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# ON INTRA-REGULAR ORDERED SEMIGROUPS

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Abstract. In this paper we give some new characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups.

## 1. Introduction

In [9, 10] S. Lajos and G. Szasz characterized the intra-regular semigroups in terms of right ideals and left ideals of semigroups. Also, in [6] N. Kehayopulu, S. Lajos and M. Tsingelis characterized the intra-regular ordered semigroups in terms of right ideals and left ideals of ordered semigroups, and they obtained many characterizations of the intra-regularity in ordered semigroups . Recently, in [11] S. K. Lee characterized the intra-regular semigroups -without order- in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, and bi-ideals and right ideals of semigroups.

In this paper we give some new characterizations of the intra-regular ordered semigroups in terms of bi-ideals and quasi-ideals, bi-ideals and left ideals, bi-ideals and right ideals of ordered semigroups.

## 2. Definitions and Lemmas

An *ordered semigroup* (: *po-semigroup*) is a semigroup $S$ with an ordered relation " $\leq$ " such that for all $x \in S$ $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$([1]).

A *po*-semigroup $S$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$ ([4]).

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Let $S$ be a *po*-semigroup. A non-empty subset $A$ of $S$ is called a *left* (resp. *right*) *ideal* of $S$ if 1) $SA \subseteq A$ (resp. $AS \subseteq A$), 2) $a \in A, b \leq a$ for $b \in S$ implies $b \in A$. If $A$ is both a left ideal and a right ideal of $S$, then $A$ is called an *ideal* of $S$ ([3]).

A non-empty subset $Q$ of $S$ is called a *quasi-ideal* of $S$ if 1) $(QS] \cap (SQ] \subseteq Q$, 2) $a \in Q, b \leq a$ for $b \in S$ implies $b \in Q$. A non-empty subset $B$ of $S$ is called a *bi-ideal* of $S$ if 1) $BSB \subseteq B$, 2) $a \in B, b \leq a$ for $b \in S$ implies $b \in B$ ([4, 5, 7]).
For an ordered semigroup $S$, we denote by $R(a)$ (resp. $L(a), Q(a), B(a)$) the right (resp. left, quasi-, bi-) ideal of $S$ generated by $a(a \in S)$.

For $H \subseteq S$, we denote $(H] := \{t \in S | t \leq h \text{ for some } h \in H\}$.

We can prove easily that for a non-empty subset of a *po*-semigroup $S$

$$R(a) = (a \cup aS], \quad L(a) = (a \cup Sa],$$

$$Q(a) = (a \cup ((aS] \cap (Sa])], \quad B(a) = (a \cup a^2 \cup aSa].$$

The ideal of $S$ generated by $a$ is the set $(a \cup Sa \cup aS \cup SaS]$ which is equal to $R(L(a)) = L(R(a))(see[6])$.

For all other definitions we refer to [2, 4, 8].

We have the following lemma (see [3]).

LEMMA. *Let $S$ be a po-semigroup. Then we have*
*1) $A \subseteq (A]$ for any $A \subseteq S$.*
*2) If $A \subseteq B$, then $(A] \subseteq (B]$.*
*3) If $A$ is some types of ideal, then $A = (A]$.*
*4) If $A$ is a subset of $S$, then $((A]] = (A]$.*
*5) $(A](B] \subseteq (AB]$ for all $A, B \subseteq S$.*
*6) For $A, B \subseteq S$, $(A \cap B] \neq (A] \cap (B]$, in general. In particular, if $A$ and $B$ are some types of ideal of $S$, then $(A \cap B] = (A] \cap (B]$.*

## 3. Main Theorems

N. Kehayopulu, S. Lajos and M. Tsingelis gave many characterizations of the intra-regular *po*-semigroups([6]). Now we give some new characterizations of the intra-regular *po*-semigroups in terms of bi-ideals, quasi-ideals, left ideals and right ideals of *po*-semigroups.

THEOREM 1. *Let $S$ be a po-semigroup. Then the followings are true.*

*(1) $S$ is intra-regular if and only if for a bi-ideal $B$ and a quasi-ideal $Q$ of $S$, we have $B \cap Q \subseteq (SBQS]$.*

*(2) $S$ is intra-regular if and only if for a bi-ideal $B$ and a quasi-ideal $Q$ of $S$, we have $B \cap Q \subseteq (SQBS]$.*

*Proof.* (1) Let $a \in B \cap Q$. Since $S$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Thus $a \leq xa^2y \leq xa(xa^2y)y = x(axa)ay^2 \in S(BSB)QS \subseteq SBQS$. Therefore, $B \cap Q \subseteq (SBQS]$.

Conversely, let $B(a)$ be a bi-ideal and $Q(a)$ a quasi-ideal generated by a in $S$. Then by hypothesis,

$$
\begin{aligned}
a \in B(a) \cap Q(a) &\subseteq (SB(a)Q(a)S] \\
&= (S(a \cup a^2 \cup aSa](a \cup ((aS] \cap (Sa]))]S] \\
&\subseteq ((Sa \cup Sa^2 \cup SaSa](a \cup (aS]]S] \\
&\subseteq ((Sa](a \cup (aS]]S] \\
&\subseteq ((Sa](aS \cup (aS^2]] \\
&\subseteq ((Sa^2S] \cup (Sa^2S^2]] \\
&\subseteq ((Sa^2S]] = (Sa^2S].
\end{aligned}
$$

Therefore, $S$ is intra-regular.

(2) Let $a \in B \cap Q$. Since $S$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y$. Thus $a \leq xa^2y \leq x(xa^2y)ay = x^2a(aya)y \in SQ(BSB)S \subseteq SQBS$. Therefore, $B \cap Q \subseteq (SQBS]$.

Conversely, let $B(a)$ be a bi-ideal and $Q(a)$ a quasi-ideal generated by a of $S$. Then by hypothesis,

$$
\begin{aligned}
a \in B(a) \cap Q(a) &\subseteq (SQ(a)B(a)S] \\
&= (S(a \cup ((aS] \cap (Sa]))](a \cup a^2 \cup aSa]S] \\
&\subseteq (S(a \cup (Sa]](aS \cup a^2S \cup aSaS]] \\
&\subseteq ((Sa \cup (S^2a]](aS]] \subseteq ((Sa](aS]] \\
&\subseteq ((Sa^2S]] = (Sa^2S].
\end{aligned}
$$

Therefore, $S$ is intra-regular. $\square$

THEOREM 2. *Let $S$ be a po-semigroup. Then the followings are true.*

*(1) $S$ is intra-regular if and only if for a left ideal $L$ and a bi-ideal $B$ of $S$, we have $L \cap B \subseteq (LBS]$.*

*(2) $S$ is intra-regular if and only if for a right ideal $R$ and a bi-ideal $B$ of $S$, we have $B \cap R \subseteq (SBR]$.*

*Proof.* (1) Let $a \in L \cap B$. Since $S$ is intra-regular, there exists $x, y \in S$ such that $a \leq xa^2y \leq x(xa^2y)ay = x^2a(aya)y \in SL(BSB)S \subseteq LBS$. Thus $L \cap B \subseteq (LBS]$.

Conversely, let $B(a)$ be a bi-ideal generating by a and $L(a)$ a left ideal generating by $a$. Then by hypothesis,

$$\begin{aligned} a \in L(a) \cap B(a) &\subseteq (L(a)B(a)S] \\ &= ((a \cup Sa](a \cup a^2 \cup aSa]S] \\ &\subseteq ((a \cup Sa](aS \cup a^2S \cup aSaS]] \\ &\subseteq ((a \cup Sa](aS]] = (a^2S \cup Sa^2S]. \end{aligned}$$

Hence $a \leq t$ for some $t \in (a^2S \cup Sa^2S]$. If $t \in a^2S$, then $a \leq a^2x$ for some $x \in S$. Thus we have $a \leq a^2x \leq a(a^2x)x = aa^2x^2 \in Sa^2S$. If $t \in Sa^2S$, it is obvious. Therefore $a \in (Sa^2S]$ for any cases. Hence $S$ is intra-regular.

(2) Let $a \in B \cap R$. Since $S$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y \leq xa(xa^2y)y = x(axa)ay^2 \in S(BSB)RS \subseteq SBR$. Thus $B \cap R \subseteq (SBR]$.

Conversely, let $B(a)$ be a bi-ideal generating by a and $R(a)$ a right-ideal generating by $a$. Then by hypothesis,

$$\begin{aligned} a \in B(a) \cap R(a) &\subseteq (SB(a)R(a)] \\ &= (S(a \cup a^2 \cup aSa](a \cup aS]] \\ &\subseteq ((Sa \cup Sa^2 \cup SaSa](a \cup aS]] \\ &\subseteq ((Sa](a \cup aS]] \\ &\subseteq ((Sa^2 \cup Sa^2S]] = (Sa^2 \cup Sa^2S]. \end{aligned}$$

Hence $a \leq t$ for some $t \in (Sa^2 \cup Sa^2S]$. If $t \in Sa^2$, then $a \leq xa^2$ for some $x \in S$. Thus we have $a \leq xa^2 \leq x(xa^2)a = x^2a^2a \in Sa^2S$. If

$t \in Sa^2S$, it is obvious. Therefore $a \in (Sa^2S]$ for any cases. Hence $S$ is intra-regular. □

Remark. A semigroup $S$ is a *po*-semigroup with the relation $i = \{(a,a) | \forall a \in S\}$. Thus we obtain Theorem 1, 2, 3 and 4 in [11] from above Theorem 1 and 2.

We can prove that the following theorem 3 from Theorem 1 and 2.

Theorem 3. *Let $S$ be a po-semigroup and $a \in S$. Then the followings are true.*

*(1) $S$ is intra-regular.*
*(2) $B(a) \cap Q(a) \subseteq (SB(a)Q(a)S]$.*
*(3) $B(a) \cap Q(a) \subseteq (SB(a)Q(a)S]$.*
*(4) $B(a) \cap Q(a) \subseteq (SQ(a)B(a)S]$.*
*(5) $B(a) \cap Q(a) \subseteq (SQ(a)B(a)S]$.*
*(6) $L(a) \cap B(a) \subseteq (L(a)B(a)S]$.*
*(7) $L(a) \cap B(a) \subseteq (L(a)B(a)S]$.*
*(8) $B(a) \cap R(a) \subseteq (SB(a)R(a)]$.*
*(9) $B(a) \cap R(a) \subseteq (SB(a)R(a)]$.*

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