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ϵ-FUZZY EQUIVALENCE RELATIONS

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Abstract. We find the ϵ-fuzzy equivalence relation generated by the union of two ϵ-fuzzy equivalence relations on a set, find the ϵ-fuzzy equivalence relation generated by a fuzzy relation on a set, and find sufficient conditions for the composition $\mu \circ \nu$ of two ϵ-fuzzy equivalence relations μ and ν to be the ϵ-fuzzy equivalence relation generated by $\mu \cup \nu$. Also we study fuzzy partitions of ϵ-fuzzy equivalence relations.

1. Introduction

The concept of a fuzzy relation was first proposed by Zadeh ([7]). Subsequently, Goguen ([1]) and Sanchez ([5]) studied fuzzy relations in various contexts. In [4] Nemitz discussed fuzzy equivalence relations, fuzzy functions as fuzzy relations, and fuzzy partitions. Murali ([3]) developed some properties of fuzzy equivalence relations and certain lattice theoretic properties of fuzzy equivalence relations. The standard definition of a reflexive fuzzy relation μ on a set X, which Murali ([3]) and Nemitz ([4]) used in their papers, is $\mu(x, x) = 1$ for all $x \in X$. Yeh ([6]) weakened the standard reflexive fuzzy relation to $\mu(x, x) \geq \epsilon > 0$, which is called an ϵ-reflexive fuzzy relation. Also Gupta et al. ([2]) proposed a generalized definition of a fuzzy equivalence relation on a set, which is called a G-reflexive fuzzy relation, and developed some properties of that relation.

We characterize the generated ϵ-fuzzy equivalence relations on sets and fuzzy partitions of ϵ-fuzzy equivalence relations. In section 2 we

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review some basic definitions and properties of fuzzy relations and ϵ-reflexive fuzzy relations. In section 3 we find the ϵ-fuzzy equivalence relation generated by the union of two ϵ-fuzzy equivalence relations on a set, find the ϵ-fuzzy equivalence relation generated by a fuzzy relation on a set, and show that if μ and ν are ϵ-fuzzy equivalence relations on a set S such that $\mu \circ \nu = \nu \circ \mu$, $\mu(x,x) \geq \nu(x,y)$, and $\nu(y,y) \geq \mu(x,y)$ for all $x, y \in S$, then $\mu \circ \nu$ is the ϵ-fuzzy equivalence relation generated by $\mu \cup \nu$. In section 4 we define a fuzzy partition based on ϵ-fuzzy equivalence relations and construct a fuzzy partition.

2. Preliminaries

In this section we recall some basic definitions and properties of fuzzy relations and ϵ-reflexive fuzzy relations.

DEFINITION 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B.

The standard definition of a reflexive fuzzy relation μ in a set X demands $\mu(x,x) = 1$. Yeh ([6]) weakened this definition as follows.

DEFINITION 2.2. A *fuzzy relation* μ in a set X is a fuzzy subset of $X \times X$. μ is *ϵ-reflexive* in X if $\mu(x,x) \geq \epsilon > 0$ for all $x \in X$. μ is *symmetric* in X if $\mu(x,y) = \mu(y,x)$ for all x, y in X. The composition $\lambda \circ \mu$ of two fuzzy relations λ, μ in X is the fuzzy subset of $X \times X$ defined by

$$(\lambda \circ \mu)(x,y) = \sup_{z \in X} \min(\lambda(x,z), \mu(z,y)).$$

A fuzzy relation μ in X is *transitive* in X if $\mu \circ \mu \subseteq \mu$. A fuzzy relation μ in X is called *ϵ-fuzzy equivalence relation* if μ is ϵ-reflexive, symmetric, and transitive.

Let $\mathcal{F}_X$ be the set of all fuzzy relations in a set X. Then it is easy to see that the composition $\circ$ is associative and $\mathcal{F}_X$ is a monoid under the operation of composition $\circ$.

DEFINITION 2.3. Let μ be a fuzzy relation in a set X. μ^{-1} is defined as a fuzzy relation in X by $\mu^{-1}(x,y) = \mu(y,x)$.

It is easy to see that $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1}$ for fuzzy relations μ and ν.

PROPOSITION 2.4. *Let μ be a fuzzy relation on a set X. Then $\cup_{n=1}^{\infty} \mu^n$ is the smallest transitive fuzzy relation on X containing μ, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.3 of [5]. □

PROPOSITION 2.5. *Let μ be a fuzzy relation on a set X. If μ is symmetric, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. See Proposition 2.4 of [5]. □

PROPOSITION 2.6. *Let μ be a fuzzy relation on a set S. If μ is ϵ-reflexive, then so is $\cup_{n=1}^{\infty} \mu^n$, where $\mu^n = \mu \circ \mu \circ \cdots \circ \mu$.*

Proof. Clearly μ is ϵ-reflexive. Suppose μ^k is ϵ-reflexive. Then

$$\begin{aligned} \mu^{k+1}(x,x) = (\mu^k \circ \mu)(x,x) &= \sup_{z \in X} \min[\mu^k(x,z),\ \mu(z,x)] \\ &\geq \min[\mu^k(x,x),\ \mu(x,x)] \geq \epsilon > 0. \end{aligned}$$

By the mathematical induction, μ^n is ϵ-reflexive for all natural numbers n. Thus $[\cup_{n=1}^{\infty} \mu^n](x,x) = \sup\ [\mu(x,x),\ (\mu \circ \mu)(x,x),\ \ldots] \geq \epsilon > 0$. Hence $\cup_{n=1}^{\infty} \mu^n$ is ϵ-reflexive. □

PROPOSITION 2.7. *Let μ and each ν_i be fuzzy relations in a set X for all $i \in I$. Then $\mu \circ (\bigcap_{i \in I} \nu_i) \subseteq \bigcap_{i \in I} (\mu \circ \nu_i)$ and $(\bigcap_{i \in I} \nu_i) \circ \mu \subseteq \bigcap_{i \in I} (\nu_i \circ \mu)$.*

Proof. Straightforward. □

3. ϵ-Fuzzy equivalence relations generated by fuzzy relations

In this section we characterize the generated ϵ-fuzzy equivalence relations on sets.

PROPOSITION 3.1. *Let μ and ν be ϵ-fuzzy equivalence relations in a set X. Then $\mu \cap \nu$ is an ϵ-fuzzy equivalence relation.*

Proof. It is clear that $\mu \cap \nu$ is ϵ-reflexive and symmetric. By Proposition 2.7, $[(\mu \cap \nu) \circ (\mu \cap \nu)] \subseteq [\mu \circ (\mu \cap \nu)] \cap [\nu \circ (\mu \cap \nu)] \subseteq [(\mu \circ \mu) \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap (\nu \circ \nu)] \subseteq [\mu \cap (\mu \circ \nu)] \cap [(\nu \circ \mu) \cap \nu] \subseteq \mu \cap \nu$. That is, $\mu \cap \nu$ is transitive. Thus $\mu \cap \nu$ is an ϵ-fuzzy equivalence relation. □

It is easy to see that even though μ and ν are ϵ-fuzzy equivalence relations, $\mu \cup \nu$ is not necessarily an ϵ-fuzzy equivalence relation. We find the ϵ-fuzzy equivalence relation generated by $\mu \cup \nu$ on a set in the following proposition.

PROPOSITION 3.2. *Let μ and ν be ϵ-fuzzy equivalence relations on a set S. Then the ϵ-fuzzy equivalence relation generated by $\mu \cup \nu$ in S is $\cup_{n=1}^{\infty} (\mu \cup \nu)^n = (\mu \cup \nu) \cup [(\mu \cup \nu) \circ (\mu \cup \nu)] \cup \dots$.*

Proof. Clearly $(\mu \cup \nu)(x, x) \geq \epsilon > 0$. That is, $\mu \cup \nu$ is ϵ-reflexive. By Proposition 2.6, $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is ϵ-reflexive. Clearly $\mu \cup \nu$ is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is symmetric. By Proposition 2.4, $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is transitive. Hence $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is an ϵ-fuzzy equivalence relation containing $\mu \cup \nu$. Let λ be an ϵ-fuzzy equivalence relation in S containing $\mu \cup \nu$. Then $\cup_{n=1}^{\infty} (\mu \cup \nu)^n \subseteq \cup_{n=1}^{\infty} \lambda^n = \lambda \cup (\lambda \circ \lambda) \cup (\lambda \circ \lambda \circ \lambda) \cup \dots \subseteq \lambda \cup \lambda \cup \dots = \lambda$. Thus $\cup_{n=1}^{\infty} (\mu \cup \nu)^n$ is the ϵ-fuzzy equivalence relation generated by $\mu \cup \nu$. □

THEOREM 3.3. *Let μ and ν be ϵ-fuzzy equivalence relations on a set S such that $\mu(x, x) \geq \nu(x, y)$ and $\nu(y, y) \geq \mu(x, y)$ for all $x, y \in S$. If $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is the ϵ-fuzzy equivalence relation on S generated by $\mu \cup \nu$.*

Proof.
$(\mu \circ \nu)(x, x) = \sup_{z \in S} \min\,[\mu(x, z), \nu(z, x)] \geq \min\,(\mu(x, x), \nu(x, x)) \geq \epsilon > 0$ for all $x \in S$. That is, $\mu \circ \nu$ is ϵ-reflexive. Since μ and ν are symmetric, $(\mu \circ \nu)^{-1} = \nu^{-1} \circ \mu^{-1} = \nu \circ \mu = \mu \circ \nu$. Thus $\mu \circ \nu$ is symmetric. Since μ and ν are transitive and the operation $\circ$ is associative, $(\mu \circ \nu) \circ (\mu \circ \nu) = \mu \circ (\nu \circ \mu) \circ \nu = \mu \circ (\mu \circ \nu) \circ \nu = (\mu \circ \mu) \circ (\nu \circ \nu) \subseteq \mu \circ \nu$. Hence $\mu \circ \nu$ is an ϵ-fuzzy equivalence relation. Since $\nu(y, y) \geq \mu(x, y)$, $(\mu \circ \nu)(x, y) = \sup_{z \in S} \min[\mu(x, z), \nu(z, y)] \geq \min(\mu(x, y), \nu(y, y)) = \mu(x, y)$. Since $\mu(x, x) \geq \nu(x, y)$, $(\mu \circ \nu)(x, y) = \sup_{z \in S} \min\,[\mu(x, z), \nu(z, y)] \geq \min\,(\mu(x, x), \nu(x, y)) = \nu(x, y)$. Thus

$(\mu \circ \nu)(x,y) \geq \max\ (\mu(x,y), \nu(x,y)) = (\mu \cup \nu)(x,y)$ for all $x, y \in S$. Thus $\mu \cup \nu \subseteq \mu \circ \nu$. Let λ be an ϵ-fuzzy equivalence relation in S containing $\mu \cup \nu$. Since λ is transitive, $\mu \circ \nu \subseteq (\mu \cup \nu) \circ (\mu \cup \nu) \subseteq \lambda \circ \lambda \subseteq \lambda$. Thus $\mu \circ \nu$ is the ϵ-fuzzy equivalence relation generated by $\mu \cup \nu$. □

THEOREM 3.4. *Let μ be a fuzzy relation on a set S. Then the ϵ-fuzzy equivalence relation generated by μ in S is $\cup_{n=1}^{\infty}(\mu \cup \mu^{-1} \cup \theta)^n$, where θ is a fuzzy relation in S such that $\theta(a,a) = \epsilon$ for all $a \in S$ and $\theta(x,y) = \theta(y,x) \leq \min\ [\mu(x,y), \mu(y,x)]$ for all $x, y \in S$ with $x \neq y$.*

Proof. $(\mu \cup \mu^{-1} \cup \theta)(a,a) \geq \theta(a,a) = \epsilon > 0$ for all $a \in S$. Thus $\mu \cup \mu^{-1} \cup \theta$ is ϵ-reflexive. Let $\mu_1 = \mu \cup \mu^{-1} \cup \theta$. By Proposition 2.6, $\cup_{n=1}^{\infty}\mu_1^n$ is ϵ-reflexive. $\mu_1(x,y) = (\mu \cup \mu^{-1} \cup \theta)(x,y) = \max[\mu(x,y), \mu^{-1}(x,y), \theta(x,y)] = \max[\mu^{-1}(y,x), \mu(y,x), \theta(y,x)] = (\mu \cup \mu^{-1} \cup \theta)(y,x) = \mu_1(y,x)$. Thus μ_1 is symmetric. By Proposition 2.5, $\cup_{n=1}^{\infty}\mu_1^n$ is symmetric. By Proposition 2.4, $\cup_{n=1}^{\infty}\mu_1^n$ is transitive. Hence $\cup_{n=1}^{\infty}\mu_1^n$ is an ϵ-fuzzy equivalence relation containing μ. Let ν be an ϵ-fuzzy equivalence relation containing μ. Then $\mu(x,y) \leq \nu(x,y)$, $\mu^{-1}(x,y) = \mu(y,x) \leq \nu(y,x) = \nu(x,y)$, and $\theta(x,y) \leq \mu(x,y) \leq \nu(x,y)$ for all $x, y \in S$ such that $x \neq y$. That is, $\nu(x,y) \geq (\mu \cup \mu^{-1} \cup \theta)(x,y)$ for all $x, y \in S$ such that $x \neq y$. $\nu(a,a) \geq \mu(a,a) = \mu^{-1}(a,a)$ for all $a \in S$. Since $\theta(a,a) = \epsilon$ and $\nu(a,a) \geq \epsilon$ for all $a \in S$, $\theta(a,a) \leq \nu(a,a)$. That is, $\mu_1(a,a) \leq \nu(a,a)$ for all $a \in S$. Thus $\mu_1 = (\mu \cup \mu^{-1} \cup \theta) \subseteq \nu$. Suppose $\mu_1^k \subseteq \nu$. Then $\mu_1^{k+1}(x,y) = (\mu_1^k \circ \mu_1)(x,y) = \sup_{z \in S} \min[\mu_1^k(x,z), \mu_1(z,y)] \leq \sup_{z \in S} \min[\nu(x,z), \nu(z,y)] = (\nu \circ \nu)(x,y)$. Since ν is transitive, $\mu_1^{k+1} \subseteq \nu \circ \nu \subseteq \nu$. By the mathematical induction, $\mu_1^n \subseteq \nu$ for $n = 1, 2 \ldots$. Hence $\cup_{n=1}^{\infty}\ \mu_1^n = \mu_1 \cup (\mu_1 \circ \mu_1) \cup (\mu_1 \circ \mu_1 \circ \mu_1) \cdots \subseteq \nu$. □

4. Partitions of ϵ-fuzzy equivalence relations

Murali([3]) studied partition of fuzzy equivalence relations. In this section we define a fuzzy partition based on ϵ-fuzzy equivalence relations and construct a fuzzy partition, which may be considered as a generalization of Murali's work.

DEFINITION 4.1. Let μ be an ϵ-fuzzy equivalence relation on a set X. For $0 < p \leq \epsilon$, a relation $\prec_p$ on X is defined by $x \prec_p y$ iff $\mu(x,y) \geq p$.

PROPOSITION 4.2. *Let μ be an ϵ-fuzzy equivalence relation on a set X. Then the relation $\prec_p$ on a set X defined in Definition 4.1 is an equivalence relation.*

Proof. Since $\mu(x,x) \geq \epsilon \geq p$, $\prec_p$ is reflexive. Suppose $x \prec_p y$. Then $\mu(x,y) \geq p$, and hence $\mu(y,x) \geq p$. Thus $\prec_p$ is symmetric. Suppose $x \prec_p y$ and $y \prec_p z$. Then $\mu(x,y) \geq p$ and $\mu(y,z) \geq p$. $\mu(x,z) = (\mu \circ \mu)(x,z) = \sup_{k \in X} \min(\mu(x,k), \mu(k,z)) \geq \min(\mu(x,y), \mu(y,z)) \geq p$. Thus $\prec_p$ is transitive. $\square$

DEFINITION 4.3. Let μ be an ϵ-fuzzy equivalence relation on a set X and let $\prec_p$ be an equivalence relation on X defined in Proposition 4.2. The equivalence class containing x is denoted by $[x]_p$. That is, $[x]_p = \{y \in X : y \prec_p x\}$ for $p \leq \epsilon$.

DEFINITION 4.4. Let μ be an ϵ-fuzzy equivalence relation on a set X and let $\prec_p$ be an equivalence relation on X defined in Proposition 4.2. A fuzzy subset $\mu_{[x]_\epsilon}$ on a set X is defined by $\mu_{[x]_\epsilon}(y) = \mu(x,y)$.

DEFINITION 4.5. Let $\{\nu_i : \in I\}$ be a collection of fuzzy sets on a set X. If $(\cup_{i \in I} \nu_i)(z) \geq p$ and $\nu_i \cap \nu_j = 0$ for all $i, j \in I$ with $i \neq j$, we call $\{\nu_i : i \in I\}$ is a *fuzzy partition* of a fuzzy set χ_X^p on X, where $\chi_X^p : X \to \mathbb{R}$ is a function defined by $\chi_X^p(x) \geq p$ for all $x \in X$.

LEMMA 4.6. *Let μ be an ϵ-fuzzy equivalence relation on a set X. Then $[x]_p \cap [y]_p = \emptyset$ for some $0 < p \leq \epsilon$ iff $(\mu_{[x]_\epsilon} \cap \mu_{[y]_\epsilon})(z) = 0$ for all $z \in X$.*

Proof. $(\to)$ Suppose $(\mu_{[x]_\epsilon} \cap \mu_{[y]_\epsilon})(z) > 0$ for some $z \in X$. Then $\mu_{[x]_\epsilon}(z) \geq p$ and $\mu_{[y]_\epsilon}(z) \geq p$ for some $p \leq \epsilon$. That is, $\mu(x,z) \geq p$ and $\mu(y,z) \geq p$. Thus $\mu(x,y) \geq (\mu \circ \mu)(x,y) = \sup_{k \in X} \min(\mu(x,k), \mu(k,y)) \geq \min(\mu(x,z), \mu(y,z)) \geq p$. That is, $x \prec_p y$. This contradicts $[x]_p \cap [y]_p = \emptyset$.

$(\to)$ Suppose $\alpha \in [x]_p \cap [y]_p$. Then $x \prec_p \alpha$ and $y \prec_p \alpha$. Since $\prec_p$ is an equivalence relation by Proposition 4.2, $x \prec_p y$. Thus $\mu(x,y) \geq p$. Since $\mu(y,y) \geq \epsilon$ and $p \leq \epsilon$, $(\mu_{[x]_\epsilon} \cap \mu_{[y]_\epsilon})(y) = \min(\mu(x,y), \mu(y,y)) \geq p$. This contradicts that $(\mu_{[x]_\epsilon} \cap \mu_{[y]_\epsilon})(z) = 0$ for all $z \in X$. $\square$

THEOREM 4.7. *Let μ be an ϵ-fuzzy equivalence relation on a set X. Let $\{[x_i]_p : x_i \in X,\ i \in I\}$ be a partition of X. Then $\{\mu_{[x_i]_\epsilon} : x_i \in X,\ i \in I\}$ is a fuzzy partition of a fuzzy set χ_X^p on X.*

Proof. Let $i, j \in I$ with $i \neq j$. Then $[x_i]_p \cap [x_j]_p = \emptyset$. By Lemma 4.6, $(\mu_{[x_i]_\epsilon} \cap \mu_{[x_j]_\epsilon})(z) = 0$ for all $z \in X$. Let $y \in X$. Then $y \in [x_k]_p$ for some $k \in I$. Since $y \prec_p x_k$, $\mu(y, x_k) \geq p$. Thus

$$(\cup_{i\in I}\ \mu_{[x_i]_\epsilon})(y) = \sup_{i\in I}\ \mu_{[x_i]_\epsilon}(y) = \sup_{i\in I}\ \mu(x_{i,y}\ ) = \mu(x_k, y) \geq p.$$

$\square$

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