 v^2\,dx\right)^{\frac{3}{2}} + \hat{C}\int_0^1 v^2\,dx\\ &\le -c_2\left(\int_0^1 v^2\,dx\right)^{\frac{3}{2}} + \hat{C}\int_0^1 v^2\,dx,\end{aligned}$$

where $\hat{d}_2$ and $\hat{C}$ are positive constants depending on d_2, a_i, b_i, and c_i $(i = 1, 2)$. If $d_2 \ge \delta > 0$ then we can take $\hat{d}_2$ independent of d_2 by choosing small $\epsilon > 0$.

Hence we conclude that there exist positive constants $M_0 = M_0(d_2, a_i, b_i, c_i, i = 1, 2)$ and τ_0 such that

$$\int_0^1 u\,dx \le M_0, \quad \int_0^1 v^2(t)\,dx \le M_0 \quad \text{for all } t \in (\tau_0, \infty). \tag{3.4}$$

If $d_2 \ge \delta > 0$ then we can take M_0 independent of d_2, that is, $M_0 = M_0(\delta, a_i, b_i, c_i, i = 1, 2)$.

Step 2. Now we estimate the L_2-norm of u using the estimate (3.4) of v. By multiplying the first equation in (1.6) by u and integrating over $[0, 1]$ we have for $t > \tau_{1,1}$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx\\ =\;& \int_0^1 u(d_1u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx}\,dx\\ &+\int_0^1 u^2(a_1 - b_1u - c_1v)\,dx\\ =\;& -\int_0^1(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2\,dx - \int_0^1 \alpha_{12}uu_xv_x\,dx\\ &+\int_0^1 u^2(a_1 - b_1u - c_1v)\,dx\\ \le\;& -d_1\int_0^1 u_x^2\,dx - 2\alpha_{11}\int_0^1 uu_x^2\,dx\\ &-\alpha_{12}\int_0^1 uu_xv_x\,dx + a_1\int_0^1 u^2\,dx.\end{aligned} \tag{3.5}$$

The mixed term of u and v in the last line of (3.5) is estimated as follows :

$$\begin{aligned}\left|\int_0^1 uu_xv_x\,dx\right| &\le \frac{\epsilon}{2}\int_0^1 uu_x^2\,dx + \frac{1}{2\epsilon}\int_0^1 uv_x^2\,dx\\ &\le \frac{\epsilon}{2}\int_0^1 uu_x^2\,dx + \frac{1}{2\epsilon}|v_x|_\infty^2\int_0^1|u|\,dx\\ &\le \frac{\epsilon}{2}\int_0^1 uu_x^2\,dx + \frac{1}{2\epsilon}M_0|v_x|_\infty^2\\ &\le \frac{\epsilon}{2}\int_0^1 uu_x^2\,dx + C\int_0^1 v_x^2\,dx + \epsilon\int_0^1 v_{xx}^2\,dx,\end{aligned}$$

for any small $\epsilon > 0$ and for some positive constant C by the inequality (2.5). Thus we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx &\le -d_1\int_0^1 u_x^2\,dx - (2\alpha_{11} - \frac{\epsilon}{2}\alpha_{12})\int_0^1 uu_x^2\,dx\\ &\quad +C\int_0^1 v_x^2\,dx + \epsilon\alpha_{12}\int_0^1 v_{xx}^2\,dx\\ &\le -d_1\int_0^1 u_x^2\,dx + C\int_0^1 v_x^2\,dx + \epsilon\alpha_{12}\int_0^1 v_{xx}^2\,dx.\end{aligned} \tag{3.6}$$

In order to deal with the v-terms in the last line of (3.6), multiply the second equation in (1.6) by $-v_{xx}$ and integrate over $[0,1]$ so that we have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^1 v_x^2\,dx\\
= &-\textstyle\int_0^1 v_{xx}(d_2v+2\alpha_{22}v^2)_{xx}\,dx\\
&-\textstyle\int_0^1 v_{xx}v(a_2+b_2u-c_2v)\,dx\\
= &-\textstyle\int_0^1 v_{xx}(d_2v_{xx}+2\alpha_{22}v_x^2+2\alpha_{22}vv_{xx})\,dx\\
&-a_2\textstyle\int_0^1 vv_{xx}\,dx-b_2\int_0^1 uvv_{xx}\,dx+c_2\int_0^1 v^2v_{xx}\,dx\\
\le &-d_2\textstyle\int_0^1 v_{xx}^2\,dx-2\alpha_{22}\int_0^1 v_x^2v_{xx}\,dx\\
&+a_2\textstyle\int_0^1 v_x^2\,dx-b_2\int_0^1 uvv_{xx}\,dx+c_2\int_0^1 v^2v_{xx}\,dx.
\end{aligned}
\tag{3.7}
$$

For the terms in the last line of (3.7) we make the following observations.

$$\textstyle\int_0^1 v_x^2v_{xx}\,dx = 0, \tag{3.8}$$

$$\textstyle\int_0^1 v^2v_{xx}\,dx = -2\int_0^1 vv_x^2\,dx \le 0. \tag{3.9}$$

From the estimate (3.4) and (2.5)

$$
\begin{aligned}
&|\textstyle\int_0^1 uvv_{xx}\,dx|\\
= &\tfrac{1}{\epsilon_1}\textstyle\int_0^1 u^2v^2\,dx+\epsilon_1\int_0^1 v_{xx}^2\,dx\\
\le &\tfrac{1}{\epsilon_1}|u^2|_\infty\textstyle\int_0^1 v^2\,dx+\epsilon_1\int_0^1 v_{xx}^2\,dx\\
\le &\tfrac{M_0}{\epsilon_1}|u^2|_\infty+\epsilon_1\textstyle\int_0^1 v_{xx}^2\,dx\\
\le &\tfrac{CM_0}{\epsilon_1}\left((1+\tfrac{1}{\epsilon_2})|u|_2^2+\epsilon_2|u_x|_2^2\right)+\epsilon_1\textstyle\int_0^1 v_{xx}^2\,dx\\
\le &\tilde{C}\textstyle\int_0^1 u^2\,dx+\epsilon\int_0^1 u_x^2\,dx+\epsilon\int_0^1 v_{xx}^2\,dx
\end{aligned}
\tag{3.10}
$$

for any small $\epsilon>0$ and some $\tilde{C}>0$ by choosing small $\epsilon_1>0$ and $\epsilon_2=o(\epsilon_1)$. Substituting the estimates (3.8), (3.10), and (3.9) into (3.7) we obtain

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^1 v_x^2\,dx \le &-(d_2-\epsilon c_2)\textstyle\int_0^1 v_{xx}\,dx\\
&+(a_2+\tilde{C}b_2)\textstyle\int_0^1 v_x^2\,dx+\epsilon b_2\int_0^1 u_x^2\,dx.
\end{aligned}
\tag{3.11}
$$

Now adding (3.6) and (3.11) we have

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^1(u^2+v_x^2)\,dx \le &-(d_1-\epsilon b_2)\textstyle\int_0^1 u_x^2\,dx\\
&-(d_2-\epsilon\alpha_{12}-\epsilon b_2)\textstyle\int_0^1 v_{xx}^2\,dx\\
&+(a_1+\tilde{C}b_2)\textstyle\int_0^1 u^2\,dx+a_2\int_0^1 v_x^2\,dx.
\end{aligned}
\tag{3.12}
$$

Let us denote $\tilde{C}_2=\min\{d_1-\epsilon b_2,\quad d_2-\epsilon\alpha_{12}-\epsilon b_2\}$ for the simplicity of notation. Here we can choose small enough $\epsilon>0$ so that $\tilde{C}_2>0$.

Especially, if $d_1, d_2 \geq \delta > 0$, then $\tilde{C}_2 \geq \frac{\delta}{2}$ for small $\epsilon > 0$. Hence we finally have for $t > \tau_1' > 0$

$$
(3.13)\quad
\begin{array}{rl}
& \frac{1}{2}\frac{d}{dt}\int_0^1 (u^2+v_x^2)\,dx \\
\leq & -\tilde{C}_2(\int_0^1 u_x^2\,dx + \int_0^1 v_{xx}^2)\,dx + C_1\int_0^1 (u^2+v_x^2)\,dx \\
\leq & -C_2((\int_0^1 u^2\,dx)^3 + (\int_0^1 v_x^2\,dx)^2) + C_1\int_0^1 (u^2+v_x^2)\,dx + C_0 \\
\leq & -C_2(\int_0^1 (u^2+v_x^2)\,dx)^2 + C_1\int_0^1 (u^2+v_x^2)\,dx + C_0
\end{array}
$$

by using the inequalities (2.2), (2.3) and the uniform bounds of $|u|_1$ and $|v|_2$ from Step 1.

Therefore we conclude that there exists a positive constant M_1 depending only on d_i, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i=1,2)$ such that

$$
(3.14)\qquad \int_0^1 u^2(t)\,dx \leq M_1 \quad \text{and} \quad \int_0^1 v_x^2(t)\,dx \leq M_1
$$

for $t \in (\tau_1, \infty)$, where τ_1 is a positive constant. If $d_1, d_2 \geq \delta > 0$, then M_1 depends only on δ, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i=1,2)$.

From the results that $|v|_2 \leq M_0$ and $|v_{xx}|_2 \leq M_1$ and the calculus inequality (2.5) we obtain the uniform pointwise bound of v : there exists a positive constant $\tilde{M}_1$ depending only on m_0, d_i, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i=1,2)$ such that

$$
(3.15)\qquad |v|_\infty \leq \tilde{M}_1 \qquad \text{for all } t \in (\tau_1, \infty).
$$

If $d_1, d_2 \geq \delta > 0$, then $\tilde{M}_1$ depends only on δ, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i=1,2)$.

Step 3. Multiplying the first equation in (1.6) by $-u_{xx}$ and integrating over $[0,1]$, we obtain

$$
(3.16)\quad
\begin{array}{rl}
& \frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2\,dx \\
= & -\int_0^1 u_{xx}(d_1 u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx}\,dx \\
& -\int_0^1 u_{xx}u(a_1 - b_1 u - c_1 v)\,dx \\
\leq & -d_1\int_0^1 (u_{xx})^2\,dx \\
& -2\alpha_{12}\int_0^1 u_x v_x u_{xx}\,dx - \alpha_{12}\int_0^1 u u_{xx} v_{xx}\,dx \\
& +a_1\int_0^1 u_x^2\,dx + b_1\int_0^1 u^2 u_{xx}\,dx + c_1\int_0^1 uvu_{xx}\,dx,
\end{array}
$$

by noticing that $\int_0^1 u_x^2 u_{xx}\,dx = 0$. For the terms in the last line of (3.16) we make the following observations.

$$
(3.17)\quad
\begin{array}{rlll}
|\int_0^1 u_x v_x u_{xx}\,dx| & \leq & |u_x|_\infty |v_x|_2 |u_{xx}|_2 & \leq C|u_x|_\infty |u_{xx}|_2 \\
& & & \leq C|u_x|_\infty^2 + \frac{1}{2}\epsilon|u_{xx}|_2^2 \\
& & & \leq C|u_x|_2^2 + \epsilon|u_{xx}|_2^2,
\end{array}
$$

$$(3.18)\qquad \begin{aligned} |\int_0^1 uu_{xx}v_{xx}\,dx| &\le |v_{xx}|_\infty |u|_2 |u_{xx}|_2 \\ &\le C|v_{xx}|_\infty |u_{xx}|_2 \\ &\le C|v_{xx}|_\infty^2 + \epsilon |u_{xx}|_2^2 \\ &\le C|v_{xx}|_2^2 + \epsilon |v_{xxx}|_2^2 + \epsilon |u_{xx}|_2, \end{aligned}$$

$$(3.19)\qquad \begin{aligned} |\int_0^1 u^2 u_{xx}\,dx| &\le C\int_0^1 u^4\,dx + \epsilon \int_0^1 u_{xx}^2\,dx \\ &\le C + C\int_0^1 u_x^2\,dx + \epsilon\int_0^1 u_{xx}^2\,dx, \end{aligned}$$

and from (3.14) and (3.15) in Step 2 we have

$$(3.20)\qquad \begin{aligned} |\int_0^1 uvu_{xx}\,dx| &\le \tilde{M}_1 \int_0^1 |uu_{xx}|\,dx \\ &\le \hat{C}\int_0^1 u^2\,dx + \epsilon\int_0^1 u_{xx}^2\,dx \\ &\le \tilde{C} + \epsilon\int_0^1 u_{xx}^2\,dx \end{aligned}$$

for any small $\epsilon > 0$ and some $\hat{C} > 0$, $\tilde{C} > 0$. Substituting (3.17), (3.18), (3.19), and (3.20) into (3.16) we have

$$(3.21)\qquad \begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\int_0^1 u_x^2\,dx \\ \le\ & -(d_1 - 4\epsilon)\int_0^1 (u_{xx})^2\,dx + (a_1 + C)\int_0^1 u_x^2\,dx \\ & + C\int_0^1 v_{xx}^2\,dx + \epsilon\int_0^1 v_{xxx}^2\,dx + C. \end{aligned}$$

Now we take the second derivative of the second equation of (1.6) with respect to x, multiply by v_{xx}, and integrate over $[0,1]$ to obtain

$$(3.22)\qquad \begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\int_0^1 v_{xx}^2\,dx &= -d_2\int_0^1 v_{xxx}^2\,dx - 6\alpha_{22}\int_0^1 v_x v_{xx} v_{xxx}\,dx \\ &\quad + \int_0^1 G_{xx} v_{xx}\,dx, \end{aligned}$$

where $G = v(a_2 + b_2 u - c_2 v)$, by noticing that

$$\begin{aligned} \int_0^1 (v^2)_{xxxx} v_{xx}\,dx &= \int_0^1 (2vv_x)_{xxx} v_{xx}\,dx \\ &= 2\int_0^1 (v_x^2 + vv_{xx})_{xx} v_{xx}\,dx \\ &= -2\int_0^1 (3v_x v_{xx} + vv_{xxx}) v_{xxx}\,dx, \end{aligned}$$

since $v_x = v_{xxx} = 0$ at $x = 0, 1$ because of the system (1.6). In the following we estimate the terms on the right hand side of (3.22).

$$(3.23)\qquad \begin{aligned} |\int_0^1 v_x v_{xx} v_{xxx}\,dx| &\le |v_{xx}|_\infty |v_x|_2 |v_{xxx}|_2 \\ &\le C|v_{xx}|_\infty |v_{xxx}|_2 \\ &\le C|v_{xx}|_\infty^2 + \tfrac{\epsilon}{2}|v_{xxx}|_2^2 \\ &\le C|v_{xx}|_2^2 + \epsilon|v_{xxx}|_2^2. \end{aligned}$$

$$(3.24)\qquad \begin{aligned} \int_0^1 G_{xx} v_{xx}\,dx &= -\int_0^1 G_x v_{xxx}\,dx \\ &\le C\int_0^1 G_x^2\,dx + \epsilon\int_0^1 v_{xxx}^2\,dx \\ &\le C\int_0^1 (u_x^2 + v_x^2)\,dx + \epsilon\int_0^1 u_{xx}^2\,dx \\ &\quad + \epsilon\int_0^1 v_{xx}^2\,dx + \epsilon\int_0^1 v_{xxx}^2\,dx, \end{aligned}$$

since

$$\begin{aligned}\int_0^1 G_x^2\,dx &\le C\int_0^1(u_x^2+v_x^2)\,dx + C\int_0^1(u^2+v^2)(u_x^2+v_x^2)\,dx\\ &\le C\int_0^1(u_x^2+v_x^2)\,dx + C(|u_x|_\infty^2+|v_x|_\infty^2)\\ &\le C\int_0^1(u_x^2+v_x^2)\,dx + \epsilon(|u_{xx}|_2^2+|v_{xx}|_2^2).\end{aligned}$$

Substituting (3.23) and (3.24) into (3.22) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 v_{xx}^2\,dx &\le -(d_2-2\epsilon)\int_0^1 v_{xxx}^2\,dx + \epsilon\int_0^1 u_{xx}^2\,dx\\ &\quad +C\int_0^1 v_{xx}^2\,dx + C\int_0^1 u_x^2\,dx + C.\end{aligned}\tag{3.25}$$

Adding (3.21) and (3.25) we have for $t > \tau_2' > 0$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1(u_x^2+v_{xx}^2)\,dx\\ \le\ & -(d_1-5\epsilon)\int_0^1(u_{xx})^2\,dx - (d_2-3\epsilon)\int_0^1 v_{xxx}^2\,dx\\ & +C\int_0^1 u_x^2\,dx + C\int_0^1 v_{xx}^2\,dx + C\\ \le\ & -C_2\left(\int_0^1 u_x^2\,dx\right)^2 - C_2\left(\int_0^1 v_{xx}^2\,dx\right)^{\frac{3}{2}}\\ & +C\int_0^1(u_x^2+v_{xx}^2)\,dx + C\\ \le\ & -C_2\left(\int_0^1(u_x^2+v_{xx}^2)\,dx\right)^{\frac{3}{2}} + C\int_0^1(u_x^2+v_{xx}^2)\,dx + C,\end{aligned}\tag{3.26}$$

where $C_2 > 0$, by using the inequalities (2.3), (2.4) and the uniform boundedness of $|u|_2$ and $|v|_2$. Especially, if $d_1, d_2 \ge \delta > 0$, then we can choose $C_2 \ge \frac{\delta}{2}$ for small $\epsilon > 0$. Therefore we conclude that there exists a positive constant M_2 depending only on d_i, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i = 1, 2)$ such that $\int_0^1 u_x^2(t)\,dx \le M_2$, and $\int_0^1 v_{xx}^2(t)\,dx \le M_2$ for $t \in (\tau_2, \infty)$, where τ_2 is a positive constant. If $d_1, d_2 \ge \delta > 0$, then M_2 depends only on δ, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i = 1, 2)$.

From the results of Step 1, Step 2 and Step 3 we have a constant M independent of T and depending only on d_i, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i = 1, 2)$ such that

$$\max\{\|u(\cdot,t)\|_{1,2},\ \|v(\cdot,t)\|_{2,2} : t \in (t_0, T)\} \le M$$

for (u, v), the maximal solution to the system (1.6). If $d_1, d_2 \ge \delta > 0$, then M depends only on δ, α_{11}, α_{12}, α_{22}, a_i, b_i, c_i, $(i = 1, 2)$. We also conclude that $T = +\infty$ from Theorem 1. □

4. Convergence results

The ordinary differential equation system (1.1) is called the kinetic system of the cross-diffusion system (1.6). Before we discuss the asymptotic behaviors of the solution to (1.6) let us observe the kinetic system of (1.6).

The asymptotic behaviors of the solutions of system (1.1) are classified into the three cases when :

(i) $-\frac{a_1 b_2}{b_1 c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$, (ii) $\frac{a_1}{c_1} < \frac{a_2}{c_2}$, (iii) $\frac{a_2}{c_2} < -\frac{a_1 b_2}{b_1 c_2}$.

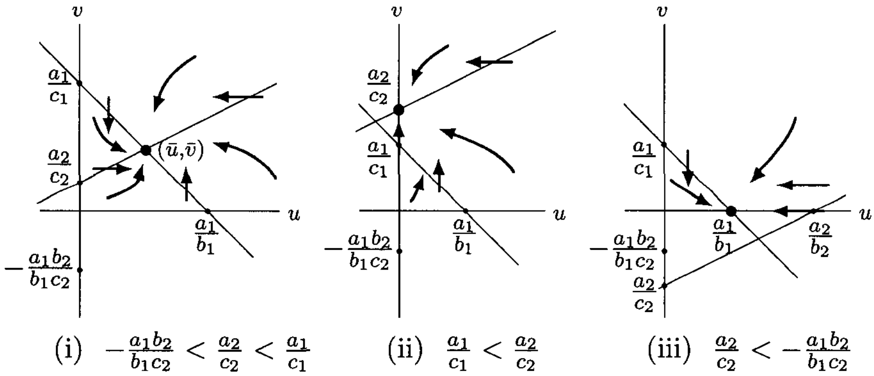

FIGURE 1. The unique nonnegative stable steady-state of the kinetic system (1.1) in each case of (i), (ii), (iii)

In each case above the kinetic system (1.1) has a unique nonnegative stable steady-state as illustrated in Figure 1. The positive steady-state $(\overline{u}, \overline{v})$ in the case (i) in Figure 1 is given by

$$(\overline{u}, \overline{v}) = \left(\frac{a_1 c_2 - a_2 c_1}{b_1 c_2 + b_2 c_1}, \frac{a_2 b_1 + a_1 b_2}{b_1 c_2 + b_2 c_1}\right).$$

The nonnegative steady-states $(\overline{u}, \overline{v})$, $(0, \frac{a_2}{c_2})$, and $(\frac{a_1}{b_1}, 0)$ of the kinetic system (1.1) are also nonnegative constant steady-states of the semilinear system (1.2) and the cross-diffusion system (1.6).

For the semilinear pre-predator system (1.2) the asymptotic behaviors of the solution have been studied by Conway and Smoller [5]. They showed under some condition the solution to (1.2) is uniformly bounded and converges to the positive constant steady-state.

In this section, we prove similar convergence results for the cross-diffusion system (1.6) in case (i). In case (ii) and (iii) we can prove convergence properties by using Liapunov functionals of different form from case (i). For this, refer the author's paper [18]. In case (i) for system (1.6) we obtain the following convergence results saying that under some condition a cross-diffusion prey-predator system has the same asymptotic property as its kinetic system :

THEOREM 8. *Suppose that* $-\frac{a_1b_2}{b_1c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$ *for the system* (1.6). *Let* $d_1, d_2 \geq \delta$ *for any positive constant* δ, *and* u_0, v_0 *be in* $W_2^2([0,1])$. *If* d_1, d_2 *satisfies that*

$$(b_2^2\alpha_{12}^2\overline{u}^2 + c_1^2\alpha_{21}^2\overline{v}^2)M^2 < 4b_2c_1\overline{u}\,\overline{v}d_1d_2, \tag{4.1}$$

where M *is the positive constant in* Theorem 2 *(independent of* d_1, d_2*), then the solution* $(u(x,t), v(x,t))$ *converges to* $(\overline{u}, \overline{v})$ *uniformly in* $[0,1]$ *as* $t \to \infty$, *and* $(\overline{u}, \overline{v})$ *is globally asymptotically stable.*

Proof. In this proof we consider the case (i) when $-\frac{a_1b_2}{b_1c_2} < \frac{a_2}{c_2} < \frac{a_1}{c_1}$. Using the functional $H(u,v)$ defined below we observe the convergence of global solutions of the cross-diffusion prey-predator system (1.6) :

$$H(u,v) = \int_0^1 \left\{b_2\left(u - \overline{u} - \overline{u}\log\tfrac{u}{\overline{u}}\right) + c_1\left(v - \overline{v} - \overline{v}\log\tfrac{v}{\overline{v}}\right)\right\}\, dx, \tag{4.2}$$

where $(\overline{u}, \overline{v}) = \left(\frac{a_1c_2 - a_2c_1}{b_1c_2 + b_2c_1}, \frac{a_2b_1 + a_1b_2}{b_1c_2 + b_2c_1}\right)$ is the positive stable steady-state of the kinetic system (k) in the case (i) as shown in Figure 1. $H(u,v)$ is always nonnegative and is zero only if $u \equiv \overline{u}$ and $v \equiv \overline{v}$. In order to prove the convergence of the solution first we observe the time derivative of $H(u(t), v(t))$ for the system (1.6) :

$$\begin{aligned}
&\frac{dH(u(t),v(t))}{dt} \\
= &\int_0^1 \{b_2(1 - \tfrac{\overline{u}}{u})u_t + c_1(1 - \tfrac{\overline{v}}{v})v_t\}\, dx \\
= &\int_0^1 \{b_2(1 - \tfrac{\overline{u}}{u})(d_1u + \alpha_{11}u^2 + \alpha_{12}uv)_{xx} \\
&\quad + c_1(1 - \tfrac{\overline{v}}{v})(d_2v + \alpha_{22}v^2)_{xx}\}\, dx \\
&\quad + \int_0^1 \{b_2(u - \overline{u})f + c_1(v - \overline{v})g\}\, dx \\
= &-\int_0^1 \{\tfrac{b_2\overline{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 + \tfrac{b_2\alpha_{12}\overline{u}}{u}u_xv_x \\
&\quad + \tfrac{c_1\overline{v}}{v^2}(d_2 + 2\alpha_{22}v)v_x^2\}\, dx \\
&-\int_0^1 \{b_1b_2(u - \overline{u})^2 + c_1c_2(v - \overline{v})^2\}\, dx,
\end{aligned} \tag{4.3}$$

where we denoted $f = a_1 - b_1u - c_1v$ and $g = a_2 + b_2u - c_2v$ and used the fact that

$$\begin{aligned}
&b_2(u - \overline{u})f + c_1(v - \overline{v})g \\
= \; &b_2(u - \overline{u})(a_1 - b_1u - c_1v) + c_1(v - \overline{v})(a_2 + b_2u - c_2v) \\
= \; &-b_2(u - \overline{u})(b_1(u - \overline{u}) + c_1(v - \overline{v})) \\
&+c_1(v - \overline{v})(b_2(u - \overline{u}) - c_2(v - \overline{v})) \\
= \; &-b_1b_2(u - \overline{u})^2 - c_1c_2(v - \overline{v})^2.
\end{aligned}$$

Now we remind the uniform boundedness result for the solution of the system (1.6) in the case $d \geq \delta > 0$ as in Theorem 2 that there exist positive constants t_0 and $M = M(\delta, \alpha_{11}, \alpha_{12}, \alpha_{21}, a_i, b_i, c_i, i = 1, 2)$ such that

$$(4.4) \qquad 0 \leq u(x,t),\ v(x;t) \leq M \qquad \text{for every } (x,t) \in [0,1] \times (t_0, \infty).$$

From the proof of Theorem 2 we can choose the constant M depending on the initial functions u_0, v_0 so that the inequalities in (4.4) hold for all $t \geq 0$. Using (4.4) and condition (4.1) in the hypothesis of the present theorem(Theorem 8) for every constant γ such that

$$0 < \gamma < \frac{b_2\overline{u}(4c_1\overline{v}d_1d_2 - b_2\alpha_{12}^2\overline{u}M^2)}{4[b_2\overline{u}(d_1 + 2\alpha_{11}M + \alpha_{12}M)M^2 + c_1\overline{v}(d_2 + 2\alpha_{22}M)M^2\}]}$$

we have the following inequality :

(4.5)

$$\begin{aligned} &\tfrac{b_2\overline{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v)u_x^2 + \tfrac{b_2\alpha_{12}\overline{u}}{u}u_xv_x + \tfrac{c_1\overline{v}}{v^2}(d_2 + 2\alpha_{22}v)v_x^2 \\ \geq\ & \gamma\{u_x^2 + v_x^2\}, \end{aligned}$$

since

$$\begin{aligned} &(\tfrac{b_2\alpha_{12}\overline{u}}{u})^2 - 4\{\tfrac{b_2\overline{u}}{u^2}(d_1 + 2\alpha_{11}u + \alpha_{12}v) - \gamma\}\{\tfrac{c_1\overline{v}}{v^2}(d_2 + 2\alpha_{22}v) - \gamma\} \\ \leq\ & \tfrac{b_2^2\alpha_{12}^2\overline{u}^2}{u^2} - \tfrac{4b_2c_1\overline{u}\,\overline{v}d_1d_2}{u^2v^2} \\ & +4\gamma\{\tfrac{b_2\overline{u}}{u^2}(d_1 + 2\alpha_{11}M + \alpha_{12}M) + \tfrac{c_1\overline{v}}{v^2}(d_2 + 2\alpha_{22}M)\} \\ \leq\ & \tfrac{1}{u^2v^2}[b_2^2\alpha_{12}^2\overline{u}^2M^2 - 4b_2c_1\overline{u}\,\overline{v}d_1d_2 \\ & +4\gamma\{b_2\overline{u}(d_1 + 2\alpha_{11}M + \alpha_{12}M)M^2 + c_1\overline{v}(d_2 + 2\alpha_{22}M)M^2\}] \\ <\ & 0. \end{aligned}$$

From (4.3) and (4.5) we have for $t \geq 0$

$$\begin{aligned} \tfrac{dH(u(t),v(t))}{dt} \leq\ & -\gamma\textstyle\int_0^1\{u_x^2 + v_x^2\}\,dx \\ & - \textstyle\int_0^1\{b_1b_2(u - \overline{u})^2 + c_1c_2(v - \overline{v})^2\}\,dx \\ \leq\ & 0. \end{aligned}$$

We notice that $\frac{dH(u(t),v(t))}{dt} = 0$ only if $u(x,t) \equiv \overline{u}$ and $v(x,t) \equiv \overline{v}$.

Thus it is shown that $H(u(t), v(t)) \searrow 0$ as $t \to \infty$. And we obtain the L_2 convergences, $|u(t) - \overline{u}|_2 \to 0$, $|v(t) - \overline{v}|_2 \to 0$ as $t \to \infty$ by using the uniform boundedness of $(u(x,t), v(x,t))$ in $[0,1]$. From Theorem 1 with the assumption that u_0, $v_0 \in W_2^2([0,1])$, we have that $\sup_{0\leq t<\infty} |u_{xx}(t)|_2 < \infty$, and $\sup_{0\leq t<\infty} |v_{xx}(t)|_2 < \infty$. Applying the calculus inequality (2.3) in Section 2 to the functions $u(x,t) - \overline{u}$ and $v(x,t) - \overline{v}$, we obtain the convergence $(u(x,t), v(x,t)) \to (\overline{u}, \overline{v})$ as $t \to \infty$ in $W_2^1([0,1])$. By using the Sobolev embedding theorem we show that $(u(x,t), v(x,t))$ converges to $(\overline{u}, \overline{v})$ uniformly in $[0,1]$ as $t \to \infty$. We also obtain that

$(\overline{u}, \overline{v})$ is locally asymptotically stable in $C([0,1])$ by using the fact that $H(u(t), v(t))$ is decreasing for $t \geq 0$. Thus we conclude that $(\overline{u}, \overline{v})$ is globally asymptotically stable. □

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