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WEAK* QUASI-SMOOTH α-STRUCTURE OF SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper we introduce the concepts of several types of weak* quasi-smooth α-compactness in terms of the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang's fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. Particularly, Gayyar, Kerre and Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth α-closure and smooth α-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and also introduced several types of α-compactness in smooth topological spaces and obtained some of their properties. In [7] we introduced the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and investigated some of their properties.

In this paper we introduce the concepts of several types of weak* quasi-smooth α-compactness in terms of the concepts of weak smooth α-closure and weak smooth α-interior of a fuzzy set in smooth topological spaces and investigate some of their properties.

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2. Preliminaries

Let X be a set and $I = [0, 1]$ be the unit interval of the real line. I^X will denote the set of all fuzzy sets of X. 0_X and 1_X will denote the characteristic functions of ϕ and X, respectively.

A smooth topological space (s.t.s.) [8] is an ordered pair (X, τ), where X is a non-empty set and $\tau : I^X \to I$ is a mapping satisfying the following conditions:

(O1) $\tau(0_X) = \tau(1_X) = 1$;

(O2) $\forall A, B \in I^X,\ \tau(A \cap B) \geq \tau(A) \wedge \tau(B)$;

(O3) for any subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$.

Then the mapping $\tau : I^X \to I$ is called a smooth topology on X. The number $\tau(A)$ is called the degree of openness of A.

A mapping $\tau^* : I^X \to I$ is called a smooth cotopology [8] if the following three conditions are satisfied:

(C1) $\tau^*(0_X) = \tau^*(1_X) = 1$;

(C2) $\forall A, B \in I^X,\ \tau^*(A \cup B) \geq \tau^*(A) \wedge \tau^*(B)$;

(C3) for every subfamily $\{A_i : i \in J\} \subseteq I^X$, $\tau^*(\cap_{i\in J} A_i) \geq \wedge_{i\in J} \tau^*(A_i)$.

If τ is a smooth topology on X, then the mapping $\tau^* : I^X \to I$, defined by $\tau^*(A) = \tau(A^c)$ where A^c denotes the complement of A, is a smooth cotopology on X. Conversely, if τ^* is a smooth cotopology on X, then the mapping $\tau : I^X \to I$, defined by $\tau(A) = \tau^*(A^c)$, is a smooth topology on X [8].

Demirci [3] introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let (X, τ) be a s.t.s. and $A \in I^X$. Then the τ-smooth closure (resp., τ-smooth interior) of A, denoted by $\bar{A}$ (resp., A^o), is defined by $\bar{A} = \cap\{K \in I^X : \tau^*(K) > 0, A \subseteq K\}$ (resp., $A^o = \cup\{K \in I^X : \tau(K) > 0, K \subseteq A\}$). Demirci [4] defined the families $W(\tau) = \{A \in I^X : A = A^o\}$ and $W^*(\tau) = \{A \in I^X : A = \overline{A}\}$, where (X, τ) is a s.t.s. Note that $A \in W(\tau)$ if and only if $A^c \in W^*(\tau)$.

Let (X, τ) and (Y, σ) be two smooth topological spaces. A function $f : X \to Y$ is called smooth continuous with respect to τ and σ [8] if $\tau(f^{-1}(A)) \geq \sigma(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is called weakly smooth continuous with respect to τ and σ [8] if $\sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0$ for every $A \in I^Y$. In this paper, a weakly smooth continuous function with respect to τ and σ is called a quasi-smooth continuous function with respect to τ and σ.

A function $f : X \to Y$ is smooth continuous with respect to τ and σ if and only if $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to τ and σ if and only if $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [8].

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to τ and σ [8] if

$$\tau(A) \leq \sigma(f(A)) \text{ (resp., } \tau^*(A) \leq \sigma^*(f(A)))$$

for every $A \in I^X$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to τ and σ [5] if

$$\sigma(A) \geq \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$$

$$\text{(resp., } \sigma(A) > \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \tau(f^{-1}(B)))$$

for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to τ and σ, then $\sigma^*(A) \geq \sigma^*(B)$ if and only if $\tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B)$ if and only if $\tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to τ and σ [5] if $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A, B \in I^X$.

Let (X, τ) be a s.t.s., $\alpha \in [0, 1)$ and $A \in I^X$. The τ-smooth α-closure (resp., τ-smooth α-interior) of A, denoted by $\overline{A}_\alpha$ (resp., A^o_α), is defined by $\overline{A}_\alpha = \cap\{K \in I^X : \tau^*(K) > \alpha\tau^*(A), A \subseteq K\}$ (resp., $A^o_\alpha = \cup\{K \in I^X : \tau(K) > \alpha\tau(A), K \subseteq A\}$) [6]. In [7] we defined the families $W_\alpha(\tau) = \{A \in I^X : A = A^o_\alpha\}$ and $W^*_\alpha(\tau) = \{A \in I^X : A = \overline{A}_\alpha\}$, where (X, τ) is a s.t.s. Note that $A \in W_\alpha(\tau) \Leftrightarrow A^c \in W^*_\alpha(\tau)$.

3. Types of weak* quasi-smooth α-compactness

In this section, we introduce the concepts of several types of weak* quasi-smooth α-compactness in smooth topological spaces and investigate some of their properties.

Definition 3.1[7]. Let (X, τ) be a s.t.s., $\alpha \in [0,1)$ and $A \in I^X$. The weak τ-smooth α-closure (resp., weak τ-smooth α-interior) of A, denoted by $wcl_\alpha(A)$ (resp., $wint_\alpha(A)$), is defined by $wcl_\alpha(A) = \cap\{K \in I^X : K \in W^*_\alpha(\tau), A \subseteq K\}$ (resp., $wint_\alpha(A) = \cup\{K \in I^X : K \in W_\alpha(\tau), K \subseteq A\}$).

We define the families $W_{w\alpha}(\tau) = \{A \in I^X : A = wint_\alpha(A)\}$ and $W^*_{w\alpha}(\tau) = \{A \in I^X : A = wcl_\alpha(A)\}$, where (X, τ) is a s.t.s. and $\alpha \in [0,1)$. Then

$A \in W_{w\alpha}(\tau) \Leftrightarrow A^c \in W^*_{w\alpha}(\tau)$,

$A \in W_\alpha(\tau) \Rightarrow A \in W(\tau) \Rightarrow A \in W_{w\alpha}(\tau)$,

$A \in W^*_\alpha(\tau) \Rightarrow A \in W^*(\tau) \Rightarrow A \in W^*_{w\alpha}(\tau)$.

Definition 3.2[7]. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0,1)$. A function $f : X \to Y$ is called weak smooth α-continuous with respect to τ and σ if $A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)$ for every $A \in I^Y$.

Definition 3.3. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0,1)$. A function $f : X \to Y$ is called weak* smooth α-continuous with respect to τ and σ if $A \in W_{w\alpha}(\sigma) \Rightarrow f^{-1}(A) \in W_{w\alpha}(\tau)$ for every $A \in I^Y$.

Let (X, τ) and (Y, σ) be two smooth topological spaces. A function $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ if and only if $A \in W^*_{w\alpha}(\sigma) \Rightarrow f^{-1}(A) \in W^*_{w\alpha}(\tau)$ for every $A \in I^Y$.

Definition 3.4. Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0,1)$. A function $f : X \to Y$ is called weak* smooth α-open (resp., weak* smooth α-closed) with respect to τ and σ if $A \in W_{w\alpha}(\tau) \Rightarrow f(A) \in W_{w\alpha}(\sigma)$ (resp., $A \in W^*_{w\alpha}(\tau) \Rightarrow f(A) \in W^*_{w\alpha}(\sigma)$) for every $A \in I^X$.

Theorem 3.5. *Let (X, τ) and (Y, σ) be two smooth topological spaces and let $\alpha \in [0,1)$. If a function $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, then $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ.*

Proof. Let $f : X \to Y$ be a weak smooth α-continuous function with respect to τ and σ. Then by Theorem 3.10[7] $f^{-1}(wint_\alpha(A)) \subseteq wint_\alpha(f^{-1}(A))$ for every $A \in I^Y$. Let $A \in W_{w\alpha}(\sigma)$, i.e, $A = wint_\alpha A$. Then $f^{-1}(A) = f^{-1}(wint_\alpha A) \subseteq wint_\alpha(f^{-1}(A))$. From the definition of weak smooth α-interior we have $wint_\alpha(f^{-1}(A)) \subseteq f^{-1}(A)$. Hence $f^{-1}(A) = wint_\alpha(f^{-1}(A)$, i.e., $f^{-1}(A) \in W_{w\alpha}(\tau)$. Therefore $f : X \to Y$ is weak* smooth α-continuous with respect to τ and σ.

□

Definition 3.6. Let $\alpha \in [0, 1)$. A s.t.s. (X, τ) is called weak* quasi-smooth nearly α-compact if for every family $\{A_i : i \in J\}$ in $W_{w\alpha}(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$.

Definition 3.7. Let $\alpha \in [0, 1)$. A s.t.s. (X, τ) is called weak* quasi-smooth almost α-compact if for every family $\{A_i : i \in J\}$ in $W_{w\alpha}(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wcl_\alpha(A_i) = 1_X$.

Note that (X, τ) is weak* quasi-smooth almost α-compact $\Rightarrow (X, \tau)$ is weak* smooth almost compact $\Rightarrow (X, \tau)$ is weak* smooth almost α-compact.

Theorem 3.8. *Let (X, τ) be a s.t.s. and let $\alpha \in [0, 1)$. If (X, τ) is weak* smooth compact, then (X, τ) is weak* quasi-smooth nearly α-compact.*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\tau)$ covering X. Since (X, τ) is weak* smooth compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} A_i = 1_X$. Since $A_i \in W_{w\alpha}(\tau)$ for each $i \in J$, $A_i = wint_\alpha(A_i)$ for each $i \in J$. From Theorem 3.3 and 3.4[7] we have $wint_\alpha(A_i) \subseteq wint_\alpha(wcl_\alpha(A_i))$ for each $i \in J$. Thus $1_X = \cup_{i\in J_0} A_i = \cup_{i\in J_0} wint_\alpha(A_i) \subseteq \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i))$, i.e., $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$. Hence (X, τ) is weak* quasi-smooth nearly α-compact.

□

Theorem 3.9. *Let $\alpha \in [0, 1)$. Then a weak* quasi-smooth nearly α-compact s.t.s. (X, τ) is weak* quasi-smooth almost α-compact.*

Proof. Let (X, τ) be a weak* quasi-smooth nearly α-compact s.t.s. Then for every family $\{A_i : i \in J\}$ in $W_{w\alpha}(\tau)$ covering X, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_X$. Since $wint_\alpha(wcl_\alpha(A_i)) \subseteq wcl_\alpha(A_i)$ for each $i \in J$ by Theorem 3.3[7], $1_X = \cup_{i \in J_0} wint_\alpha(wcl_\alpha(A_i)) \subseteq \cup_{i \in J_0} wcl_\alpha(A_i)$. Thus $\cup_{i \in J_0} wcl_\alpha(A_i) = 1_X$. Hence (X, τ) is weak* quasi-smooth almost α-compact.

□

THEOREM 3.10. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective and weak smooth α-continuous function with respect to τ and σ. If (X, τ) is weak* quasi-smooth almost α-compact, then so is (Y, σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\sigma)$ covering Y, i.e., $\cup_{i \in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)$. Since f is weak smooth α-continuous with respect to τ and σ, f is weak* smooth α-continuous with respect to τ and σ by Theorem 3.5. Hence $f^{-1}(A_i) \in W_{w\alpha}(\tau)$ for each $i \in J$. Since (X, τ) is weak* quasi-smooth almost α-compact, there exists a finite subset J_0 of J such that $\cup_{i \in J_0} wcl_\alpha(f^{-1}(A_i)) = 1_X$. From the surjectivity of f we have $1_Y = f(1_X) = f(\cup_{i \in J_0} wcl_\alpha(f^{-1}(A_i))) = \cup_{i \in J_0} f(wcl_\alpha(f^{-1}(A_i)))$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, from Theorem 3.10[7] we have $wcl_\alpha(f^{-1}(A)) \subseteq f^{-1}(wcl_\alpha(A))$ for every $A \in I^Y$. Hence $1_Y = \cup_{i \in J_0} f(wcl_\alpha(f^{-1}(A_i))) \subseteq \cup_{i \in J_0} f(f^{-1}(wcl_\alpha(A_i))) = \cup_{i \in J_0} wcl_\alpha(A_i)$, i.e., $\cup_{i \in J_0} wcl_\alpha(A_i) = 1_Y$. Thus (Y, σ) is weak* quasi-smooth almost α-compact.

□

THEOREM 3.11. *Let (X, τ) and (Y, σ) be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective, weak smooth α-continuous and weak smooth α-open function with respect to τ and σ. If (X, τ) is weak* quasi-smooth nearly α-compact, then so is (Y, σ).*

Proof. Let $\{A_i : i \in J\}$ be a family in $W_{w\alpha}(\sigma)$ covering Y, i.e., $\cup_{i \in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)$. Since f is weak smooth α-continuous with respect to τ and σ, f is weak* smooth α-continuous with respect to τ and σ by Theorem 3.5. Hence $f^{-1}(A_i) \in W_{w\alpha}(\tau)$ for each $i \in J$. Since (X, τ) is weak* quasi-smooth nearly

α-compact, there exists a finite subset J_0 of J such that $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(f^{-1}(A_i))) = 1_X$. From the surjectivity of f we have

$$\begin{aligned}1_Y &= f(1_X) = f(\cup_{i\in J_0} wint_\alpha(wcl_\alpha(f^{-1}(A_i))))\\ &= \cup_{i\in J_0} f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))).\end{aligned}$$

Since $f : X \to Y$ is weak smooth α-open with respect to τ and σ, from Theorem 3.12[7] we have

$$f(wint_\alpha(wcl_\alpha(f^{-1}(A_i)))) \subseteq wint_\alpha(f(wcl_\alpha(f^{-1}(A_i))))$$

for each $i \in J$. Since $f : X \to Y$ is weak smooth α-continuous with respect to τ and σ, from Theorem 3.10[7] we have $wcl_\alpha(f^{-1}(A_i)) \subseteq f^{-1}(wcl_\alpha(A_i))$ for each $i \in J$. Hence we have

$$\begin{aligned}1_Y &= \cup_{i\in J_0} f(wint_\alpha(wcl_\alpha(f^{-1}(A_i))))\\ &\subseteq \cup_{i\in J_0} wint_\alpha(f(wcl_\alpha(f^{-1}(A_i))))\\ &\subseteq \cup_{i\in J_0} wint_\alpha(f(f^{-1}(wcl_\alpha(A_i))))\\ &= \cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)).\end{aligned}$$

Thus $\cup_{i\in J_0} wint_\alpha(wcl_\alpha(A_i)) = 1_Y$. Hence (Y, σ) is weak* quasi-smooth nearly α-compact.

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