Kangweon-Kyungki Math. Jour. **14** (2006), No. 2, pp. 281–289

SOME GEOMETRIC APPLICATIONS OF RESISTANT LENGTH OF CURVE FAMILIES (I)

Bohyun Chung* and Wansoo Jung

Abstract. We introduce the resistant length and examine its properties. We also consider the geometric applications of resistant length to the boundary behavior of analytic functions, conformal mappings and derive the theorem in connection with the cluster sets, purely geometric problems. The method of resistant length leads a simple proofs of theorems. So it shows us the usefulness of the method of resistant length.

1. Introduction

Throughout this paper, $\mathbb{C}$ will denote complex plane, D is a domain in $\mathbb{C}$, ∂D is a boundary of D, and $cl(D)$ is a closure of D.

There is a physical interpretation of resistant length. Think of the curve family Γ as representing a system of homogenous electric wires. Then the resistant length $\lambda(\Gamma)$ represents the resistance of Γ.

Using the concept of the resistant length, in [5], we established the following theorems for analytic functions. A purely function-theoretic proof of theorem A is difficult. The use of resistant length makes the proof trivial.

Theorem 1.1. [5]. *Let Q be a general quadrilateral of area M. Let a be the length of the shortest arc in Q connecting one pair of opposite sides. Let b be the length of the shortest arc in Q connecting the other pair of sides. Then*

$$a \cdot b \leq M.$$

Received November 29, 2006.
2000 Mathematics Subject Classification: 30C62, 30C85, 30D40.
Key words and phrases: resistant length, conformal mapping.
* Corresponding author.
This work was supported by 2005 sabbatical year system of Soonchunhyang University.

The following theorem 1.2 applies the resistant length to the analytic function defined on the domain with a number of holes. So it shows us the high usefulness of the method of resistant length.

Theorem 1.2. [5]. *Let $f(z)$ be a bounded single-valued analytic function in the complement of E, where E is a totally disconnected compact set of positive capacity in $\mathbb{C}$. Then it is not the case that for each z in E, except for those z in a set of capacity zero, there exist two curves in the complement of E at z on which $f(z)$ has the limits ω_1 and ω_2, $(\omega_1 \neq \omega_2)$.*

The purpose of this paper is to apply the resistant length of a curve family in the complex plane to the boundary behavior of analytic functions of a complex variable.

2. Resistant length

Let Γ be a family whose elements γ are locally rectifiable curves (simply, curves or arcs) in D, and let $\rho(z)$ be a non-negative Borel measurable function defined on $\mathbb{C}$. Every curve γ has a well-defined

$$L(\gamma, \rho) = \int_\gamma \rho(z)\,|dz|, \quad z = x + iy$$

which may be infinite, and D has a

$$A(D, \rho) = \iint_D \rho(z)^2\,dx\,dy \neq 0, \infty. \tag{1}$$

In order to define an invariant which depends on the whole set Γ, we introduce

$$L(\Gamma, \rho) = \underset{\gamma\in\Gamma}{inf}\, L(\gamma, \rho),$$

where we agree that $L(\Gamma, \rho) = \infty$ in case Γ is empty.

To obtain a quantity that does not change when the weight function ρ is multiplied by a constant, we form the homogeneous expression $\dfrac{L(\Gamma, \rho)^2}{A(D, \rho)}$.

DEFINITION 2.1. [1]. The *resistant length*(or extremal length) of Γ in D is defined as

$$\lambda(\Gamma) = \lambda_D(\Gamma) = \sup_{\rho} \frac{L(\Gamma,\rho)^2}{A(D,\rho)} \ .$$

where ρ is subject to the condition $0 < A(D,\rho) < \infty$, obviously $0 \leq \lambda(\Gamma) \leq \infty$.

REMARK 1. (i)[1] $\lambda_D(\Gamma)$ depends only on Γ and not on D. Accordingly, we shall simplify the notation to $\lambda(\Gamma)$.

(ii)[9] Since almost every curve in $\mathbb{C}$ is rectifiable, the non-rectifiable curves of a family Γ have no influence on the resistant length of Γ. Accordingly, we shall simplify the terminology to curve or arc.

We introduce the following propositions which are frequently used in our paper. The conformal invariance of resistant length is an immediate consequence of the definition.

PROPOSITION 2.2. [9]. (Conformal invariance of resistant length) *Let $z^* = f(z)$ be a 1-1 conformal mapping on D upon a domain D^* and Γ a family of curves on D, then*

$$\lambda(\Gamma) = \lambda[f(\Gamma)].$$

PROPOSITION 2.3. [1]. (Comparison principle of resistant length) *For two curve families Γ_1, Γ_2, if every $\gamma_2 \in \Gamma_2$ contains a $\gamma_1 \in \Gamma_1$, then*

$$\lambda(\Gamma_1) \leq \lambda(\Gamma_2).$$

Indeed, both resistant lengths can be evaluated with respect to the same D. For any ρ in D it is clear that $L(\Gamma_2,\rho) \geq L(\Gamma_1,\rho)$. These minimum lengths are compared with the same $A(D,\rho)$.

REMARK 2. (i)[1] Briefly, the set Γ_2 of fewer or longer curves has the larger resistant length. Observe that $\Gamma_2 \subset \Gamma_1$ implies $\Gamma_1 < \Gamma_2$.

(ii) The above Proposition 2.3 reflect the fact that systems of fewer or longer wires have greater resistance (smaller conductance).

PROPOSITION 2.4. [9]. *Suppose there exist disjoint open sets G_n containing the curves in Γ_n. If $\cup_n \Gamma_n \subset \Gamma$, then*

$$\sum_n \frac{1}{\lambda(\Gamma_n)} \leq \frac{1}{\lambda(\Gamma)} \ .$$

We discuss the following theorems which are frequently used in our paper.

Theorem 2.5. *Let R be a rectangle of sides a and b. Let Γ be the family of arcs in R which joins the sides of length b. Then*

$$\lambda(\Gamma) = \frac{a}{b}\,.$$

Proof. For any $\rho(z)$, we have

$$\int_0^a \rho(z)\,dx \geq L(\Gamma,\rho), \qquad \iint_R \rho(z)\,dx\,dy \geq b\,L(\Gamma,\rho)$$

Then, by the Schwarz inequality,

$$\begin{aligned} b^2\,[L(\Gamma,\rho)]^2 &\leq ab\iint_R \rho^2\,dx\,dy \\ &= ab\,A(R,\rho). \end{aligned}$$

This proves $\lambda(\Gamma) \leq \frac{a}{b}\,.$

For $\rho = 1$, we have

$$L(\Gamma,1) = a, \quad A(R,1) = ab.$$

Thus $\lambda(\Gamma) \geq \frac{a}{b}\,.$ □

Theorem 2.6. *Let Δ be the annulus $\Delta = \{z\,|\,a < |z| < b\}$. Let Γ be the family of arcs in Δ which joins the two contours. Then*

$$\lambda(\Gamma) = \frac{1}{2\pi}\log\frac{b}{a}\,.$$

Proof. For any $\rho(z)$, we have

$$\int_a^b \rho\,dr \geq L(\Gamma,\rho), \qquad \iint_\Delta \rho\,dr\,d\theta \geq 2\pi L(\Gamma,\rho)$$

Then, by the Schwarz inequality,

$$\begin{aligned} 4\pi^2\,L(\Gamma,\rho)^2 &\leq [\iint_\Delta \rho\,dr\,d\theta]^2 \\ &\leq [\iint_\Delta \rho^2\,\frac{1}{r}\,dr\,d\theta][\iint_\Delta r\,dr\,d\theta] \\ &= [2\pi\log\frac{b}{a}][\iint_\Delta \rho^2 r\,dr\,d\theta]. \end{aligned}$$

This proves $\lambda(\Gamma) \leq \frac{1}{2\pi} \log \frac{b}{a}$.

For $\rho = \frac{1}{r}$, we have

$$L(\Gamma, \frac{1}{r}) = \log \frac{b}{a}, \quad A(\Delta, \frac{1}{r}) = 2\pi \log \frac{b}{a}.$$

Thus $\lambda(\Gamma) \geq \frac{1}{2\pi} \log \frac{b}{a}$. □

3. Applications

Using the following definitions and lemma, we have the theorem.

Definition 3.1. [11]. If every component of a set is a point, the set is called *totally disconnected.*

For example, $\{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$ is a totally disconnected compact set.

Lemma 3.2. [7]. *For any totally disconnected compact set E in $\mathbb{C}$, there exists a Jordan domain D such that the Jordan curve J bounding D passes every point of E.*

Definition 3.3. [9]. By an *arbitrary function* g we mean a (single-valued) function whose domain is a subset of $\mathbb{C}$ and whose range is on the Riemann sphere Ω. Let Λ be a curve at $z_0 \in cl(D)$, then the *cluster set* of arbitrary function g at z_0 along Λ, denoted by $C_\Lambda(g, z_0)$, is defined to be the set of all points $\omega \in \Omega$ with the property that, for some sequence of points $\{z_n\}$ on Λ for which

$$\lim_{n\to\infty} z_n = z_0, \quad \text{we have} \quad \lim_{n\to\infty} g(z_n) = \omega.$$

A value ω is called a *cluster value* of g at z_0 along Λ. It follows readily that $C_\Lambda(g, z_0)$ is a nonempty closed subset of Ω.

Theorem 3.4. *Let E be as in Theorem 1.2, and let D be a Jordan domain such that the Jordan curve bounding D passes every point of E. Let $N(z_0)$ denote a neighborhood of some point z_0 in E, and u a some harmonic function on D. If u is a bounded function in $N(z_0) \cap D$, then it is not the case that z_0 in E, there exist two curves in D at z_0 on which u has the cluster values ω_1 and ω_2, $(\omega_1 \neq \omega_2)$.*

Proof. Since u is harmonic on Jordan domain D, there exists a v the harmonic conjugate of u on D. Hence we let $f(z)$ denote a function satisfying

$$f(z) = \exp(u + iv).$$

Then $f(z)$ is a single-valued analytic function on D and $f(z)$ is bounded on $N(z_0) \cap D$. Hence applying Theorem 1.2 to $f(z)$, we obtain the above consequence for $u(x, y) = Re\, f(z)$. □

There are a number of geometric applications of resistant length. The simplest example concerns the ring domain. In our discussion we will need the following.

Definition 3.5. [1]. Let D be a simply connected domain in $\mathbb{C}$. A *crosscut* of D is a Jordan curve γ in D which in both directions tends to a boundary point.

Lemma 3.6. [10]. *Let R be a ring domain in $\mathbb{C}$ and let R_0 and R_1 denote the bounded component and unbounded component of R^c the complement of R, respectively. Let ∂R_0 and ∂R_1 denote the two components of the boundary of R, and let Γ_R be the family of all curves in R connecting ∂R_0 and ∂R_1. Then*

$$\lambda(\Gamma_R) = \infty$$

if and only if R_0 consists of a single point.

Lemma 3.7. [3]. *Let R, R_0, R_1 and Γ_R be as in Lemma 3.6. We say the closed curve γ in R separates R_0 and R_1 if γ has non-zero winding number about the points of R_0. Let Γ_S be the family of all closed curves in R which separates R_0 and R_1. Then*

$$\lambda(\Gamma_R) \cdot \lambda(\Gamma_S) = 1.$$

We say that $\lambda(\Gamma_S)$ is the *conjugate resistant length* of $\lambda(\Gamma_R)$.

Theorem 3.8. *Let $R, \partial R_0$ and ∂R_1 be as in Lemma 3.6. Let a be the length of the shortest arc in D connecting ∂R_0 and ∂R_1. Let b be the length of the Jordan curve, ∂R_0. Then*

$$a \cdot b \leq S,$$

where S is the area of R.

Proof. The purely geometric proof of this theorem is difficult. The use of resistant length, however, makes the proof trivial.

Let Γ_R and Γ_S be as in Lemmas 3.6 and 3.7 respectively. Then by Lemma 3.7,

$$\lambda(\Gamma_R) \cdot \lambda(\Gamma_S) = 1.$$

On the other hand, we choose the non-negative Borel measurable function $\rho = 1$, then $\lambda(\Gamma_R)$ and $\lambda(\Gamma_S)$ have the following lower bounds respectively. That is,

$$\begin{aligned} \frac{a^2}{S} \cdot \frac{b^2}{S} &= \frac{L(\Gamma_R, 1)^2}{A(D,1)} \cdot \frac{L(\Gamma_S, 1)^2}{A(D,1)} \\ &\leq \lambda(\Gamma_R) \cdot \lambda(\Gamma_S) \\ &= 1. \end{aligned}$$

and the theorem follows at once. □

The following theorem concerns the general quadrilateral.

THEOREM 3.9. *Suppose that we have a set of n disjoint general quadrilaterals Q_k, for $k = 1, 2, \ldots, n$, that are contained in the annulus $\Delta = \{z \,|\, r < |z| < R\}$, $(0 < r < R,\ R \neq \infty)$ and that are bounded by Jordan curves each of which has an arc, in common with each of the circles $\{z \,|\, |z| = r\}$ and $\{z \,|\, |z| = R\}$. (The Q_k can be regarded as strips extending from the inner to the outer circle.) If these domains Q_k are mapped onto rectangles B_k with sides equal respectively to a_k and b_k in such a way that the arcs referred to are mapped into sides of lengths a_k, then*

$$\sum_{k=1}^{n} \frac{a_k}{b_k} \leq \frac{2\pi}{\log(R/r)} \tag{2}$$

with equality holding only if the Q_k are domains of the form $\{z \,|\, r < |z| < R,\ \phi_k < \arg z < \phi_{k+1}\}$ completely filling the annulus.

Proof. The method of resistant length considered leads to a simple proof of the inequality(2).

We can map an arbitrary general quadrilateral conformally onto a rectangle, ([6, p.15]). Let $w = f_k(z)$ be 1-1 conformal mappings on Q_k upon B_k respectively. Let Γ be the family of arcs in Δ which join the two boundary circles, and let Γ_k be the family of arcs in Q_k which join the two sides of $Q_k \subset \partial\Delta$. Then by the conformal invariance of resistant length(Proposition 2.2) and Theorem 2.5,

$$\lambda(\Gamma_k) = \lambda[f_k(\Gamma_k)] = \frac{b_k}{a_k}\,. \tag{3}$$

By the hypothesis, there exist disjoint open sets $Q_k(k = 1, 2, \ldots, n)$ containing Γ_k and $\cup_k\Gamma_k \subset \Gamma$. Hence by Proposition 2.4,

$$\sum_{k=1}^{n} \frac{1}{\lambda(\Gamma_k)} \;\leq\; \frac{1}{\lambda(\Gamma)}\,. \tag{4}$$

Therefore by Theorem 2.6, (3) and (4), we obtain (2).

The proof is complete. $\square$

Now, we will alternatively prove the well-known result by making use of the method of resistant length. In particular, this method shortens the length of proof significantly as we shall see by comparing the following proof with that of Theorem 14.22 in [8].

THEOREM 3.10. [8]. *Let* $\Delta(r, R) = \{z \,|\, r < |z| < R\}$, $(0 < r < R, R \neq \infty)$. *Then* $\Delta_1(r_1, R_1)$ *and* $\Delta_2(r_2, R_2)$ *are conformally equivalent if and only if*

$$\frac{R_1}{r_1} = \frac{R_2}{r_2} \tag{5}$$

Proof.(Method of resistant length) Since the proof of sufficient conditions is trivial, we discuss the proof of necessary conditions. Let Γ_Δ be the family of arcs in $\Delta(r, R)$ which joins the two contours. Then by Theorem 2.6,

$$\lambda(\Gamma_\Delta) = \frac{1}{2\pi}\log\frac{R}{r}\,. \tag{6}$$

Suppose that $\Delta_1(r_1, R_1)$ and $\Delta_2(r_2, R_2)$ are conformally equivalent and let f be a 1-1 conformal mapping on $\Delta_1(r_1, R_1)$ upon $\Delta_2(r_2, R_2)$. Then by the conformal invariance of resistant length(proposition 2.2),

$$\lambda(\Gamma_{\Delta_1}) = \lambda[f(\Gamma_{\Delta_1})] = \lambda(\Gamma_{\Delta_2}). \tag{7}$$

Hence by (6), (7), we obtain (5).

The proof is now complete. $\square$

References

[1] L. V. Ahlfors, *Conformal Invariants. Topics in Geometric Function Theory*, McGraw-Hill, New York (1973)

[2] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand (1987)

[3] L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton Math. Ser., **26** (1960)

[4] Bohyun Chung, *Some results for the resistant lengths of curve families* (II), J. Appl. Math. and Computing., 15(No. 1-2) (2004), 495–502

[5] Wansoo Jung and Bohyun Chung, *Some results related to extremal length* (II), Jour. Chungcheong Math. Soc., 16(No. 1)(2003), 49–60

[6] Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Springer-Verlag, New York (1973)

[7] K. Matsumoto, *On some boundary problems in the theory of conformal mappings of Jordan domains*, Nagoya. Math. J., **24** (1964), 129–141.

[8] W. Rudin, *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New Delhi, (1974)

[9] L. Sario and K. Oikawa, *Capacity Functions*, Springer-Verlag, New York, (1969)

[10] J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings*, Springer-Verlag, New York, (1971)

[11] V. Wolontis, *Properties of conformal invariants*, Amer. J. Math., **74** (1962), 587–606.

Bohyun Chung
Mathematics Section
College of Science and Technology
Hongik University
Chochiwon 339–701, Korea
E-mail: `bohyun@hongik.ac.kr`

Wansoo Jung
Department of Mathematics
Soonchunhyang University
Asan 336–745, Korea
E-mail: `jungws@sch.ac.kr`