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WEAK QUASI-SMOOTH α-COMPACTNESS IN SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and obtain some of their structural properties. We also introduce the concepts of several types of weak quasi-smooth α-compactness in terms of the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang's fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. Particularly, Gayyar, Kerre and Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and also introduced several types of α-compactness in smooth topological spaces and obtained some of their properties.

In this paper, we introduce the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set and obtain some of their structural properties. We also introduce the concepts of several types of weak quasi-smooth α-compactness in terms of the concepts of weak

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smooth α-closure and weak smooth α-interior of a fuzzy set and investigate some of their properties.

2. Preliminaries

Let X be a set and $I = [0,1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

A smooth topological space (s.t.s.) [7] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;
(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;
(O3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$,

$$\tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i).$$

Then the mapping $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^* : I^X \to I$ is called a smooth cotopology [7] iff the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;
(C2) $\forall A, B \in I^X,\ \tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;
(C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \tau^*(A_i)$.

If τ is a smooth topology on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a smooth cotopology on X. Conversely, if τ^* is a smooth cotopology on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a smooth topology on X [7].

For the s.t.s. (X, τ) and $\alpha \in [0,1]$, the family $\tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\}$ defines a Chang's fuzzy topology (CFT) on X [2]. The family of all closed fuzzy sets with respect to τ_α is denoted by τ_α^* and we have $\tau_\alpha^* = \{A \in I^X : \tau^*(A) \geq \alpha\}$. For $A \in I^X$ and $\alpha \in [0,1]$, the τ_α-closure (resp., τ_α-interior) of A, denoted by $cl_\alpha(A)$ (resp., $int_\alpha(A)$), is defined by $cl_\alpha(A) = \cap\{K \in \tau_\alpha^* : A \subseteq K\}$ (resp., $int_\alpha(A) = \cup\{K \in \tau_\alpha : K \subseteq A\}$).

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let (X, τ) be a s.t.s. and $A \in I^X$. Then the τ-smooth closure (resp., τ-smooth interior) of A, denoted by $\overline{A}$ (resp., A^o), is defined by $\overline{A} = \cap\{K \in I^X : \tau^*(K) > 0, A \subseteq K\}$ (resp., $A^o = \cup\{K \in I^X : \tau(K) > 0, K \subseteq A\}$).

Let (X, τ) and (Y, σ) be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to τ and σ [7] iff $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is called weakly smooth continuous with respect to τ and σ [7] iff $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$. In this paper, a weakly smooth continuous function is called a quasi-smooth continuous function.

A function $f : X \to Y$ is smooth continuous with respect to τ and σ iff $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to τ and σ iff $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [7].

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to τ and σ [7] if and only if $\tau(A) \leq \sigma(f(A))$ (resp., $\tau^*(A) \leq \sigma^*(f(A)))$ for every $A \in I^X$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to τ and σ [5] if and only if $\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$ (resp., $\sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B)))$ for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to τ and σ, then $\sigma^*(A) \geq \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B) \Leftrightarrow \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B)))$ for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to τ and σ [5] iff $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B)))$ for every $A, B \in I^X$.

3. weak smooth α-closure and weak smooth α-interior

In this section, we introduce the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and investigate some of their properties.

Definition 3.1[6]. Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. The τ-smooth α-closure (resp., τ-smooth α-interior) of A, denoted by $\overline{A}_\alpha$ (resp., A^o_α), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A), A \subseteq K\}$ (resp., $A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha\tau(A), K \subseteq A\}$).

Demirci [4] defined the families $W(\tau) = \{A \in I^X : A = A^o\}$ and $W^*(\tau) = \{A \in I^X : A = \overline{A}\}$, where (X,τ) is a s.t.s. Note that $A \in W(\tau) \Leftrightarrow A^c \in W^*(\tau)$.

We define the families $W_\alpha(\tau) = \{A \in I^X : A = A^o_\alpha\}$ and $W^*_\alpha(\tau) = \{A \in I^X : A = \overline{A}_\alpha\}$, where (X,τ) is a s.t.s. and $\alpha \in [0,1)$. Note that $A \in W_\alpha(\tau) \Leftrightarrow A^c \in W^*_\alpha(\tau)$.

Definition 3.2. Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. The weak τ-smooth α-closure (resp., weak τ-smooth α-interior) of A, denoted by $wcl_\alpha(A)$ (resp., $wint_\alpha(A)$), is defined by $wcl_\alpha(A) = \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\}$ (resp., $wint_\alpha(A) = \cup\{K \in I^X : K \in W_\alpha(\tau), K \subseteq A\}$).

Theorem 3.3. *Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. Then*
(a) $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha$,
(b) $A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$.

Proof. (a) Let $K \in I^X$ and $A \subseteq K$. Then $\tau^*(K) > \alpha\tau^*(A) \Rightarrow \tau^*(K) > 0$ and $\tau^*(K) > 0 \Rightarrow K = \overline{K}_\alpha$, i.e., $K \in W^*_\alpha(\tau)$ by Theorem 3.6[6]. From the definitions of $\overline{A}_\alpha$, $\overline{A}$ and $wcl_\alpha(A)$ we have $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha$.

(b) Let $K \in I^X$ and $K \subseteq A$. Then $\tau(K) > \alpha\tau(A) \Rightarrow \tau(K) > 0$ and $\tau(K) > 0 \Rightarrow K = K^o_\alpha$, i.e., $K \in W_\alpha(\tau)$ by Theorem 3.6[6]. From the definition of A^o_α, A^o and $wint_\alpha(A)$ we have $A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$. □

Theorem 3.4. *Let (X,τ) be a s.t.s., $\alpha \in [0,1)$ and $A, B \in I^X$. Then*
(a) $A \subseteq B \Rightarrow wcl_\alpha(A) \subseteq wcl_\alpha(B)$,
(b) $A \subseteq B \Rightarrow wint_\alpha(A) \subseteq wint_\alpha(B)$,
(c) $(wcl_\alpha(A))^c = wint_\alpha(A^c)$,
(d) $wcl_\alpha(A) = (wint_\alpha(A^c))^c$,
(e) $(wint_\alpha(A))^c = wcl_\alpha(A^c)$,

(f) $wint_\alpha(A) = (wcl_\alpha(A^c))^c$.

Proof. (a) and (b) follow directly from Definition 3.2.

(c) From Definition 3.2 we have

$$\begin{aligned}(wcl_\alpha(A))^c &= (\cap\{K \in I^X : K \in W_\alpha^*(\tau), A \subseteq K\})^c \\ &= \cup\{K^c : K \in I^X, K^c \in W_\alpha(\tau), K^c \subseteq A^c\} \\ &= \cup\{U \in I^X : U \in W_\alpha(\tau), U \subseteq A^c\} \\ &= wint_\alpha(A^c).\end{aligned}$$

(d), (e) and (f) can be easily obtained from (c).

□

THEOREM 3.5. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A, B \in I^X$. Then*

(a) $wcl_\alpha(0_X) = 0_X$,

(b) $A \subseteq wcl_\alpha(A)$,

(c) $wcl_\alpha(A) = wcl_\alpha(wcl_\alpha(A))$,

(d) $wcl_\alpha(A) \cup wcl_\alpha(B) \subseteq wcl_\alpha(A \cup B)$,

(e) $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(A) \cap wcl_\alpha(B)$.

Proof. (a) By Theorem 3.4[6], $\overline{(0_X)}_\alpha = 0_X$, i.e., $0_X \in W_\alpha^*(\tau)$. From Definition 3.2 we have $wcl_\alpha(0_X) = 0_X$.

(b) follows directly from Definition 3.2.

(c) From (b) we have $wcl_\alpha(A) \subseteq wcl_\alpha(wcl_\alpha(A))$. From Definition 3.2 we have

$$\begin{aligned}wcl_\alpha(wcl_\alpha(A)) &= \cap\{K \in I^X : K \in W_\alpha^*(\tau), wcl_\alpha(A) \subseteq K\} \\ &= \cap\{K \in I^X : K \in W_\alpha^*(\tau), \cap\{U \in I^X : U \in W_\alpha^*(\tau), \\ &\qquad A \subseteq U\} \subseteq K\} \\ &\subseteq \cap\{K \in I^X : K \in W_\alpha^*(\tau), A \subseteq K\} \\ &= wcl_\alpha(A).\end{aligned}$$

Hence $wcl_\alpha(A) = wcl_\alpha(wcl_\alpha(A))$.

(d) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $wcl_\alpha(A) \subseteq wcl_\alpha(A \cup B)$ and $wcl_\alpha(B) \subseteq wcl_\alpha(A \cup B)$ by Theorem 3.4. Hence $wcl_\alpha(A) \cup wcl_\alpha(B) \subseteq wcl_\alpha(A \cup B)$.

(e) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(A)$ and $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(B)$ by Theorem 3.4. Hence $wcl_\alpha(A \cap B) \subseteq wcl_\alpha(A) \cap wcl_\alpha(B)$.

□

THEOREM 3.6. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A, B \in I^X$. Then*

(a) $wint_\alpha(1_X) = 1_X$,

(b) $wint_\alpha(A) \subseteq A$,

(c) $wint_\alpha(A) = wint_\alpha(wint_\alpha(A))$,

(d) $wint_\alpha(A) \cup wint_\alpha(B) \subseteq wint_\alpha(A \cup B)$,

(e) $wint_\alpha(A \cap B) \subseteq wint_\alpha(A) \cap wint_\alpha(B)$.

Proof. The proof is similar to the proof of Theorem 3.5.

□

THEOREM 3.7. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. Then*

(a) $\tau^*(A) > 0 \Rightarrow wcl_\alpha(A) = A$,

(b) $\tau(A) > 0 \Rightarrow wint_\alpha(A) = A$.

Proof. Let $\tau^*(A) > 0$. Then $\overline{A}_\alpha = A$, i.e., $A \in W^*_\alpha(\tau)$ by Theorem 3.6[6]. Hence $A \in \{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\}$. By Definition 3.2, $wcl_\alpha(A) \subseteq A$. By Theorem 3.3, $A \subseteq wcl_\alpha(A)$. Hence $wcl_\alpha(A) = A$.

(b) Let $\tau(A) > 0$. Then $A^o_\alpha = A$, i.e., $A \in W_\alpha(\tau)$ by Theorem 3.6[6]. Hence $A \in \{K \in I^X : K \in W_\alpha(\tau), K \subseteq A\}$. By Definition 3.2, $A \subseteq wint_\alpha(A)$. By Theorem 3.3, $wint_\alpha(A) \subseteq A$. Hence $wint_\alpha(A) = A$.

□

THEOREM 3.8. *Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. Then*

(a) if there exists a $\beta \in (\alpha\tau^*(A), 1]$ such that $A = cl_\beta(A)$, then $A = wcl_\alpha(A) = \overline{A} = \overline{A}_\alpha$,

(b) if there exists a $\beta \in (\alpha\tau(A), 1]$ such that $A = int_\beta(A)$, then $A = wint_\alpha(A) = A^o = A^o_\alpha$.

Proof. (a) If there exists a $\beta \in (\alpha\tau^*(A), 1]$ such that $A = cl_\beta(A)$, then $A \subseteq wcl_\alpha(A) \subseteq \overline{A} \subseteq \overline{A}_\alpha = \cap_{\beta>\alpha\tau^*(A)} cl_\beta(A) \subseteq cl_\beta(A) = A$ by Theorem 3.8[6] and 3.3. Hence $A = wcl_\alpha(A) = \overline{A} = \overline{A}_\alpha$.

(b) If there exists a $\beta \in (\alpha\tau(A), 1]$ such that $A = int_\beta(A)$, then $A = int_\beta(A) \subseteq \cup_{\beta>\alpha\tau(A)} int_\beta(A) = A^o_\alpha \subseteq A^o \subseteq wint_\alpha(A) \subseteq A$ by Theorem 3.8[6] and 3.3. Hence $A = wint_\alpha(A) = A^o = A^o_\alpha$.

□

Definition 3.9. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth α-continuous with respect to τ and σ iff $A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)$ for every $A \in I^Y$.

Theorem 3.10. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, then*

(a) $f(wcl_\alpha(A)) \subseteq wcl_\alpha(f(A))$ for every $A \in I^X$,
(b) $wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))$ for every $A \in I^Y$,
(c) $f^{-1}(wint_\alpha(A)) \subseteq wint_\alpha(f^{-1}(A))$ for every $A \in I^Y$.

Proof. (a) For every $A \in I^X$, we have

$$\begin{aligned}
&f^{-1}(wcl_\alpha(f(A))) = f^{-1}(\cap\{U \in I^Y : U \in W^*_\alpha(\sigma), f(A) \subseteq U\}) \\
&\supseteq f^{-1}(\cap\{U \in I^Y : f^{-1}(U) \in W^*_\alpha(\tau), A \subseteq f^{-1}(U)\}) \\
&= \cap\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W^*_\alpha(\tau), A \subseteq f^{-1}(U)\} \\
&\supseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\} \\
&= wcl_\alpha(A).
\end{aligned}$$

Hence $f(wcl_\alpha(A)) \subseteq wcl_\alpha(f(A))$.

(b) For every $A \in I^Y$, we have

$$\begin{aligned}
&f^{-1}(wcl_\alpha(A)) = f^{-1}(\cap\{U \in I^Y : U \in W^*_\alpha(\sigma), A \subseteq U\}) \\
&\supseteq f^{-1}(\cap\{U \in I^Y : f^{-1}(U) \in W^*_\alpha(\tau), f^{-1}(A) \subseteq f^{-1}(U)\}) \\
&= \cap\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W^*_\alpha(\tau), \\
&\qquad f^{-1}(A) \subseteq f^{-1}(U)\} \\
&\supseteq \cap\{K \in I^X : K \in W^*_\alpha(\tau), f^{-1}(A) \subseteq K\} \\
&= wcl_\alpha(f^{-1}(A)).
\end{aligned}$$

(c) For every $A \in I^Y$, we have

$$f^{-1}(wint_\alpha(A)) = f^{-1}(\cup\{U \in I^Y : U \in W_\alpha(\sigma), U \subseteq A\})$$
$$\subseteq f^{-1}(\cup\{U \in I^Y : f^{-1}(U) \in W_\alpha(\tau), f^{-1}(U) \subseteq f^{-1}(A)\})$$
$$= \cup\{f^{-1}(U) \in I^X : U \in I^Y, f^{-1}(U) \in W_\alpha(\tau),$$
$$f^{-1}(U) \subseteq f^{-1}(A)\}$$
$$\subseteq \cup\{K \in I^X : K \in W_\alpha(\tau), K \subseteq f^{-1}(A)\}$$
$$= wint_\alpha(f^{-1}(A)).$$

□

Definition 3.11. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth α-open (resp., weak smooth α-closed) with respect to τ and σ iff $A \in W_\alpha(\tau) \Rightarrow f(A) \in W_\alpha(\sigma)$ (resp., $A \in W^*_\alpha(\tau) \Rightarrow f(A) \in W^*_\alpha(\sigma)$) for every $A \in I^X$.

Theorem 3.12. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is weak smooth α-open with respect to τ and σ, then $f(wint_\alpha(A)) \subseteq wint_\alpha(f(A))$ for every $A \in I^X$.*

Proof. For every $A \in I^X$, we have

$$\begin{aligned} f(wint_\alpha(A)) &= f(\cup\{U \in I^X : U \in W_\alpha(\tau), U \subseteq A\}) \\ &\subseteq f(\cup\{U \in I^X : f(U) \in W_\alpha(\sigma), f(U) \subseteq f(A)\}) \\ &= \cup\{f(U) \in I^Y : U \in I^X, f(U) \in W_\alpha(\sigma), f(U) \subseteq f(A)\} \\ &\subseteq \cup\{K \in I^Y : K \in W_\alpha(\sigma), K \subseteq f(A)\} \\ &= wint_\alpha(f(A)). \end{aligned}$$

□

4. Several types of weak quasi-smooth α-compactness

In this section, we introduce the concepts of several types of weak quasi-smooth α-compactness in smooth topological spaces and investigate some of their properties.

Definition 4.1. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak smooth α-compact iff every family in $W_\alpha(\tau)$ covering X has a finite subcover.

Definition 4.2. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak quasi-smooth nearly α-compact iff for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X, there exists a finite subset J_0 of J such that

$$\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X.$$

Definition 4.3. Let $\alpha \in [0,1)$. A s.t.s. (X,τ) is called weak quasi-smooth almost α-compact iff for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_X$.

Theorem 4.4. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective and weak smooth α-continuous function with respect to τ and σ. If (X,τ) is weak smooth α-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $\cup_{i\in J} f^{-1}(A_i) = f^{-1}(1_Y) = 1_X$. Since $f : X \to Y$ is weak smooth α-continuous, $\{f^{-1}(A_i) : i \in J\} \subseteq W_\alpha(\tau)$. Since (X,τ) is weak smooth α-compact, there exists a finite subset $J_0 \subseteq J$ such that $\cup_{i\in J_0} f^{-1}(A_i) = 1_X$. From the surjectivity of f, we have $1_Y = f(1_X) = f(\cup_{i\in J_0} f^{-1}(A_i)) = \cup_{i\in J_0} f(f^{-1}(A_i)) = \cup_{i\in J_0} A_i$. Therefore (Y,σ) is weak smooth α-compact.

□

Theorem 4.5. *Let $\alpha \in [0,1)$. Then a weak smooth α-compact s.t.s. (X,τ) is weak quasi-smooth nearly α-compact.*

Proof. Let (X,τ) be a weak smooth α-compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$. Since $A_i \in W_\alpha(\tau)$ for each $i \in J$, $A_i = (A_i)^o_\alpha$ for each $i \in J$. Hence $A_i = wint_\alpha(A_i)$ for each $i \in J$ by Theorem 3.3. From Theorem 3.3 and 3.4 we have $wint_\alpha(A_i) \subseteq wint_\alpha(wcl_\alpha(A_i))$ for each $i \in J$. Thus

$$1_X = \cup_{i\in J_0} A_i = \cup_{i\in J_0} wint_\alpha(A_i) \subseteq \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)),$$

that is, $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$. Hence (X,τ) is weak quasi-smooth nearly α-compact.

□

THEOREM 4.6. *Let $\alpha \in [0,1)$. Then a weak quasi-smooth nearly α-compact s.t.s. (X,τ) is weak quasi-smooth almost α-compact.*

Proof. Let (X,τ) be a weak quasi-smooth nearly α-compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$. Since $wint_\alpha(wcl_\alpha(A_i)) \subseteq wcl_\alpha(A_i)$ for each $i \in J$ by Theorem 3.3, $1_X = \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) \subseteq \cup_{i\in J_0} wcl_\alpha(A_i)$. Thus $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_X$. Hence (X,τ) is weak quasi-smooth almost α-compact.

□

THEOREM 4.7. *Let (X,τ) and (Y,σ) be two smooth topological spaces, $\alpha \in [0,1)$ and $f : X \to Y$ a surjective and weak smooth α-continuous function with respect to τ and σ. If (X,τ) is weak quasi-smooth almost α-compact, then so is (Y,σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering Y, i.e., $\cup_{i\in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i\in J} f^{-1}(A_i)$. Since f is weak smooth α-continuous with respect to τ and σ, $f^{-1}(A_i) \in W_\alpha(\tau)$ for each $i \in J$. Since (X,τ) is weak quasi-smooth almost α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wcl_\alpha(f^{-1}(A_i)) = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i\in J_0} wcl_\alpha(f^{-1}(A_i))) = \cup_{i\in J_0} f(wcl_\alpha(f^{-1}(A_i)))$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, from Theorem 3.10 we have $wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))$ for every $A \in I^Y$. Hence

$$\begin{aligned} 1_Y &= \cup_{i\in J_0} f(wcl_\alpha(f^{-1}(A_i))) \subseteq \cup_{i\in J_0} f(f^{-1}(wcl_\alpha(A_i))) \\ &= \cup_{i\in J_0} wcl_\alpha(A_i), \end{aligned}$$

that is, $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_Y$. Thus (Y,σ) is weak quasi-smooth almost α-compact.

□

THEOREM 4.8. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective, weak smooth α-continuous and weak smooth α-open function with respect to τ and σ. If (X, τ) is weak quasi-smooth nearly α-compact, then so is (Y, σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering Y, i.e., $\cup_{i \in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)$. Since f is weak smooth α-continuous with respect to τ and σ, $f^{-1}(A_i) \in W_\alpha(\tau)$ for each $i \in J$. Since (X, τ) is weak quasi-smooth nearly α-compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} wint_\alpha(wcl_\alpha(f^{-1}(A_i))) = 1_X$. From the surjectivity of f we have

$$\begin{aligned} 1_Y = f(1_X) &= f(\cup_{i \in J_0} wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) \\ &= \cup_{i \in J_0} f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))). \end{aligned}$$

Since $f : X \to Y$ is weak smooth α-open with respect to τ and σ, from Theorem 3.12 we have

$$f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) \subseteq wint_\alpha(f(wcl_\alpha(f^{-1}(A_i))))$$

for each $i \in J$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, from Theorem 3.10 we have $wcl_\alpha(f^{-1}(A_i)) \subseteq f^{-1}(wcl_\alpha(A_i))$ for each $i \in J$. Hence we have

$$\begin{aligned} 1_Y &= \cup_{i \in J_0} f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) \\ &\subseteq \cup_{i \in J_0} wint_\alpha(f(wcl_\alpha(f^{-1}(A_i)))) \\ &\subseteq \cup_{i \in J_0} wint_\alpha(f(f^{-1}(wcl_\alpha(A_i)))) \\ &= \cup_{i \in J_0} wint_\alpha(wcl_\alpha(A_i)). \end{aligned}$$

Thus $\cup_{i \in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_Y$. Hence (Y, σ) is weak quasi-smooth nearly α-compact.

□

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