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ON WEAKENED FORMS OF (θ, s)-CONTINUITY

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Abstract. The weakened forms of the (θ, s)-continuous function are introduced and their basic properties are investigated in concern with the other weakened continuous function. The open property of a function and the extremal disconnectedness of the spaces are crucial tools for the survey of these functions.

1. Introduction

Joseph and Kwack [2] introduced the concept of (θ, s)-continuous function to investigate S-closed spaces due to Thompson [6]. T. Noiri and Saeid Jafari [5] obtained several properties of this function and the relationships between (θ, s)-continuity, contra-continuity, regular set-connectedness and other related functions. In section 3 of present paper we introduce a slightly weakened function of (θ, s)-continuity, namely a weakly (θ, s)-continuous function and study the properties of this function related to the paper of T. Noiri and Saeid Jafari. In section 4 we investigate the relationships between some weakened continuous functions such as weakly quasi-, and precontinuous functions. In section 5 we define an s-quasi-continuity(resp. weak s-quasi-continuity) which implies quasi-continuity(resp. weak quasi-continuity) and will see that under certain conditions the weakly (θ, s)-, weakly quasi-, weakly s-quasi-continuous and precontinuous functions are equivalent. In sections 4,5 we will see that the open property of the functions and the extremal disconnectedness of the spaces play the essential role for a study of the

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equivalence of these functions. In the last section the s-closed spaces related to functions mentioned above are studied.

2. Preliminaries

Throughout the present paper, X and Y are always topological spaces. For a subset S of X we denote the interior and the closure of S by Int(S) and Cl(S), respectively. S is said to be *semi-open* [3] if $S \subset$ Cl(Int(S)). The family of all semi-open sets of X is denoted by $SO(X)$. We set $SO(X, x) = \{S | x \in S, S \in SO(X)\}$. A point $x \in X$ is said to be *θ-semi-cluster point* of a subset T of X if $\mathrm{Cl}(S) \cap T \neq \emptyset$ for every $S \in SO(X, x)$. The set of all θ-semi-cluster point of S called *θ-semi-closure* of S and denoted by θ-sCl(S). S is called *θ-semi-closed* if $S = \theta$-sCl(S). The complement of a θ-semi-closed set is called *θ-semi-open.* S is said to be *regular open* (resp. *regular closed*) if $S =$Int(Cl(S))(resp. $S =$Cl(Int(S))).

A function $f : X \to Y$ is said to be *weakly continuous at* $x \in X$ if to each open set V in Y containing $f(x)$ there is an open set U containing x such that $U \subseteq f^{-1}(Cl(V))$. $f : X \to Y$ is said to be *weakly continuous* if f is weakly continuous at every point of X. A function $f : X \to Y$ is said to be *quasi-continuous (resp. weakly quasi-continuous) at* $x \in X$ if to each open set V in Y containing $f(x)$ and to each open set U in X containing x there is a non empty open set G contained in U such that $G \subseteq f^{-1}(V)$ (resp. $G \subseteq f^{-1}(Cl(V))$). $f : X \to Y$ is said to be *quasi-continuous* (resp. *weakly quasi-continuous*) if it is quasi-continuous (resp. weakly quasi-continuous) at each point $x \in X$. $f : X \to Y$ is said to be *semi-continuous at* $x \in X$ if for each open set V in Y containing $f(x)$ $f^{-1}(V)$ is semi-open. $f : X \to Y$ is said to be *semi-continuous* if it is semi-continuous at each point $x \in X$. Observe that the quasi-continuity and the semi-continuity are equivalent concepts.

3. Weakly (θ, s)-continuous functions

Recall that a function $f : X \to Y$ is called a (θ, s)-continuous if for each $x \in X$ and each $S \in SO(Y, f(x))$, there exists an open set U in X containing x such that $f(U) \subset Cl(S)$. Next we introduce a weakened form of (θ, s)-continuity and obtain partially similar results of the section 1 in [5].

DEFINITION 3.1. *A function $f : X \to Y$ is called a weakly (θ, s)-continuous if for each $x \in X$ and each $S \in SO(Y, f(x))$, there exists $R \in SO(X, x)$ such that $f(R) \subset Cl(S)$.*

The following example shows that the weak (θ, s)-continuity does not imply the (θ, s)-continuity.

EXAMPLE 3.1. *A function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} 1 \text{ if } x \in [0, \infty) \\ 0 \text{ if } x \in (-\infty, 0) \end{cases}$ where on both $\mathbb{R}$ the natural topologies are given. Then f is weakly (θ, s)-continuous, but not (θ, s)-continuous at $x = 0$.*

THEOREM 3.1. *The following are equivalent for a function $f : X \to Y$*
(a) f is weakly-(θ, s)-continuous.
(b) $f^{-1}(Cl(S))$ is semi-open for a semi-open set S in Y.
(c) $f^{-1}(R)$ is semi-open for a regular closed set R in Y.
(d) $f^{-1}(T)$ is semi-closed for a regular open set T in Y.
(e) $f^{-1}(S)$ is semi-open for a θ-semi-open set S in Y.

Proof. $(a) \longrightarrow (b)$: Let S be a semi-open set in Y and $x \in f^{-1}(\text{Cl}(S))$. Let W be an open set in X containing x. Then $f(x) \in \text{Cl}(S)$ and there exists an $R \in \text{SO}(X, x)$ such that $R \subset f^{-1}(\text{Cl}(S))$. Thus $W \cap R \subset f^{-1}(\text{Cl}(S))$. Since $W \cap R$ is a non empty semi-open set, there exists non empty open set G such that $G \subset W \cap R \subset f^{-1}(\text{Cl}(S))$. Hence $\emptyset \neq G \subset W \cap \text{Int}(f^{-1}(\text{Cl}(S)))$. Therefore $x \in \text{Cl}(\text{Int}(f^{-1}(\text{Cl}(S))))$.

$(b) \longrightarrow (c)$: It is obvious from the fact that every regular closed set is semi-open.

$(c) \longrightarrow (d)$: Straightforward.

$(d) \longrightarrow (e)$: It is obvious from the fact that every θ-semi-open set is a union of regular closed sets.

$(e) \longrightarrow (a)$: For each $R \in \text{SO}(Y, f(x))$, $\text{Cl}(R)$ is θ-semi-open in Y and by (e) $f^{-1}(\text{Cl}(R))$ is semi-open. This means that f is weakly (θ, s)-continuous. □

T. Noiri and S. Jafari([5]) defined a space X as an *s-Uryshon space* if for each pair of distinct points x and y in X, there exist $S \in SO(X, x)$ and $R \in SO(X, y)$ such that $\text{Cl}(S) \cap \text{Cl}(R) = \emptyset$.

DEFINITION 3.2. *A space X is called a semi T_2-space(Maheswari and Prasad([4]) if for each pair of distinct points x and y in X, there exist $S \in SO(X, x)$ and $R \in SO(X, y)$ such that $S \cap R = \emptyset$.*

Here we introduce just two results about the weakly (θ, s)-continuous functions and the semi-T_2 spaces related to the theorems 3.2 and 3.4 in [5]. The proofs of these theorems are very similar to those of them, thus omitted.

THEOREM 3.2. *If $f : X \to Y$ is weakly-(θ, s)-continuous injection and Y is s-Urysohn, then X is a semi-T_2-space.*

THEOREM 3.3. *If $f : X \to Y$ are weakly (θ, s)-continuous functions and Y is s-Urysohn, then $E = \{(x, y)|x, y \in X, f(x) = f(y)\}$ is semi-closed.*

4. The relationships between functions

THEOREM 4.1. *Every weakly (θ, s)-continuous function is weakly quasi-continuous.*

Proof. Let $f : X \to Y$ be weakly (θ, s)-continuous function. Let $x \in X$ and V be an open set in Y containing $f(x)$. There exists $S \in SO(X, x)$ such that $S \subseteq f^{-1}(Cl(V))$. Then for each open set U in X containing x, $U \cap S \neq \emptyset$ that is semi-open. Hence there exists a non empty open G such that $G \subset U \cap S \subset f^{-1}(Cl(V))$. □

The following example shows that the converse of the theorem above does not hold.

EXAMPLE 4.1. *A function* $f : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} x+1 \text{ if } x \in [0, \infty) \\ x \text{ if } x \in (-\infty, 0) \end{cases}$ *where on both $\mathbb{R}$ the natural topologies are given. Then f is weakly quasi-continuous, but not weakly (θ, s)-continuous at $x = 0$.*

LEMMA 4.1. *If $f : X \to Y$ is an open function, we have $f^{-1}(Cl(S)) \subset Cl(f^{-1}(S)$ for all $S \subset Y$.*

THEOREM 4.2. *Let $f : X \to Y$ be open. The following are equivalent.*
(a) f is weakly quasi-continuous
(b) $f^{-1}(Cl(S))$ is semi-open for each semi-open $S \subset Y$.
(c) $f^{-1}(Cl(V))$ is semi-open for each open $V \subset Y$.

Proof. $(a) \longrightarrow (b)$: Let $S \subseteq Y$ be semi-open. We show
$$f^{-1}(Cl(S)) \subseteq Cl(Int(f^{-1}(Cl(S)))).$$

Let $y \in f^{-1}(Cl(S))$ and W be an open subset of X containing y. Then $f(y) \in Cl(S) = Cl(Q)$ where Q is an open set in Y. Thus $f(W) \cap Q \neq \emptyset$. There exists $z \in W$ such that $f(z) \in Q$. By the weak quasi-continuity of f there exists non empty open $G \subseteq W$ such that $G \subseteq f^{-1}(Cl(Q)) = f^{-1}(Cl(S))$. Therefore $G \subseteq Int(f^{-1}(Cl(S)))$ that means $W \cap Int(f^{-1}(Cl(S))) \neq \emptyset$.

$(b) \longrightarrow (c)$: Obvious.

$(c) \longrightarrow (a)$: This follows easily from the fact that the intersection of semi-open set and open set is semi-open. □

By theorem 3.1, 4.2 and 4.1 we obtain

Theorem 4.3. *Let $f : X \to Y$ be open. Then the following are equivalent.*

(a) f is weakly (θ, s)-continuous.

(b) f is weakly quasi-continuous.

Recall that a function $f : X \to Y$ is said to be *precontinuous at* $x \in X$ if for each neighborhood V of $f(x)$ in Y, $Cl(f^{-1}(V))$ is a neighborhood of x. $f : X \to Y$ is called *precontinuous* if it is precontinuous at each $x \in X$. To obtain relations between the weakly (θ, s), weakly quasi-and precontinuity we need the following.

Theorem 4.4. *If $f : X \to Y$ is precontinuous, then $Cl(f^{-1}(V))$ is semi open for each open $V \subseteq Y$*

Proof. Let $V \subseteq Y$ be open. We show $Cl(f^{-1}(V)) \subseteq Cl(Int(Cl(f^{-1}(V))))$. Let $x \in Cl(f^{-1}(V))$ and W be an open set in X containing x. Then $W \cap f^{-1}(V) \neq \emptyset$, thus there exists $y \in W$ such that $f(y) \in V$. Hence there exists an open set T containing y such that $T \subseteq Cl(f^{-1}(V))$ and also $W \cap T \subseteq Cl(f^{-1}(V))$. Therefore $W \cap Int(Cl(f^{-1}(V)) \neq \emptyset$ that means $x \in Cl(Int(Cl(f^{-1}(V))))$. □

Definition 4.1. *If $Cl(A \cap B) = Cl(A) \cap Cl(B)$ for every $A, B \subset X$, we say that X satisfies the closure equality.*

Theorem 4.5. *Let X satisfy the closure equality. If $f : X \to Y$ precontinuous and open, then $f^{-1}(Cl(S)) = Cl(f^{-1}(S))$ for each semi-open $S \subset Y$.*

Proof. Let $S \in$SO(X). By the lemma 4.1 it is enough to show $f^{-1}(Cl(S)) \supset Cl(f^{-1}(S))$. Let $x \in Cl(f^{-1}(S))$ and $f(x) \in W$, W is open in Y. We will show $W \cap S \neq \emptyset$. By the precontinuity of f there exists an open set U in X containing x such that $U \subset Cl(f^{-1}(W))$. From $\emptyset \neq U \cap f^{-1}(S) \subset Cl(f^{-1}(W)) \cap Cl(f^{-1}(S)) = Cl(f^{-1}(W \cap S))$ follows $f^{-1}(W \cap S) \neq \emptyset$ that means $W \cap S \neq \emptyset$. □

Due to the theorems 4.4, 4.5 and 4.2 we have the following theorem.

THEOREM 4.6. *Let X satisfy the closure equality. If $f : X \to Y$ precontinuous and open, then f is weakly quasi-continuous.*

THEOREM 4.7. *Let X be extremally disconnected. If $f : X \to Y$ is weakly quasi-continuous and open function, f is precontinuous.*

Proof. Let $x \in X$. If V is an open neighborhood of $f(x)$ in Y, then from the open property of f follows $x \in f^{-1}(V) \subset f^{-1}(Cl(V)) \subset Cl(f^{-1}(V))$. Due to theorem 4.2, there exists an open T in X such that $Cl(f^{-1}(Cl(V))) = Cl(T)$ that is open in X. Therefore $x \in Cl(T) \subset Cl(f^{-1}(V))$. □

By theorems 4.3, 4.6 and 4.7 we obtain

THEOREM 4.8. *Let X satisfy the closure equality and be extremally disconnected. If $f : X \to Y$ is open, the following are equivalent.*
(a) f is weakly (θ, s)-continuous.
(b) f is weakly quasi-continuous.
(c) f is precontinuous.

THEOREM 4.9. *Let $f : X \to Y$ be precontinuous and open. If Y is extremally disconnected, $Cl(f^{-1}(S))$ is semi-open for every semi-open set $S \subset Y$.*

Proof. Let $S \subseteq Y$ be semi-open. We show

$$\mathrm{Cl}(f^{-1}(S)) \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(S)))).$$

Let $x \in \mathrm{Cl}(f^{-1}(S))$ and $x \in W, W \subseteq X$ be open . Then $W \cap f^{-1}(S) \neq \emptyset$, thus there exists $y \in W$ such that $f(y) \in S$. There exists an open set T in Y such that $\mathrm{Cl}(S) = \mathrm{Cl}(T)$ which is also open in Y. By precontinuity of f there exists an open set Q in X containing y such that $Q \subseteq Cl(f^{-1}(\mathrm{Cl}(S)))$. By openness of f $Cl(f^{-1}(\mathrm{Cl}(S))) = \mathrm{Cl}(f^{-1}(S))$. Hence $\emptyset \neq W \cap Q \subseteq \mathrm{Int}(\mathrm{Cl}(f^{-1}(S)))$. Therefore $W \cap \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(S)))) \neq \emptyset$ that means $x \in \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(f^{-1}(S))))$. □

5. s-quasi-continuous functions

Definition 5.1. *A function $f : X \to Y$ is called s-quasi-continuous at x(resp. weakly s-quasi-continuous at x) if for each open subset V of Y containing $f(x)$ and for each $S \in SO(X, x)$, there exists a non empty $R \in SO(X)$ such that $R \subset f^{-1}(V) \cap S$(resp.$R \subset f^{-1}(Cl(V)) \cap S$).*

f is said to be s-quasi-continuous (resp. weakly s-quasi-continuous), if it is s-quasi-continuous at each x of X(resp. weakly s-quasi-continuous at each x of X).

It is easy to see that the s-quasi-continuity (resp. weak s-quasi-continuity) implies the quasi-continuity(resp. weak quasi-continuity), but the same function of example 4.1 shows that the converse of both cases does not hold.

Theorem 5.1. *If a function $f : X \to Y$ is quasi-continuous(resp. weakly quasi-continuous) and X is extremally disconnected, then f is s-quasi-continuous(resp. weakly s-quasi-continuous).*

Proof. We only prove the case of weak quasi-continuity. Let $x \in X$ and V be an open set in Y containing $f(x)$. Let $S \in \mathrm{SO}(X, x)$. There exists an open T in X such that $\mathrm{Cl}(T) = \mathrm{Cl}(S)$ which is also open in X. Hence there exists an non empty open set $G \subset \mathrm{Cl}(T)$ such that $G \subset f^{-1}(\mathrm{Cl}(V))$. From $G \subset \mathrm{Cl}(S)$ follows $\emptyset \neq G \cap S \subset f^{-1}(\mathrm{Cl}(V)) \cap S$ where $G \cap S$ is semi-open. □

Theorem 5.2. *Let X be extremally disconnected. Then the following are equivalent.*

(a) $f : X \to Y$ is quasi-continuous(resp. weakly quasi-continuous).

(b) f is s-quasi-continuous(resp. weakly s-quasi-continuous)

Due to theorem 4.3, if the condition of openness of a function f is added in theorem 5.2, we have

Theorem 5.3. *Let $f : X \to Y$ be open and X be extremally disconnected. Then the following are equivalent.*

(a) f is weakly (θ, s)-continuous.

(b) f is weakly quasi-continuous.

(c) f is weakly s-quasi-continuous.

THEOREM 5.4. *If the conditions of the theorem 4.8 are satisfied, then the following are equivalent.*
(a) f is weakly (θ, s)-continuous.
(b) f is weakly quasi-continuous.
(c) f is weakly s-quasi-continuous.
(d) f is precontinuous.

THEOREM 5.5. *If a function $f : X \to Y$ is weakly s-quasi-continuous and Y is extremally disconnected, there exists an open T such that $Cl(T) = Cl(f^{-1}(Cl(S)))$ for all $S \in SO(Y)$.*

Proof. Let $S \in$SO(Y). Define $\mathcal{R} = \bigcup\{R | R \in \text{SO}(X), R \subset f^{-1}(\text{Cl}(S))\}$ and it is enough to show that $\text{Cl}(\mathcal{R}) = \text{Cl}(f^{-1}(\text{Cl}(S)))$.
Let $x \in \text{Cl}(f^{-1}(\text{Cl}(S)))$ and $x \in U$ which is open in X. Then $U \cap f^{-1}(\text{Cl}(S)) \neq \emptyset$. Hence there exists $y \in U$ such that $f(y) \in \text{Cl}(S)$. Since Cl(S) is open in Y and containing $f(y)$, there exists non empty $R \in \text{SO}(X)$ such that $R \subset U \cap f^{-1}(\text{Cl}(S))$. From this follows $R \subset \mathcal{R}$. Thus $R \subset U \cap \mathcal{R}$ which is not empty. Hence $x \in \text{Cl}(\mathcal{R})$. The other inclusion is obvious. □

6. $s-$closed spaces

DEFINITION 6.1. *A topological space X is said to be s-closed(Di Maio and Noiri[1])(resp. S-closed) if for every semi-open cover $\{S_\lambda | \lambda \in \Lambda\}$ of X there exists a finite subfamily Λ_0 such that $X = \bigcup\{sCl(S_\lambda) | \lambda \in \Lambda_0\}(resp. X = \bigcup\{Cl(S_\lambda) | \lambda \in \Lambda_0\})$ where*

$$sCl(S) = \bigcap\{C \mid C \subset X, C \text{ semi-closed and } S \subset C\}$$

for $S \subset X$.

LEMMA 6.1. *Let $f : X \to Y$ be open and X be extremally disconnected. If f is weakly quasi-continuous, then $f(Cl(S)) \subset Cl(f(S))$ for each $S \in SO(X)$.*

Proof. Let $S \in SO(X)$ and $x \in \text{Cl}(S)$. Let W be open set in Y containing $f(x)$. By the weak quasi-continuity of f and the extremal disconnectedness of X there exists a non empty open subset G of $Cl(S)$ such that $f(G) \subset Cl(W)$, thus also $f(G \cap S) \subset Cl(W)$. Since $G \cap S$ is semi-open, there exists a non empty open subset G' of $G \cap S$(note that $G \cap S \neq \emptyset$). Hence $f(G') \subset Cl(W)$ and from openness of f follows $f(G') \cap W \neq \emptyset$. Therefore $f(S) \cap W \neq \emptyset$. □

THEOREM 6.1. *Let $f : X \to Y$ be open, surjective and weakly quasi-continuous function. Let X be extremally disconnected. If X is s-closed, Y is S-closed.*

Proof. Let $(S_\lambda)_{\lambda\in\Lambda}$ be a semi-open cover of Y. Then $(Cl(S_\lambda))_{\lambda\in\Lambda}$ is also a semi-open cover of Y. By theorem 4.2 $(f^{-1}(Cl(S_\lambda)))_{\lambda\in\Lambda}$ is a semi-open cover of X, thus there exists a finite $\Lambda_0 \subset \Lambda$ such that $(sCl(f^{-1}(Cl(S_\lambda))))_{\lambda\in\Lambda_0}$ is a cover of X. Due to Di Maio and Noiri([1]) $sCl(f^{-1}(Cl(S_\lambda))) = Cl(f^{-1}(Cl(S_\lambda)))$ for each $\lambda \in \Lambda$. Since f is open, the following inclusions hold.

$$f(Cl(f^{-1}(Cl(S_\lambda)))) \subset Cl(f(f^{-1}(Cl(S_\lambda)))) \subset Cl(S_\lambda)$$

for each $\lambda \in \Lambda$. Therefore $(Cl(S_\lambda))_{\lambda\in\Lambda_0}$ is a cover of Y. □

REMARK 6.1. Due to theorem 4.3, the weak quasi-continuity of f can be replaced by the weak (θ, s)-continuity. Due to theorem 4.8 or 5.4, if the condition closure equality of X is added in theorem 6.1, the weak quasi-continuity of f can be replaced by the precontinuity or by the weak s-quasi-continuity.

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