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SEMIGROUP OF LIPSCHITZ OPERATORS

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Abstract. Lipschitzian semigroup is a semigroup of Lipschitz operators which contains C_0 semigroup and nonlinear semigroup. In this paper, we establish the cannonical exponential formula of Lipschitzian semigroup from its Lie generator and the approximation theorem by Laplace transform approach to Lipschitzian semigroup.

1. Introduction

Let X and Y be Banach spaces over the field K, and let $C \subset X$ and $D \subset Y$. A function $T : C \to D$ is said to be a Lipschitz operator if there exists a constant M such that

$$||Tx - Ty|| \leq M||x - y||$$

for all $x,\ y \in C$.

Let $Lip(C, D)$ be the space of Lipschitz operators from C to D and let the Lipschitz seminorm of $T \in Lip(C, D)$ be

$$||T||_{Lip} = \sup\{||Tx - Ty||/||x - y|| : x, y \in C,\ x \neq y\}.$$

Then $(Lip(C, D),\ ||\cdot||_{Lip})$ is a seminormed linear space. If T is linear, then $||T||_{Lip}$ is the usual operator norm of T.

Let $x_0 \in C$ and let $Lip_{x_0}(C, D) = \{T \in Lip(C, D) : Tx_0 = 0\}$. It was known in [4] that $Lip_{x_0}(C, D)$ is a Banach space.

Definition 1. An operator family $\{T(t) : t \geq 0\} \subset Lip(C, C)$ is called a Lipschitzian semigroup on C if the following conditions hold.

(i) $T(0) = I$, the identity operator on C.

(ii) $T(t)T(s) = T(t + s)$ for $t,\ s \geq 0$.

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(iii) For each $x \in C$, $T(t)x$ is continuous in $t \geq 0$.

The Lipschitzian semigroup $\{T(t) : t \geq 0\}$ is said to be exponentially bounded if there exist constants ω and $M \geq 1$ such that $||T(t)||_{Lip} \leq Me^{\omega t}$ for all $t \geq 0$.

Next we define a Lie generator of Lipscitzian semigroup (see [2]).

Definition 2. The Lie generator of Lipschitzian semigroup $\{T(t) : t \geq 0\}$ is the linear operator $B : D(B) \subset Lip(C, K) \to Lip(C, K)$ consisting of all $(f,\ g)$ such that $f, g \in Lip(C, K)$ and

$$g(x) = \lim_{t \to 0^+} \frac{1}{t}(f(T(t)x) - f(x))$$

for all $x \in C$.

In this paper our purpose is to develop the Laplace transform approach to a Lipschitzian semigroup. We show that the Lie generator can be represented by the Laplace transform of Lipschitzian semigroup and Lipschitzian semigroup may be recovered from its Lie generator by the cannonical exponential formula. The approximation of Lipschitzian semigroups can be shown by the equicontinuity. In this paper, C is closed and a Lipschitzian semigroup is exponentially bounded.

2. Semigroup of Lipschitz operators

First we will show that $(\lambda I - B)^{-1}$ can be represented by the Laplace transform.

Lemma 1. *Let B be the Lie generator of Lipschitzian semigroup $\{T(t) : t \geq 0\}$ satisfying $||T(t)||_{Lip} \leq Me^{\omega t}$ for all $t \geq 0$. Suppose that x_0 is the fixed point of $\{T(t) : t \geq 0\}$, that is, $T(t)x_0 = x_0$ for all $t \geq 0$. Then for all $\lambda > \omega$,*

(i) *$(\lambda I - B)^{-1}$ is a bounded linear operator on $Lip_{x_0}(C, K)$ with $||(\lambda I - B)^{-1}|| \leq M/(\lambda - \omega)$.*
(ii) *$((\lambda I - B)^{-1} f)(x) = \int_0^\infty e^{-\lambda t} f(T(t)x)dt$ for $f \in Lip_{x_0}(C, K)$ and $x \in C$.*
(iii) *For $f \in D(B) \cap Lip_{x_0}(C, K)$, $\lim_{\lambda \to \infty} \lambda(\lambda I - B)^{-1} f = f$.*

Proof. Let $\lambda > \omega$. Since $\{T(t) : t \geq 0\}$ has a fixed point x_0 and $f \in Lip_{x_0}(C, K)$, we can define a linear operator $R(\lambda) : Lip_{x_0}(C, K) \to Lip_{x_0}(C, K)$ by

$$(R(\lambda)f)(x) = \int_0^\infty e^{-\lambda t} f(T(t)x)dt.$$

Then

$$\begin{aligned} |(R(\lambda)f)(x) - (R(\lambda)f)(y)| &= \left| \int_0^\infty e^{-\lambda t}(f(T(t)x) - f(T(t)y))dt \right| \\ &\leq \int_0^\infty e^{-\lambda t} ||f||_{Lip} ||T(t)||_{Lip} ||x - y|| dt \\ &\leq M||f||_{Lip}||x - y|| \int_0^\infty e^{-(\lambda-\omega)t} dt \\ &= \frac{M||f||_{Lip}}{\lambda - \omega} ||x - y||. \end{aligned}$$

Hence $||R(\lambda)f||_{Lip} \leq M||f||_{Lip}/(\lambda - \omega)$ and so $||R(\lambda)|| \leq M/(\lambda - \omega)$.

Let $f \in Lip_{x_0}(C, K)$ and $x \in C$. Then

$$\begin{aligned} &(R(\lambda)f)(T(t)x) - (R(\lambda)f)(x) \\ &= \int_0^\infty e^{-\lambda s} f(T(s+t)x)ds - \int_0^\infty e^{-\lambda s} f(T(s)x)ds \\ &= e^{\lambda t} \int_t^\infty e^{-\lambda s} f(T(s)x)ds - \int_0^\infty e^{-\lambda s} f(T(s)x)ds \\ &= (e^{\lambda t} - 1) \int_0^\infty e^{-\lambda s} f(T(s)x)ds \\ &\quad - e^{\lambda t} \int_0^t e^{-\lambda s} f(T(s)x)ds. \end{aligned}$$

Hence

$$\lim_{t \to 0^+} \frac{(R(\lambda)f)(T(t)x) - (R(\lambda)f)(x)}{t} = \lambda(R(\lambda)f)(x) - f(x)$$

for all $x \in C$. That is, $R(\lambda)f \in D(B)$ and $(\lambda I - B)R(\lambda)f = f$.

Let $f \in D(B) \cap Lip_{x_0}(C,K)$. Since $(Bf)(T(t)x) = \lim_{s\to 0^+} \frac{1}{s}(f(T(t+s)x) - f(T(t)x)) = \frac{d}{dt}f(T(t)x)$,

$$\begin{aligned}(R(\lambda)Bf)(x) &= \int_0^\infty e^{-\lambda t}(Bf)(T(t)x)dt \\ &= \int_0^\infty e^{-\lambda t}\frac{d}{dt}f(T(t)x))dt \\ &= \lambda\int_0^\infty e^{-\lambda t}f(T(t)x)dt - f(x) \\ &= \lambda(R(\lambda)f)(x) - f(x).\end{aligned}$$

So we have $R(\lambda)(\lambda I - B)f = f$ for all $f \in Lip_{x_0}(C,K) \cap D(B)$. Therefore we have

$$((\lambda I - B)^{-1}f)(x) = (R(\lambda)f)(x) = \int_0^\infty e^{-\lambda t}f(T(t)x)dt$$

for all $f \in Lip_{x_0}(C,K)$ and $x \in C$.

Let $f \in D(B) \cap Lip_{x_0}(C,K)$. Since $\lambda(\lambda I - B)^{-1}f = (\lambda I - B)^{-1}Bf + f$,

$$\begin{aligned}||\lambda(\lambda I - B)^{-1}f - f||_{Lip} &= ||(\lambda I - B)^{-1}Bf||_{Lip} \\ &\leq \frac{M}{\lambda - \omega}||Bf||_{Lip}.\end{aligned}$$

Hence $\lim_{\lambda\to\infty}\lambda(\lambda I - B)^{-1}f = f$. □

LEMMA 2. *Let B be the Lie generator of Lipschitzian semigroup $\{T(t) : t \geq 0\}$ satisfying $||T(t)||_{Lip} \leq Me^{\omega t}$ for all $t \geq 0$. Let $R(\lambda) = (\lambda I - B)^{-1}$ for $\lambda > \omega$. Suppose that x_0 is the fixed point of $\{T(t) : t \geq 0\}$. Then for all $f \in Lip_{x_0}(C,K)$ and $x \in C$*

$$(R(\lambda)f)(T(h)x) - (R(\lambda)f)(x) = ((\lambda R(\lambda) - I)f_h)(x),$$

where $f_h \in Lip_{x_0}(C,K)$ is defined as $f_h(x) = \int_0^h f(T(t)x))dt$.

Proof. By integration by parts, we obtain

$$\begin{aligned}((\lambda R(\lambda) &- I)f_h)(x) \\ &= \lambda\int_0^\infty e^{-\lambda t}(f_h(T(t)x)) - f_h(x))dt \\ &= \lambda\int_0^\infty e^{-\lambda t}(\int_0^h f(T(t+s)x)ds - \int_0^h f(T(s)x)ds)dt \\ &= \lambda\int_0^\infty e^{-\lambda t}(\int_h^{t+h} f(T(s)x)ds - \int_0^t f(T(s)x)ds)dt \\ &= \int_0^\infty e^{-\lambda t}(f(T(t+h)x) - f(T(t)x))dt \\ &= (R(\lambda)f)(T(h)x) - (R(\lambda)f)(x).\end{aligned}$$

□

Now we can prove the cannonical exponential formula.

THEOREM 1. *Let $\{T(t) : t \geq 0\}$ be a Lipschitzian semigroup on C satisfying $||T(t)||_{Lip} \leq Me^{\omega t}$ for all $t \geq 0$ and let B be its Lie generator. Suppose that $\{T(t) : t \geq 0\}$ has a fixed point $x_0 \in C$ and $Lip_{x_0}(C, K) \subset \overline{D(B)}$. Then for all $x \in C$ and $f \in Lip_{x_0}(C, K)$,*

$$f(T(t)x) = \lim_{n\to\infty}\left((I - \frac{t}{n}B)^{-n}f\right)(x), \quad t \geq 0.$$

Proof. Let $\lambda > \omega$. Then $((\lambda I - B)^{-1}f)(x)$ is the Laplace transform of $f(T(t)x)$ and so we have [1, Theorem 1.5.1]

$$\frac{d^n}{d\lambda^n}((\lambda I - B)^{-1}f)(x) = \int_0^\infty e^{-\lambda t}(-t)^n f(T(t)x)dt.$$

By induction on n, we obtain

$$\begin{aligned}&((\lambda I - B)^{-n}f)(x) \\ &= \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_n+\cdots+t_1)}f(T(t_n + \cdots + t_1)x)dt_n \cdots dt_1 \\ &= \int_0^\infty e^{-\lambda t}\frac{t^{n-1}}{(n-1)!}f(T(t)x)dt\end{aligned}$$

Thus we have

$$((\lambda I - B)^{-(n+1)}f)(x) = \frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}((\lambda I - B)^{-1}f)(x).$$

By the Post-Widder inversion formula,

$$\begin{aligned} f(T(t)x) &= \lim_{n\to\infty}(-1)^n \frac{1}{n!}(\frac{n}{t})^{n+1}\frac{d^n}{d\lambda^n}((\lambda I - B)^{-1}f)(x)\Big|_{\lambda=\frac{n}{t}} \\ &= \lim_{n\to\infty}((I-\frac{t}{n}B)^{-(n+1)}f)(x). \end{aligned}$$

By Lemma 1, $\lim_{n\to\infty}((I-\frac{t}{n}B)^{-1}f)(x) = f(x)$ for all $f \in Lip_{x_0}(C,K) \cap D(B)$. Let $f \in Lip_{x_0}(C,K)$. Since $Lip_{x_0}(C,K) \subset \overline{D(B)}$, there exists $\{f_k\}$ in $Lip_{x_0}(C,K) \cap D(B)$ such that $\lim_{k\to\infty} f_k = f$. So

$$\begin{aligned} &|((I-\frac{t}{n}B)^{-1}f)(x) - f(x)| \\ \le\ & |((I-\frac{t}{n}B)^{-1}f)(x) - ((I-\frac{t}{n}B)^{-1}f_k)(x)| \\ & + |((I-\frac{t}{n}B)^{-1}f_k)(x) - f_k(x)| + |f_k(x) - f(x)| \\ \le\ & (||(I-\frac{t}{n}B)^{-1}|| + 1)||f-f_k||_{Lip}||x-x_0|| \\ & + |((I-\frac{t}{n}B)^{-1}f_k)(x) - f_k(x)|. \end{aligned}$$

Hence we obtain $\lim_{n\to\infty}((I-\frac{t}{n}B)^{-1}f)(x) = f(x)$ for $f \in Lip_{x_0}(C,K)$ and so the result follows. □

Before establishing the approximation theorem of Lipschitzian semigroups, we present approximation theorem of Laplace transforms given in [5].

Theorem 2. *For each $m \in N$, let $f_m \in C([0,\infty), X)$ satisfying $||f_m(t)|| \le Me^{\omega t}$, $m \in N$, $t \ge 0$, and let $F_m(\lambda) = \int_0^\infty e^{-\lambda t} f_m(t)dt$, $\lambda > \omega$. Then the following assertions are equivalent.*

- (i) *$\{f_m : m \in N\}$ is equicontinuous at each point $t \in [0,\infty)$ and $\lim_{m\to\infty} F_m(\lambda)$ exists for $\lambda > \omega$.*
- (ii) *$\lim_{m\to\infty} f_m(t)$ exists for $t \ge 0$ and the convergence is uniform on bounded t-intervals.*

Theorem 3. *Let each B_n be the Lie generator of Lipschitzian semigroup $\{T_n(t) : t \ge 0\}$ satisfying $||T_n(t)||_{Lip} \le Me^{\omega t}$ for all $t \ge 0$. Let $R_n(\lambda) = (\lambda I - B_n)^{-1}$ for $\lambda > \omega$. Suppose that $\{T_n(t) : t \ge 0\}$ has a fixed point $x_0 \in C$ and $\lim_{n\to\infty} R_n(\lambda)f = R_0(\lambda)f$ for all $f \in Lip_{x_0}(C,K)$ and*

$\lambda > \omega$. *If* $Lip_{x_0}(C,K) \subset \overline{D(B)}$, *then*

$$\lim_{n\to\infty} f(T_n(t)x) = f(T_0(t)x)$$

for all $f \in Lip_{x_0}(C,K)$ *and* $x \in C$, *and the convergence is uniform on bounded* t*-intervals.*

Proof. Let $f \in Lip_{x_0}(C,K)$ and $\lambda > \omega$. By Theorem 2, it is enough to show that $\{f(T_n(t)x)\}$ is equicontinuous. By Lemma 2, we have for $0 < s < t$

$$\begin{aligned}
&|(R_0(\lambda)f)(T_n(t)x) - (R_0(\lambda)f)(T_n(s)x)| \\
&\le |(R_0(\lambda)f)(T_n(t)x) - (R_n(\lambda)f)(T_n(t)x)| \\
&\quad +|(R_n(\lambda)f)(T_n(t)x) - (R_n(\lambda)f)(T_n(s)x)| \\
&\quad +|(R_n(\lambda)f)(T_n(s)x) - (R_0(\lambda)f)(T_n(s)x)| \\
&\le ||R_0(\lambda)f - R_n(\lambda)f||_{Lip}||T_n(t)||_{Lip}||x - x_0|| \\
&\quad +||R_n(\lambda)f - R_0(\lambda)f||_{Lip}||T_n(s)||_{Lip}||x - x_0|| \\
&\quad +|((\lambda R_n(\lambda) - I)f_t^{(n)})(x) - ((\lambda R_n(\lambda) - I)f_s^{(n)})(x)|,
\end{aligned}$$

where $f_h^{(n)}(x) = \int_0^h f(T_n(t)x)dt$.

Note that

$$\begin{aligned}
&|((\lambda R_n(\lambda) - I)f_t^{(n)})(x) - ((\lambda R_n(\lambda) - I)f_s^{(n)})(x)| \\
&= |\lambda \int_0^\infty e^{-\lambda r}(\int_s^t f(T_n(\alpha + r)x)d\alpha)dr - \int_s^t f(T(r)x)dr| \\
&\le Me^{\omega t}||f||_{Lip}||x - x_0||(\frac{\lambda}{\lambda - \omega} + 1)(t - s).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&|(R_0(\lambda)f)(T_n(t)x) - (R_0(\lambda)f)(T_n(s)x)| \\
&= M(e^{\omega t} + e^{\omega s})||x - x_0||||R_n(\lambda)f - R_0(\lambda)f||_{Lip} \\
&\quad +(\frac{\lambda}{\lambda - \omega} + 1)||f||_{Lip}Me^{\omega t}||x - x_0||(t - s)
\end{aligned}$$

and so $\{(R(\lambda)f)(T_n(t)x)\}$ is equicontinuous at t.

Next we will show that $\{f(T(t)x)\}$ is equicontinuos at t. Let $f \in Lip_{x_0}(C,K) \cap D(B)$. Then

$$\begin{aligned} &|f(T_n(t)x) - f(T_n(s)x)| \\ \leq\ & |f(T_n(t)x) - (\lambda R_0(\lambda)f)(T_n(t)x)| \\ &+\lambda|R(\lambda)f(T_n(t)x) - R(\lambda)f(T_n(s)x)| \\ &+|(\lambda R_0(\lambda)f)(T_n(s)x) - f(T_n(s)x)|. \end{aligned}$$

Let $\epsilon > 0$ be given. By Lemma 1, there exists λ_1 such that

$$\begin{aligned} &|f(T_n(t)x) - (\lambda_1 R_0(\lambda_1)f)(T_n(t)x)| \\ &+|(\lambda_1 R_0(\lambda_1)f)(T_n(s)x) - f(T_n(s)x)| < \frac{\epsilon}{2} \end{aligned}$$

By the equicontinuity of $\{R_0(\lambda)f(T_n(t)x)\}$, there exists $\delta > 0$ such that $|R_0(\lambda)f(T_n(t)x) - R_0(\lambda)f(T_n(s)x)| < \epsilon/(2\lambda_1)$ for $|s-t| < \delta$. Thus $|f(T_n(t)x) - f(T_n(s)x)| < \epsilon$ for $f \in Lip_{x_0}(C,K) \cap D(B)$.

Since $Lip_{x_0}(C,K) \subset \overline{D(B)}$, $\{f(T_n(x))\}$ is equicontinuos at t for $f \in Lip_{x_0}(C,K)$. □

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