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PRICING CONVERTIBLE BONDS WITH KNOWN INTEREST RATE

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Abstract. In this paper, using the Black-Scholes analysis, we will derive the partial differential equation of convertible bonds with both non-stochastic and stochastic interest rate. We also find numerical solutions of convertible bonds equation with known interest rate using the finite element method.

1. Introduction

Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates. In last two decades, the value of trading in interest rate derivatives in both the over-counter and exchange-traded markets increased very quickly. Many new products were developed to meet particular needs of end users. A key challenge for derivatives traders is to find good, robust procedures for pricing and hedging these products. Warrants are long-term call options issued by firm that give the holder the right to purchase the firm's common stock at a predetermined price(the exercise price), on or before an expiration date. Convertible bonds are hybrid instruments, having characteristics of both debt and equity. Like straight bonds, convertible bonds are entitled to receive coupons and principal payments. However, convertible bondholder has the option to forgo these rights by converting their bonds into stock at a prespecified rate. In its simplest form, a convertible bond can be decomposed into a straight bonds and a warrant.

The theory of option and warrant pricing has only of late been placed on a sound theoretical basis in a context of security market equilibrium[2, 7]; closed form expressions have been derived by Black-Scholes[2] and

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Merton[8] for the value of an option when the underlying stock pays no dividend or the option is protected against dividends, and when the stock pays a continuous dividend which is proportional to the market value of the stock. Cox and Ross[5] has extended this option pricing model to take account of jumps in security returns, and the basic option pricing model has been shown to obtain under certain assumptions, even in the absence of continuous trading opportunities[9]. More recently, Schwartz developed algorithms to solve the relevant dynamic programming problem when the stock does pay dividends and the option is not protected against dividend payments, so that the possibility of exercise prior to maturity must be considered for an American type option. Merton[8] has considered the related problem of valuing callable warrants on non-dividend paying stocks : callable warrants differ form convertible bonds in having no coupon payments.

In recent years there have been a number of attempts to extend the models so that they involve two or more factors. A number of researchers have investigated the properties of two-factor equilibrium models. Brennan and Schwartz[3, 4] developed a model where the process for the short rate reverts to a long rate. Another two-factor model, proposed by Longstaff and Schwartz[7], starts with a general equilibrium model of the economy and derives a term structure model where there is stochastic volatility.

In section 2, using the Black-Scholes analysis, we derive the partial differential equation of convertible bonds under the assumption of the known interest rate. In section 3, we also derive convertible bonds equation with stochastic interest rate. In section 4, we find the numerical solutions of the convertible bond with known interest rate using the finite element method[1, 6].

2. Convertible bonds with known interest rate

In this section, we make preparations for the valuation of convertible bonds with assumption of known interest rate, these bonds are very similar to American vanilla option. We illustrate the ideas with constant interest rate and, at the section 3, we briefly bring together convertible bonds and stochastic interest rates in a two-factor model. A convertible bond has many of the same characteristics as an ordinary bond but with the additional feature that the bond may, at any time of the owner's choosing, be exchanged for a specified asset. This exchange is called conversion. The convertible bond on an underlying asset (with price S) returns Z, say, at time T unless at some previous time the owner has converted the bond into m(conversion ratio) of the underlying asset. The bond may also pay a coupon to the holder. Since the bond price depends on the value of that asset we have

$$V = V(S,t)$$

the contract value now depends on an asset price and the time to maturity. Repeating the Black-Scholes analysis, with a portfolio Π consisting of one convertible bond and $-\Delta$ assets under $dS = \mu S dt + \sigma S dX$

$$\Pi = V - \Delta S$$

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dX$$

where dX is the Wiener process and μ is the drift rate and σ is the volatility for the stock, respectively. Let we include a coupon payment of $K(S,t)dt$ on the bond and a dividend payment of $D(S,t)dt$ on the asset, then we find that the change in the value of the portfolio is

$$\begin{aligned} d\Pi &= dV - \Delta dS + (K(S,t) - \Delta D(S,t))dt \\ &= \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dX \\ &\quad - \Delta \mu S dt - \Delta \sigma S dX + (K(S,t) - \Delta D(S,t))dt. \end{aligned}$$

We can choose

$$\Delta = \frac{\partial V}{\partial S}$$

then

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + K(S,t) - \Delta D(S,t)\right) dt.$$

The return on this risk-free portfolio is equal that from a bank deposit and so

$$d\Pi = r\Pi dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S,t))\frac{\partial V}{\partial S} - rV + K(S,t) = 0$$

where r is the interest rate. This partial differential equation is recognized as the basic Black-Scholes equation but with the addition of the coupon payment term. The final condition is

$$V(S,T) = Z.$$

Recalling that the bond may be converted into m assets we have the constraint

$$V(S,t) \geq mS.$$

In addition to this constraint, we require the continuity of V and $\partial V/\partial S$. Thus the convertible bond is similar to an American option problem. It is interesting to note that the final data itself does not satisfy the pricing constraint. Thus, although the value at maturity may be Z the value just before is

$$\max(mS, Z).$$

Boundary conditions are

$$V(S,t) \sim mS \qquad \text{as} \qquad S \to \infty$$

and

$$V(0,t) = Z\exp\{-r(T-t)\}$$

this last condition assumes that it is not optimal to exercise when $S = 0$.

THEOREM 1. *If the variable S satisfies*

$$dS = \sigma S dX + \mu S dt$$

where dX is Wiener process and μ and σ are constant. Then the function $V(S,t)$ satisfies following partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (rS - D(S,t))\frac{\partial V}{\partial S} - rV + K(S,t) = 0$$

provided

$$V(S,t) = \begin{cases} Z & \text{if } \ t = T \\ \max\{mS, Z\} & \text{if } \ 0 < t < T \\ mS & \text{if } \ S \to \infty \\ Z\exp\{-r(T-t)\} & \text{if } \ S = 0 \end{cases}$$

$$V(S,t) \geq mS.$$

3. Convertible bonds with stochastic interest rate

When interest rates are stochastic, a convertible bond has a value of the form

$$V = V(S, r, t)$$

with dependence on T suppressed. The value of the convertible bond is now a function of both S, r and t. We assume that the asset price is governed by the standard model

$$dS = \mu S dt + \sigma S dX_1 \tag{1}$$

and the interest rate by

$$dr = u(r,t)dt + \omega(r,t)dX_2. \tag{2}$$

Since we are only modelling the convertible bond, and do not intend finding explicit solutions, we allow u and ω to be any functions of r and t. Observe that in (1) and (2) the Wiener processes have been given subscripts. This is because we are allowing S and r to be governed by two different random variables ; this is a two-factor model. Thus, although dX_1 and dX_2 are both drawn from normal distributions with zero mean and variance dt, they are not necessarily the same random variable. They are , however, correlated by

$$E[dX_1 dX_2] = \rho dt$$

with $-1 \leq \rho(S, r, t) \leq 1$.

In order to manipulate $V(S, r, t)$ we need to know how Itô's lemma applies to functions of two random variables. As might be expected, the usual Taylor series expansion together with a few rules of thumb results in the correct expression for the small change in any function of both S and r.

REMARK. The Wiener processes of (1) and (2) have the following properties

- $\triangleright$ $dX_1^2 = dt$
- $\triangleright$ $dX_2^2 = dt$
- $\triangleright$ $dX_1 dX_2 = \rho dt$.

Proof. By the Winner process

$$\delta X = \varepsilon \sqrt{\delta t}$$

where ε is drawn from a standardized normal distribution.

$$\begin{aligned} E[\delta X^2] &= \delta t E[\varepsilon^2] \\ &= \delta t \left(Var[\varepsilon] + (E[\varepsilon])^2\right) \\ &= \delta t \end{aligned}$$

$Var[\delta X^2] = Var[\varepsilon^2 \delta t] = \delta t^2 Var[\varepsilon^2]$ tends to zero as $\delta t \to 0$. Thus δX^2 is non-stochastic and $dX_1^2 = dt$. And

$$\begin{aligned} E[dX_1 dX_2] &= \rho dt \\ Var[dX_1 dX_2] &= E[(dX_1 dX_2)^2] - (E[dX_1 dX_2])^2 \\ &= \delta t^2 - \rho^2 \delta t^2 \\ &= (1-\rho)\delta t^2 \end{aligned}$$

$Var[dX_1 dX_2]$ tends to zero as $\delta t \to 0$. Thus $dX_1 dX_2$ also non-stochastic and $dX_1 dX_2 = \rho dt$. □

Applying Taylor's theorem to $V(S+dS, r+dr, t+dt)$ we find that

$$V(S+dS, r+dr, t+dt) = \sum_{i=0}^{\infty} \frac{1}{i!}\left(dS\frac{\partial}{\partial S} + dr\frac{\partial}{\partial r} + dt\frac{\partial}{\partial t}\right)^i V(S,r,t)$$

$$\begin{aligned} dV = &\frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial t}dt \\ &+ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 + \frac{\partial^2 V}{\partial S \partial r}dS dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}dr^2 + \cdots \end{aligned}$$

To leading order,

$$\begin{aligned} dS^2 &= \sigma^2 S^2 dX_1^2 = \sigma^2 S^2 dt \\ dr^2 &= \omega^2 dX_2^2 = \omega^2 dt \end{aligned}$$

and

$$dS dr = \sigma S \omega dX_1 dX_2 = \rho \sigma S \omega dt \ .$$

Thus Itô's lemma for the two random variables governed by (1) and (2) becomes

$$\begin{aligned} dV = &\frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial t}dt \\ &+ \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}dt + \rho\sigma S\omega\frac{\partial^2 V}{\partial S \partial r}dt + \frac{1}{2}\omega^2\frac{\partial^2 V}{\partial r^2}dt \ . \end{aligned}$$

Now we come to the pricing of the convertible bond. Let us construct a portfolio consisting of one bond with maturity T_1, $-\Delta_2$ bonds with maturity date T_2 and $-\Delta$ of the underlying asset. Thus

$$\begin{aligned}\Pi &= V_1 - \Delta_2 V_2 - \Delta S\\ d\Pi &= \left(\frac{\partial V_1}{\partial t} - \frac{\partial V_2}{\partial t}\right) dt + \left(\frac{\partial V_1}{\partial S} - \Delta_2 \frac{\partial V_2}{\partial S} - \Delta\right) dS\\ &\quad + \left(\frac{\partial V_1}{\partial r} - \Delta_2 \frac{\partial V_2}{\partial r}\right) dr + \frac{1}{2}\sigma^2 S^2 \left(\frac{\partial^2 V_1}{\partial S^2} - \Delta_2 \frac{\partial^2 V_2}{\partial S^2}\right) dt\\ &\quad + \rho\sigma S\omega \left(\frac{\partial^2 V_1}{\partial S \partial r} - \Delta_2 \frac{\partial^2 V_2}{\partial S \partial r}\right) dt + \frac{1}{2}\omega^2 \left(\frac{\partial^2 V_1}{\partial r^2} - \Delta_2 \frac{\partial^2 V_2}{\partial r^2}\right) dt.\end{aligned}$$

We can choose

$$\begin{aligned}\frac{\partial V_1}{\partial S} - \Delta_2 \frac{\partial V_2}{\partial S} - \Delta &= 0\\ \frac{\partial V_1}{\partial r} - \Delta_2 \frac{\partial V_2}{\partial r} &= 0\end{aligned}$$

so we find

$$\Delta_2 = \frac{\partial V_1 / \partial r}{\partial V_2 / \partial r}$$

and

$$\Delta = \frac{\partial V_1}{\partial S} - \frac{\partial V_1 / \partial r}{\partial V_2 / \partial r} \frac{\partial V_2}{\partial S}$$

eliminates risk from the portfolio. Now the portfolio is risk-free,

$$d\Pi = r\Pi dt$$

$$\begin{aligned}d\Pi &= r\left(V_1 - \Delta_2 V_2 - \Delta S\right) dt\\ &= \left(rV_1 - rS\frac{\partial V_1}{\partial S}\right) dt - \Delta_2 \left(rV_2 - rS\frac{\partial V_2}{\partial S}\right) dt\\ &= \left(\frac{\partial V_1}{\partial t} - \Delta_2 \frac{\partial V_2}{\partial t}\right) dt + \frac{1}{2}\sigma^2 S^2 \left(\frac{\partial^2 V_1}{\partial S^2} - \Delta_2 \frac{\partial^2 V_2}{\partial S^2}\right) dt\\ &\quad + \rho\sigma S\omega \left(\frac{\partial^2 V_1}{\partial S \partial r} - \Delta_2 \frac{\partial^2 V_2}{\partial S \partial r}\right) dt + \frac{1}{2}\omega^2 \left(\frac{\partial^2 V_1}{\partial r^2} - \Delta_2 \frac{\partial^2 V_2}{\partial r^2}\right) dt.\end{aligned}$$

Gathering together all V_1 terms on the left-hand side and all V_2 terms on the right-hand side we find that

$$\left(\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_1}{\partial S^2}+\rho\sigma S\omega\frac{\partial^2 V_1}{\partial S\partial r}+\frac{1}{2}\omega^2\frac{\partial^2 V_1}{\partial r^2}-rV_1+rS\frac{\partial V_1}{\partial S}\right)$$
$$=\Delta_2\left(\frac{\partial V_2}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_2}{\partial S^2}+\rho\sigma S\omega\frac{\partial^2 V_2}{\partial S\partial r}+\frac{1}{2}\omega^2\frac{\partial^2 V_2}{\partial r^2}-rV_2+rS\frac{\partial V_2}{\partial S}\right)$$

Let

$$I_a=\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_1}{\partial S^2}+\rho\sigma S\omega\frac{\partial^2 V_1}{\partial S\partial r}+\frac{1}{2}\omega^2\frac{\partial^2 V_1}{\partial r^2}-rV_1+rS\frac{\partial V_1}{\partial S}$$
$$I_b=\frac{\partial V_2}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_2}{\partial S^2}+\rho\sigma S\omega\frac{\partial^2 V_2}{\partial S\partial r}+\frac{1}{2}\omega^2\frac{\partial^2 V_2}{\partial r^2}-rV_2+rS\frac{\partial V_2}{\partial S}$$

then

$$I_a=\frac{\partial V_1/\partial r}{\partial V_2/\partial r}I_b$$
$$\frac{I_a}{\partial V_1/\partial r}=\frac{I_b}{\partial V_2/\partial r}.$$

This is one equation in two unknowns. However, the left-hand side is a function of T_1 and the right-hand side is a function of T_2. The only way for this to be possible is for both side to be independent of the maturity date. Thus, dropping the subscript from V,

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S\omega\frac{\partial^2 V}{\partial S\partial r}+\frac{1}{2}\omega^2\frac{\partial^2 V}{\partial r^2}-rV+rS\frac{\partial V}{\partial S}=\frac{\partial V}{\partial r}\cdot a(S,r,t)$$

for some function $a(S,r,t)$. It is convenient to write

$$a(S,r,t)=\omega(r,t)\lambda(S,r,t)-u(r,t)$$

for given $\omega(r,t)$ (nonzero) and $u(r,t)$, this is always possible. The function $\lambda(S,r,t)$ is the market price of risk.

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S\omega\frac{\partial^2 V}{\partial S\partial r}+\frac{1}{2}\omega^2\frac{\partial^2 V}{\partial r^2}+rS\frac{\partial V}{\partial S}+(u-\omega\lambda)\frac{\partial V}{\partial r}-rV=0.$$

This is the convertible bond pricing equation. Note that it contains the known interest rate problem

- ▷ $u=0=\omega$: Black-Scholes equation.
- ▷ $\partial/\partial S=0$: The simple bond problem (zero-coupon bond).

More generally, when the underlying asset pays dividends and the bond pays a coupon we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S\omega \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}\omega^2 \frac{\partial^2 V}{\partial r^2}$$
$$+ (rS - D)\frac{\partial V}{\partial S} + (u - \omega\lambda)\frac{\partial V}{\partial r} - rV + K = 0.$$

Since this is a diffusion equation with two 'space-like' state variables S and r—that is, there are double derivatives of V with respect to each of S and r, as well as a cross-term-we need to impose boundary conditions on the edge of the (S, r) space. In other words, we must prescribe $V(0, r, t)$ and $V(\infty, r, t)$ for all t, $V(S, \infty, t)$ for all S and t and a second boundary condition on a fixed r boundary, again for all S and r.

Some of these boundary conditions are very obvious and others are result of insisting that V remain finite. For example, for a convertible bond with on call feature we have

$$V(S, r, t) \sim mS \quad \text{as} \quad S \to \infty$$

$V(0, r, t)$ is given by the solution of the simple bond problem (no convertibility and stochastic interest rates).

$$V(S, r, t) \to 0 \quad \text{as} \quad r \to \infty$$

and the last boundary condition, to be applied on the lower r boundary, is equivalent to finiteness of V.

THEOREM 2. *If the variable S and r satisfy the equation (1) and (2) respectively, then the function $V(S, r, t)$ satisfies following equation*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S\omega \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}\omega^2 \frac{\partial^2 V}{\partial r^2}$$
$$+ (rS - D)\frac{\partial V}{\partial S} + (u - \omega\lambda)\frac{\partial V}{\partial r} - rV + K = 0$$

provided

$$V(S, r, t) = \begin{cases} \textit{bond conditions} & \textit{if} \quad S = 0 \\ \textit{American call option conditions} & \textit{if} \quad r = 0 \\ mS & \textit{if} \quad S \to \infty \\ \max\{mS, \textit{bond price}\} & \textit{if} \quad 0 < t < T \\ 0 & \textit{if} \quad r \to \infty \end{cases}$$

$V(S, r, t) \geq mS$.

4. Numerical solutions for convertible bonds

In this section we find the numerical solutions of convertible bonds under the assumption of the known interest rate using the finite element method. Suppose the convertible bond pricing equation is as following

$$\frac{\partial V(S,t)}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V(S,t)}{\partial S^2}+(r-d)\,S\frac{\partial V(S,t)}{\partial S}-rV(S,t)+K=0 \tag{3}$$

where r, d(dividend yield) and K are constants and

$$a \le S \le b\ ,\ \ 0 \le t \le T.$$

Boundary conditions are

$$\begin{aligned} V(S,T) &= \max\{mS, Z\} \\ V(0,t) &= Z\exp\{-r(T-t)\} \\ \frac{\partial V}{\partial S}(b,t) &= m \end{aligned}$$

with constraint

$$V(S,t) \ge mS.$$

Now we represent the differential term of time to difference of time, then the equation (3) becomes

$$\begin{aligned} &\left(\frac{1}{\Delta t}+r\right)V(S,t_{n-1})-\frac{1}{2}\sigma^2 S^2 V''(S,t_{n-1}) \\ &-(r-d)SV'(S,t_{n-1})-\frac{1}{\Delta t}V(S,t_n)-K=0 \end{aligned}$$

where $\Delta t = t_i - t_{i-1}$ $(0 = t_0 < t_1 < \cdots < t_n = T)$ and

$$\frac{\partial V(S,t)}{\partial S}=V'(S,t)\ \ ,\ \ \frac{\partial^2 V(S,t)}{\partial S^2}=V''(S,t).$$

Define

$$\begin{aligned} u(S) &:= V(S,t_{n-1}) \\ \overline{u}(S) &:= V(S,t_n) \end{aligned}$$

then the equation (3) becomes

$$\left(\frac{1}{\Delta t}+r\right)u(S)-\frac{1}{2}\sigma^2S^2u''(S)-(r-d)Su'(S)-\frac{1}{\Delta t}\overline{u}\,(S)-K=0. \tag{4}$$

Since t is not a variable from now on, u is a function with only variable S.

Suppose $\Psi(S)\,(\in C^1\,[a,b])$ is a test function with compact supprt on $[a,b]$, we multiple the test function to both side of the equation (4) and integrate

$$\begin{aligned}&\left(\frac{1}{\Delta t}+r\right)\int_a^b u(S)\Psi(S)dS-\frac{1}{2}\sigma^2\int_a^b S^2u''(S)\Psi(S)dS\\&-(r-d)\int_a^b Su'(S)\Psi(S)dS-\frac{1}{\Delta t}\int_a^b \overline{u}\,(S)\,\Psi(S)dS\\&-K\int_a^b \Psi(S)dS=0. \end{aligned}\tag{5}$$

Because equation (5) is zero for all test function $\Psi(S)$, equation (3) and (5) have the same solution $u(S)$. We find the solution $u(S)$ of the equation (3) by solving the integral equation (5) with proper basis functions and test functions. First of all, we define two spaces

$$P=\left\{f\,(S)\mid f\,(S)=\sum_{k=0}^{2}c_kS^k,\ c_k\in\mathbb{R},\ a\le S\le b\right\}$$
$$Q=\{g\,(S)\in P\mid g\,(a)=0=g\,(b)\,,\ g\,(a_i)=1\}$$

where $a_i=a+ih$ $(i=0,1,2,\cdots,N)$ and $h=(b-a)/2k$ $(N=2k)$. We can choose the test function $\Psi_i(S)$ and the basis function $\Psi_j(S)$ on Q as following

Case I. i is even

$$\begin{aligned}\Psi_{i-2}(S)&=\frac{1}{2h^2}\,(S-a_{i-1})\,(S-a_i)\\ \Psi_{i-1}(S)&=-\frac{1}{h^2}\,(S-a_{i-2})\,(S-a_i)\\ \Psi_{i\_l}(S)&=\frac{1}{2h^2}\,(S-a_{i-2})\,(S-a_{i-1})\end{aligned}$$

when

$$a_{i-2}\le S\le a_i$$

$$\Psi_{i\_r}(S) = \frac{1}{2h^2}(S - a_{i+1})(S - a_{i+2})$$
$$\Psi_{i+1}(S) = -\frac{1}{h^2}(S - a_i)(S - a_{i+2})$$
$$\Psi_{i+2}(S) = \frac{1}{2h^2}(S - a_i)(S - a_{i+1})$$

when

$$a_i \le S \le a_{i+2}$$

Case II. i is odd

$$\Psi_{i-1}(S) = \frac{1}{2h^2}(S - a_i)(S - a_{i+1})$$
$$\Psi_{i\_0}(S) = -\frac{1}{h^2}(S - a_{i-1})(S - a_{i+1})$$
$$\Psi_{i+1}(S) = \frac{1}{2h^2}(S - a_{i-1})(S - a_i)$$

when

$$a_{i-1} \le S \le a_{i+1}.$$

Finding analytic solutions of equation (3) is very difficult, so we look for the numerical solutions. Suppose the approximated solution of $u(S)$ is $u_h(S)$,

$$u(S) \simeq u_h(S) = \sum_{j=0}^{N} \alpha_j^{n-1} \Psi_j(S)$$

But we know the solution when $j = 0$,

$$u_h(S) = \sum_{j=1}^{N} \alpha_j^{n-1} \Psi_j(S)$$

Our goal is to find the coefficient α_j^{n-1}. To simplify the notation, we define

$$(f, g)_h := \int_a^b f_h(S)\, g(S)\, dS$$

Now we calculate each terms of equation (5) using chosen test and basis functions.

$$(u, \Psi_i)_h = \sum_{j=1}^{N-1} \alpha_j^{n-1} \int_a^b \Psi_j(S)\Psi_i(S) dS$$

CaseI. i is even

$$
\begin{aligned}
&= \sum_{j=i-2}^{i} \alpha_j^{n-1} \int_{a_{i-2}}^{a_i} \Psi_j(S)\Psi_i(S)dS \\
&\quad + \sum_{j=i}^{i+2} \alpha_j^{n-1} \int_{a_i}^{a_{i+2}} \Psi_j(S)\Psi_i(S)dS \\
&= -\frac{h}{15}\alpha_{i-2}^{n-1} + \frac{2h}{15}\alpha_{i-1}^{n-1} + \frac{8h}{15}\alpha_i^{n-1} + \frac{2h}{15}\alpha_{i+1}^{n-1} - \frac{h}{15}\alpha_{i+2}^{n-1}
\end{aligned}
$$

CaseII. i is odd

$$
\begin{aligned}
&= \sum_{j=i-1}^{i+1} \alpha_j^{n-1} \int_{a_{i-1}}^{a_{i+1}} \Psi_j(S)\Psi_i(S)dS \\
&= \frac{2h}{15}\alpha_{i-1}^{n-1} + \frac{16h}{15}\alpha_i^{n-1} + \frac{2h}{15}\alpha_{i+1}^{n-1}
\end{aligned}
$$

CaseIII. i is N

$$
\begin{aligned}
&= \sum_{j=N-2}^{N} \alpha_j^{n-1} \int_{a_{N-2}}^{a_N} \Psi_j(S)\Psi_i(S)dS \\
&= -\frac{h}{15}\alpha_{N-2}^{n-1} + \frac{2h}{15}\alpha_{N-1}^{n-1} + \frac{4h}{15}\alpha_N^{n-1}.
\end{aligned}
$$

$$
\begin{aligned}
\left(S^2 u'', \Psi_i\right)_h &= \left[u'(S)S^2\Psi_i(S)\right]_a^b \\
&\quad - \sum_{j=1}^{N-1} \alpha_j^{n-1} \int_a^b \Psi_j'(S)\left\{2S\Psi_i(S) + S^2\Psi_i'(S)\right\} dS
\end{aligned}
$$

CaseI. i is even

$$
\begin{aligned}
&= - \sum_{j=i-2}^{i} \alpha_j^{n-1} \int_{a_{i-2}}^{a_i} \Psi_j'(S) \left\{ 2S\Psi_i(S) + S^2 \Psi_i'(S) \right\} dS \\
&\quad - \sum_{j=i}^{i+2} \alpha_j^{n-1} \int_{a_i}^{a_{i+2}} \Psi_j'(S) \left\{ 2S\Psi_i(S) + S^2 \Psi_i'(S) \right\} dS \\
&= - \frac{1}{30h} \left(5a^2 + 10aih + h^2 \left(4 + 5i^2 \right) \right) \alpha_{i-2}^{n-1} \\
&\quad + \frac{4}{15h} \left(5a^2 + 10aih + h^2 \left(1 + 5i^2 \right) \right) \alpha_{i-1}^{n-1} \\
&\quad - \frac{1}{15h} \left(35a^2 + 70aih + h^2 \left(4 + 35i^2 \right) \right) \alpha_{i}^{n-1} \\
&\quad + \frac{4}{15h} \left(5a^2 + 10aih + h^2 \left(1 + 5i^2 \right) \right) \alpha_{i+1}^{n-1} \\
&\quad - \frac{1}{30h} \left(5a^2 + 10aih + h^2 \left(4 + 5i^2 \right) \right) \alpha_{i+2}^{n-1}
\end{aligned}
$$

CaseII. i is odd

$$
\begin{aligned}
&= - \sum_{j=i-1}^{i+1} \alpha_j^{n-1} \int_{a_{i-1}}^{a_{i+1}} \Psi_j'(S) \left\{ 2S\Psi_i(S) + S^2 \Psi_i'(S) \right\} dS \\
&= \frac{4}{15h} \left(5a^2 + 10aih + h^2 \left(1 + 5i^2 \right) \right) \alpha_{i-1}^{n-1} \\
&\quad - \frac{8}{15h} \left(5a^2 + 10aih + h^2 \left(1 + 5i^2 \right) \right) \alpha_{i}^{n-1} \\
&\quad + \frac{4}{15h} \left(5a^2 + 10aih + h^2 \left(1 + 5i^2 \right) \right) \alpha_{i+1}^{n-1}
\end{aligned}
$$

CaseIII. i is N

$$
\begin{aligned}
&= mb^2 - \sum_{j=N-2}^{N} \alpha_j^{n-1} \int_{a_{N-2}}^{a_N} \Psi_j'(S) \left\{ 2S\Psi_N(S) + S^2 \Psi_N'(S) \right\} dS \\
&= mb^2 - \frac{1}{30h} \left(5a^2 + 10aNh + h^2 \left(4 + 5N^2 \right) \right) \alpha_{N-2}^{n-1} \\
&\quad + \frac{4}{15h} \left(5a^2 + 10aNh + h^2 \left(1 + 5N^2 \right) \right) \alpha_{N-1}^{n-1} \\
&\quad - \frac{1}{30h} \left(35a^2 + 70aNh + h^2 \left(4 + 35N^2 \right) \right) \alpha_{N}^{n-1}
\end{aligned}
$$

$$(Su', \Psi_i)_h = \sum_{j=1}^{N-1} \alpha_j^{n-1} \int_a^b S\Psi_j'(S)\Psi_i(S)dS$$

CaseI. i is even

$$\begin{aligned}
&= \sum_{j=i-2}^{i} \alpha_j^{n-1} \int_{a_{i-2}}^{a_i} S\Psi_j'(S)\Psi_i(S)dS \\
&\quad + \sum_{j=i}^{i+2} \alpha_j^{n-1} \int_{a_i}^{a_{i+2}} S\Psi_j'(S)\Psi_i(S)dS \\
&= \frac{1}{30}\left(5a + h\left(-4 + 5i\right)\right)\alpha_{i-2}^{n-1} \\
&\quad - \frac{2}{15}\left(5a + h\left(-2 + 5i\right)\right)\alpha_{i-1}^{n-1} \\
&\quad - \frac{4h}{15}\alpha_i^{n-1} \\
&\quad + \frac{2}{15}\left(5a + h\left(2 + 5i\right)\right)\alpha_{i+1}^{n-1} \\
&\quad - \frac{1}{30}\left(5a + h\left(4 + 5i\right)\right)\alpha_{i+2}^{n-1}
\end{aligned}$$

CaseII. i is odd

$$\begin{aligned}
&= \sum_{j=i-1}^{i+1} \alpha_j^{n-1} \int_{a_{i-1}}^{a_{i+1}} S\Psi_j'(S)\Psi_i(S)dS \\
&= -\frac{2}{15}\left(5a + h\left(-2 + 5i\right)\right)\alpha_{i-1}^{n-1}
\end{aligned}$$

$$\begin{aligned}
&\quad - \frac{8h}{15}\alpha_i^{n-1} \\
&\quad + \frac{2}{15}\left(5a + h\left(2 + 5i\right)\right)\alpha_{i+1}^{n-1}
\end{aligned}$$

CaseIII. i is N

$$= \sum_{j=N-2}^{N} \alpha_j^{n-1} \int_{a_{N-2}}^{a_N} S\Psi_j'(S)\Psi_i(S)dS$$

$$= \frac{1}{30}\left(5a + h\left(-4+5N\right)\right)\alpha_{N-2}^{n-1}$$

$$- \frac{2}{15}\left(5a + h\left(-2+5N\right)\right)\alpha_{N-1}^{n-1}$$

$$+ \frac{1}{30}\left(15a + h\left(-4+15N\right)\right)\alpha_{N}^{n-1}.$$

$$(1, \Psi_i)_h = \int_a^b \Psi_i(S)dS$$

CaseI. i is even

$$= \int_{a_{i-2}}^{a_{i+2}} \Psi_i(S)dS$$

$$= \frac{2h}{3}$$

CaseII. i is odd

$$= \int_{a_{i-1}}^{a_{i+1}} \Psi_i(S)dS$$

$$= \frac{4h}{3}$$

CaseIII. i is N

$$= \int_{a_{N-2}}^{a_N} \Psi_i(S)dS$$

$$= \frac{h}{3}.$$

For all cases i, the equation (5) becomes

$$\left(\frac{1}{\Delta t} + r\right)(u, \Psi_i)_h - \frac{1}{2}\sigma^2\left(S^2u'', \Psi_i\right)_h - (r-d)\left(Su', \Psi_i\right)_h$$

$$- \frac{1}{\Delta t}\left(\overline{u}, \Psi_i\right)_h - K\left(1, \Psi_i\right)_h = 0. \qquad (6)$$

In Table is shown that the value of a convertible bonds with $Z = 1$, $n = 1$, $r = 0.1$, $\sigma = 0.25$ and with one year before maturity. In both cases there are no coupon payments. We can find that the price of convertible bonds with no dividend(d) is higher than the price with $d = 0.05$. It can be shown that an increase in d makes early exercise more likely.

Stock Price	CB Price($d = 0.00$)	CB Price($d = 0.05$)
0.0	0.90483742	0.90483742
0.2	0.90484194	0.90484194
0.4	0.90486434	0.90485225
0.6	0.90869386	0.90720473
0.8	0.94723899	0.93631915
1.0	1.05458760	1.03230021
1.2	1.21677246	1.20003931
1.4	1.40457068	1.40000000
1.6	1.60116540	1.60000000
1.8	1.80030572	1.80000000
2.0	2.00014964	2.00000000

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