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STUDY ON THE TENSOR PRODUCT SPECTRUM

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Abstract. We will introduce tensor product spectrums on the tensor product spaces. And we will show that $\sigma[P(T_1, T_2, \ldots, T_n)] = P[(\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n)] = \sigma(T_1, T_2, \ldots, T_n)$.

1. Introduction

Let $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ denote tensor product space of the complex Banach space $X_i, 1 \leq i \leq n$ and $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ the completion of $X_1 \otimes X_2 \otimes \cdots \otimes X_n$.

Let $BL(X)$ denote the class of the bounded linear operators on the complex Banach space X and $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ be a tensor product operator on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$. Gelfand's theorem means that if η is an abelian Banach algebra with unit containing the elements $T_1, T_2, \ldots, T_n$ and $P(z_1, z_2, \ldots, z_n)$ is a polynomial in n variables, then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(P) = \sigma[P(T_1, T_2, \ldots, T_n)]$ if and only if there exists a homomorphism $h : \eta \to \mathbb{C}$ such that $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n) = h[P(T_1, T_2, \ldots, T_n)]$, where $\sigma(P)$ is the spectrum of a polynomial $P([8, 13])$.

The spectral mapping theorem means that when $A : X \to X$ is a bounded linear operator acting on the complex Banach space X and $\mathfrak{F}(A)$ denote the family of all complex-valued functions f that are analytic on some open neighborhood of $\sigma(A)$ and when $f \in \mathfrak{F}(A)$, then $\sigma(f(A)) = \{f(\lambda) | \lambda \in \sigma(A)\} = f([\sigma(A)])$ ([8, 13-16]).

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2. Main result

Let $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a polynomial in n variables such that $P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C}$ for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$. Then we obtain the following result.

Theorem 2.1. *Let X_i be a complex Banach space and $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ the completion of the tensor product space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm.*
Let $T_i \in BL(X_i)$ and if $P(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a polynomial in n variables such that $P(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C}$ for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$.
Then $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n)$.

Proof. In[2], Brown and Pearcy show that $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2)\cdots\sigma(T_n)$, where $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ is the tensor product operator of $T_1, T_2, \cdots, T_n$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$. In [1], Martin Schechter show that $\sigma[P(T_1, T_2, \ldots, T_n)] = P[\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n)] = \sigma(T_1)\sigma(T_2)\cdots\sigma(T_n)$.
In [4, Corollary3], B.P.Rynne show that the spectrum of the operator $T_1 \otimes T_2 \otimes \cdots \otimes T_n$ on $X = X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ is given by $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \{\lambda_1 \lambda_2 \cdots \lambda_n \in \mathbb{C} | \lambda_i \in \sigma(T_i), 1 \leq i \leq n\}$. That implies; $\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n) = \sigma(T_1)\sigma(T_2)\cdots\sigma(T_n)$. □

Let η be an abelian Banach algebra with unit, containing the elements $T_1, T_2, \ldots, T_n$ and let $h : \eta \to \mathbb{C}$ be a homomorphism. Then we can obtain the following result.

Lemma 2.2. *Let η be an abelian Banach algebra with unit, containing the elements $T_1, T_2, \ldots, T_n$. If $h : \eta \to \mathbb{C}$ is a homomorphism, and $P(T_1, T_2, \ldots, T_n)$ is a polynomial in n variables, then $h[P(T_1, T_2, \ldots, T_n)] = P[h(T_1, T_2, \ldots, T_n)]$.*

Proof. Let $h[P(T_1, T_2, \ldots, T_n)] = h[k_1(T_1, T_2, \ldots, T_n) + k_2(T_1, T_2, \ldots, T_n)^2 + \cdots + k_n(T_1, T_2, \ldots, T_n)^n]$ for scalars $k_i, 1 \leq i \leq n$.
Since h is a homomorphism, $h[k_1(T_1, T_2, \ldots, T_n) + k_2(T_1, T_2, \ldots, T_n)^2 + \cdots + k_n(T_1, T_2, \ldots, T_n)^n] = k_1 h(T_1, T_2, \ldots, T_n) + k_2 h(T_1, T_2, \ldots, T_n)^2 + \cdots + k_n h(T_1, T_2, \ldots, T_n)^n = P[h(T_1, T_2, \ldots, T_n)]$. This means that
$h[P(T_1, T_2, \ldots, T_n)] = P[h(T_1, T_2, \cdots, T_n)]$. □

Let $P(B_1, B_2, \ldots, B_n)$ be a polynomial in n variables such that $P(B_1, B_2,$

$\ldots, B_n) = B_1 \times B_2 \times \cdots \times B_n = \{z = (a_1, a_2, \ldots, a_n) | a_i \in B_i, 1 \leq i \leq n\}$.
Let us state the following result.

THEOREM 2.3. *Let X_i be a complex Banach space and $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ the completion of the tensor product $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm and let $T_i \in BL(X_i), 1 \leq i \leq n$.*
Suppose that η is an abelian Banach algebra with unit, containing the elements $T_1, T_2, \ldots, T_n$ and $\lambda_i = h(T_i)$ for some homomorphism, h for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$. Then $\sigma[P(T_1, T_2, \ldots, T_n)] = P[(\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n))] = \sigma(T_1, T_2, \ldots, T_n)$.

Proof. By [7, Theorem 2.5], $\sigma[P(T_1, T_2, \ldots, T_n)] = P[(\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_n))]$.
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma[P(T_1, T_2, \ldots, T_n)]$. Then by Gelfand's theorem, there exists a homomorphism $h : \eta \to \mathbb{C}$ such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = h[P(T_1, T_2, \ldots, T_n)]$.
Since $\lambda_i = h(T_i)$ for some homomorphism h, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = h[P(T_1, T_2, \ldots, T_n)] \in \sigma(T_1, T_2, \ldots, T_n)$ ([7]).
Similarly, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma(T_1, T_2, \ldots, T_n)$, then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \sigma[P(T_1, T_2, \ldots, T_n)]$. This means that $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma(T_1, T_2, \ldots, T_n)$. □

The projection property of the joint spectrum is given by the relation $\pi[\sigma(T_1, T_2, \ldots, T_n)] = \sigma(T_1, T_2, \ldots, T_k), k < n$, where π is the projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$ given by $\pi(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$.
Then we can obtain the following result from theorem 2.3.

COROLLARY 2.4. *Let X_i be a complex Banach space and $X_1 \overline{\otimes} X_2 \overline{\otimes} \cdots \overline{\otimes} X_n$ the completion of the tensor product space $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ with respect to some cross norm. Suppose that η is an abelian Banach algebra with unit containing the elements $T_1, T_2, \ldots, T_n$ for $T_i \in BL(X_i), 1 \leq i \leq n$ and $\lambda_i = h(T_i)$ for some homomorphism h for $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$. If π is the projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$ given by $\pi(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_1, \lambda_2, \ldots, \lambda_k), k < n$, then $\pi[\sigma(T_1, T_2, \ldots, T_n)] = \sigma(T_1, T_2, \ldots, T_k), k < n$.*

Proof. By Theorem 2.3, $\sigma[P(T_1, T_2, \ldots, T_n)] = \sigma(T_1, T_2, \ldots, T_n)$ holds. Since π is the projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$, we have $\pi[\sigma(P(T_1, T_2, \ldots, T_n))] = \sigma(T_1, T_2, \ldots, T_k), k < n$. □

Let us state the following result by the spectral mapping theorem.

THEOREM 2.5. *Let H_j be complex Hilbert spaces, $1 \leq j \leq n$ and $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ and let $T_j \in BL(H_j)$, $\widetilde{T_j} = I_1 \otimes I_2 \otimes \cdots \otimes T_j \otimes I_{j+1} \otimes \cdots \otimes I_n$, where I_j is the identity operator on H_j, $1 \leq j \leq n$ and $\widetilde{T} = (\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_n}) \in BL(H)$.*
If f is an analytic function on an open neighborhood G of $\sigma(\widetilde{T}, H) = \sigma(\widetilde{T})$ into $\mathbb{C}^n$, then $\sigma(f(\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_n})) = f(\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n, H))$.

Proof. Since f is any analytic function on an open neighborhood G of $\sigma(\widetilde{T}, H)$ in to $\mathbb{C}^n$, by spectral mapping theorem, $\sigma(f(\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_n})) = f(\sigma(\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_n}))$ holds. In [3], A.T. Dash and M. Schechter have shown that $\sigma(\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_n}) = \sigma(T_1, H_1) \times \sigma(T_2, H_2) \times \cdots \times \sigma(T_n, H_n)$. This implies that $f(\sigma(\widetilde{T_1}, \widetilde{T_2}, \ldots, \widetilde{T_n})) = f(\sigma(T_1, H_1) \times \sigma(T_2, H_2) \times \cdots \times \sigma(T_n, H_n))$. In [6, 2.6 Corollary], Zoia Ceausescu and F.H.Vasilescu have shown that $\sigma(T_1, H_1) \times \sigma(T_2, H_2) \times \cdots \times \sigma(T_n, H_n) = \sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n, H)$. This means that $f(\sigma(T_1, H_1) \times \sigma(T_2, H_2) \times \cdots \times \sigma(T_n, H_n)) = f(\sigma(T_1 \otimes T_2 \otimes \cdots \otimes T_n, H))$.
We obtain the desired result. □

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