Kangweon-Kyungki Math. Jour. **14** (2006), No. 2, pp. 291–302

SUFFICIENT CONDITIONS FOR STRONG CONVERGENCE OF WEIGHTED SUMS OF INDEPENDENT FUZZY RANDOM VARIABLES

Young Nam Hyun, Sang Yeol Joo* and Seung Soo Lee

Abstract. In this paper, we give sufficient conditions for a.s. convergence of weighted sums of integrable fuzzy random variables. As a result, we obtain strong laws of large numbers for weighted sums of independent fuzzy random variables.

1. Introduction

In the past 20 years, there have been increasing interests in limit theorems for fuzzy random variables because of its usefulness in several applied fields. Among others, laws of large numbers for fuzzy random variables have been led to the need in order to ensure consistency in estimation problems of statistical inference for fuzzy probability models.

Strong laws of large numbers for sums of independent fuzzy random variables have been studied by several researchers. Klement et al. [9] introduced L^1-metric d_1 and uniform metric d_∞ on the space of fuzzy sets (cf. Section 2) and proved SLLN for i.i.d. fuzzy random variables in the sense of d_1. Inoue [3] extended their result [9] to the case of independent and level-wise tight fuzzy random variables. Colubi et al. [1] obtained SLLN for i.i.d. fuzzy random variables with respect to d_∞ by approximation method. Molchanov [10] gave a short proof of SLLN for i.i.d. fuzzy random variables in the sense of d_∞. This was also proved by Joo and Kim [6], independently. Recently, Joo [4] obtained a SLLN for independent and convexly tight fuzzy random variables by using the Skorokhod metric d_s which was introduced by Joo and Kim [5].

Received December 27, 2006.
2000 Mathematics Subject Classification: Primary 58B34, 58J42, 81T75.
Key words and phrases: fuzzy random variables, strong laws of large number, weighted sums.
* Corresponding author.

The purpose of this paper is to find sufficient conditions for strong convergence of weighted sums of $\{\tilde{X}_n\}$ in the following;

$$\lim_{n\to\infty} d_\infty \left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i\right) = 0 \text{ a.s.}$$

2. Preliminaries

We describe some preliminary results for fuzzy numbers. Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0,1]$ with the following properties ;

(a) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.
(b) $\tilde{u}$ is upper semicontinuous.
(c) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.
(d) $\tilde{u}$ is a convex, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ \text{supp } \tilde{u}, & \alpha = 0, \end{cases}$$

then, it follows that $\tilde{u}$ is a fuzzy number if and only if $L_1\tilde{u} \neq \phi$ and $L_\alpha\tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$. From this characterization of fuzzy numbers, a fuzzy number $\tilde{u}$ is completely determined by the closed intervals $L_\alpha\tilde{u} = [u^l_\alpha, u^r_\alpha]$.

THEOREM 2.1. *For $\tilde{u} \in F(R)$, the followings hold;*

(a) *u^l is a bounded increasing function on* [0,1].
(b) *u^r is a bounded decreasing function on* [0,1].
(c) $u^l_1 \leq u^r_1$.
(d) *u^l and u^r are left continuous on* [0,1] *and right continuous at 0.*
(e) *If v^l and v^r satisfy the above* (a)-(d), *then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha\tilde{v} = [v^l_\alpha, v^r_\alpha]$.*

Proof. See Goetschel and Voxman [2]. □

The above theorem implies that we can identify a fuzzy number $\tilde{u}$ with the family of closed interval $\{[u^l_\alpha, u^r_\alpha] : 0 \leq \alpha \leq 1\}$, where u^l and u^r satisfy (a)-(d) of Theorem 2.1.

The linear structure on $F(R)$ is defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \tilde{0}, & \lambda = 0, \end{cases}$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$,

Then it is well-known that if

$$\tilde{u} = \{[u^l_\alpha,\ u^r_\alpha] : 0 \leq \alpha \leq 1\}$$

and

$$\tilde{v} = \{[v^l_\alpha,\ v^r_\alpha] : 0 \leq \alpha \leq 1\},$$

then

$$\tilde{u} \oplus \tilde{v} = \{[u^l_\alpha + v^l_\alpha,\ u^r_\alpha + v^r_\alpha) : 0 \leq \alpha \leq 1\},$$

and

$$\lambda\tilde{u} = \begin{cases} \{[\lambda u^l_\alpha, \lambda u^r_\alpha] & : \ 0 \leq \alpha \leq 1\}, \quad \lambda \geq 0, \\ \{[\lambda u^r_\alpha, \lambda u^l_\alpha] & : \ 0 \leq \alpha \leq 1\}, \quad \lambda < 0. \end{cases}$$

We can define L_1-metric d_1 and uniform metric d_∞ on $F(R)$ as follows;

$$d_1(\tilde{u}, \tilde{v}) = \int_0^1 \max\left(|u^l_\alpha - v^l_\alpha|, |u^r_\alpha - v^r_\alpha|\right) d\alpha$$

$$\begin{aligned} d_\infty(\tilde{u}, \tilde{v}) &= \sup_{0\leq\alpha\leq1} \max\left(\left|u^l_\alpha - v^l_\alpha\right|, |u^r_\alpha - v^r_\alpha|\right) \\ &= \max\left(\sup_{0\leq\alpha\leq1} \left|u^l_\alpha - v^l_\alpha\right|, \sup_{0\leq\alpha\leq1} |u^r_\alpha - v^r_\alpha|\right). \end{aligned}$$

The norm of $\tilde{u} \in F(R)$ is defined by

$$||\tilde{u}|| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u^l_0|, |u^r_0|).$$

It is well-known that $(F(R), d_1)$ is separable but is not complete, and that $(F(R), d_\infty)$ is complete but is not separable(For details, see Klement et al [9]). Joo and Kim [5] introduced the Skorokhod metric d_s on $F(R)$ which makes it a separable metric space as follows :

DEFINITION 2.2. Let T denote the class of strictly increasing, continuous mapping of [0,1] onto itself. For $\tilde{u}, \tilde{v} \in F(R)$, we define

$$\begin{aligned} d_s(\tilde{u}, \tilde{v}) &= \inf\{\epsilon > 0 : \textit{ there exists a } t \in\ T \textit{ such that} \\ &\quad \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \textit{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\}, \end{aligned}$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

It follows immediately that d_s is a metric on $F(R)$ and $d_s(\tilde{u}, \tilde{v}) \le d_\infty(\tilde{u}, \tilde{v})$. The metric d_s will be called the Skorokhod metric. It is well-known that $(F(R), d_s)$ is separable and topologically complete. Also, d_∞-convergence implies d_s-convergence and d_s-convergence implies d_1-convergence. But the converses are not true. (For details, see Joo and Kim [5])

3. Main Results

Throughout this paper, let$(\Omega, \mathcal{F}, P)$ be a probability space. A fuzzy number valued function

$$\tilde{X} : \Omega \to F(R),\ \tilde{X} = \{[X_\alpha^l, X_\alpha^r] : 0 \le \alpha \le 1\}$$

is called a fuzzy random variable if for each $\alpha \in [0, 1], X_\alpha^l$ and X_α^r are random variables in the usual sense. It is well-known that $\tilde{X}$ is a fuzzy random variables if and only if $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable(See Kim [7]). So we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated.

A fuzzy random variable $\tilde{X}$ is called integrable if $E||\tilde{X}|| < \infty$. The expectation of integrable fuzzy random variables $\tilde{X}$ is a fuzzy number defined by

$$E\tilde{X} = \{[EX_\alpha^l, EX_\alpha^r] : 0 \le \alpha \le 1\}.$$

Let $\{\tilde{X}_n\}$ be a sequence of integrable fuzzy random variables and $\{\lambda_{ni}\}$ be a Toeplitz sequence, i.e. $\{\lambda_{ni}\}$ is a double array of real numbers satisfying

(a) For each i, $\lim_{n\to\infty} \lambda_{ni} = 0$;
(b) There exists $C > 0$ such that $\sum_{i=1}^{\infty} |\lambda_{ni}| \le C$ for each n.

For our purpose, we write $\tilde{X}_n = \{[X^l_{n,\alpha}, X^r_{n,\alpha}] : 0 \le \alpha \le 1\}$ and assume the following condition:

(**3**.**1**) : For each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that for all n,

$$\max\left(\max_{1\le k\le m} E\left|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}\right|, \max_{1\le k\le m} E\left|X^r_{n,\alpha^+_{k-1}} - X^r_{n,\alpha_k}\right|\right) < \epsilon.$$

The next lemma implies that if $\{\tilde{X}_n\}$ is identically distributed, then it satisfies the condition (3.1)

LEMMA 3.1. (a) *Let* $E||\tilde{X}|| < \infty$. *Then for each* $\epsilon > 0$, *there exists a partition* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ *of* $[0,1]$ *such that*

$$\max\left(\max_{1\le k\le m} E\left|X^l_{\alpha^+_{k-1}} - X^l_{\alpha_k}\right|, \max_{1\le k\le m} E\left|X^r_{\alpha^+_{k-1}} - X^r_{\alpha_k}\right|\right) < \epsilon.$$

(b) *If* $\{\tilde{X}_n\}$ *is identically distributed and* $E||\tilde{X}_1|| < \infty$, *then it satisfies the condition* (3.1)

Proof. (a): We first prove that for each $\epsilon > 0$, there exists a partition $\mathbb{P}_l : 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_s = 1$ of $[0,1]$ such that

$$\max_{1\le k\le s} E\left|X^l_{\alpha^+_{k-1}} - X^l_{\alpha_k}\right| < \epsilon.$$

Suppose that this assertion does not hold. Then we can find a $\epsilon_0 > 0$ such that for any partition $\mathbb{P}_l : 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_s = 1$ of $[0,1]$, there is a α_k in $\mathbb{P}_l$ satisfying

$$E\left|X^l_{\alpha^+_{k-1}} - X^l_{\alpha_k}\right| \le \epsilon_0.$$

We consider a sequence $\{\mathbb{P}_n\}$ of partitions of $[0,1]$ such that

(a) $\mathbb{P}_{n+1}$is a refinement of $\mathbb{P}_n$;
(b) $||\mathbb{P}_n|| \to 0$, where $||\mathbb{P}_n||$ denotes the longest length of largest subinterval of $||\mathbb{P}_n||$.

For each n, let α_{k_n} be the largest point of $\mathbb{P}_n$ such that

$$E\left|X^l_{\alpha^+_{k_n-1}} - X^l_{\alpha_{k_n}}\right| \ge \epsilon_0. \qquad (1)$$

Then $\{\alpha_{k_n}\}$ is monotone decreasing since $\mathbb{P}_{n+1}$ is a refinement of $\mathbb{P}_n$. By (1), we can choose a sequence $\{\beta_n\}$ with $\alpha_{k_n-1} < \beta_n < \alpha_{k_n}$ such that for all n,

(2) $$E\left|X^l_{\beta_n} - X^l_{\alpha_{k_n}}\right| \geq \epsilon_0/2.$$

Since $\{\alpha_{k_n}\}$ is monotone decreasing and $||\mathbb{P}_n|| \to 0$, it follows that

$$\left|X^l_{\beta_n} - X^l_{\alpha_{k_n}}\right| \to 0 \text{ pointwise.}$$

Then by the Lebesgue dominated convergence theorem,

$$E\left|X^l_{\beta_n} - X^l_{\alpha_{k_n}}\right| \to 0,$$

which is impossible by (2).

Similarly, it can be proved that for each $\epsilon > 0$, there exists a partition $\mathbb{P}_r : 0 = \beta_0 < \beta_1 < \cdots < \beta_t = 1$ of $[0,1]$ such that

$$\max_{1\leq k\leq t} E\left|X^r_{\beta^+_{k-1}} - X^r_{\beta_k}\right| < \epsilon.$$

Then $\mathbb{P} = \mathbb{P}_l \cup \mathbb{P}_r$ is the desired partition of $[0,1]$.

(b): It follows from the fact that

$$\max\left(\max_{1\leq k\leq m} E\left|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}\right|, \max_{1\leq k\leq m} E\left|X^r_{n,\alpha^+_{k-1}} - X^r_{n,\alpha_k}\right|\right)$$
$$= \max\left(\max_{1\leq k\leq m} E\left|X^l_{1,\alpha^+_{k-1}} - X^l_{1,\alpha_k}\right|, \max_{1\leq k\leq m} E\left|X^r_{1,\alpha^+_{k-1}} - X^r_{1,\alpha_k}\right|\right).$$

□

Now we need the concept of independence for fuzzy random variables. By the way, there are several definitions of independence for fuzzy random variables. In this paper, we adopt the following notion of independence :

Definition 3.2. Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables.

(a) $\{\tilde{X}_n\}$ is called level-wise independent if for each $\alpha \in [0,1]$, a sequence of random vectors $\{(X^l_{n,\alpha}, X^r_{n,\alpha})\}$ is independent.

(b) $\{\tilde{X}_n\}$ is called independent if a sequence of σ-fields $\{\sigma(\tilde{X}_n)\}$ is independent, where $\sigma(\tilde{X})$ is the smallest σ-field of subsets of Ω such that $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable.

Note that $\sigma(\tilde{X}) = \sigma(\{X^l_\alpha, X^r_\alpha : 0 \leq \alpha \leq 1\})$ because $\tilde{X} : \Omega \to (F(R), d_s)$ is measurable if and only if for each $\alpha \in [0,1]$, X^l_α and X^r_α are measurable.

THEOREM 3.3. *Let $\{\tilde{X}_n\}$ be a sequence of independent fuzzy random variables satisfying* (3.1). *Suppose that there exists a nonnegative random variable ξ with $E\xi^{1+\frac{1}{\gamma}} < \infty$ for some $\gamma > 0$ such that for each n,*

$$P\left(||\tilde{X}_n|| \geq \lambda\right) \leq P(\xi \geq \lambda) \ for\ all\ \lambda > 0. \tag{3}$$

If $\{\lambda_{ni}\}$ is a Toeplitz sequence satisfying $\max_{1\leq i\leq n}|\lambda_{ni}| = O(n^{-\gamma})$, *then*

$$\lim_{n\to\infty} d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i\right) = 0 \ \ a.s.$$

Proof. Since $\{\lambda_{ni}\}$ is a Toeplitz sequence, there exists a positive number C such that

$$\sum_{i=1}^{\infty} |\lambda_{ni}| \leq C \text{ for all } n.$$

Let $\epsilon > 0$ be given. By the condition(3.1), we choose $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ of $[0,1]$ such that

$$\max_{1\leq k\leq m} E\left|X^l_{n,\alpha^+_{k-1}} - X^l_{n,\alpha_k}\right| < \epsilon/C. \tag{4}$$

Then

$$\begin{aligned}
&\sup_{\alpha_{k-1}<\alpha\leq\alpha_k}\left|\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha} - EX^l_{i,\alpha})\right| \\
\leq\ & \sup_{\alpha_{k-1}<\alpha\leq\alpha_k}\left|\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha} - X^l_{i,\alpha_k})\right| + \left|\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k})\right| \\
&+ \sup_{\alpha_{k-1}<\alpha\leq\alpha_k}\left|\sum_{i=1}^n \lambda_{ni}E(X^l_{i,\alpha_k} - EX^l_{i,\alpha})\right| \\
\leq\ & \sum_{i=1}^n |\lambda_{ni}|\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}\right| + \left|\sum_{i=1}^n \lambda_{ni}(X^l_{i,\alpha_k} - EX^l_{i,\alpha_k})\right| \\
&+ \sum_{i=1}^n |\lambda_{ni}|\left|EX^l_{i,\alpha_k} - EX^l_{i,\alpha^+_{k-1}}\right| \\
=\ & \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}$$

For (I), we first note that for all i,

$$P\left\{\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}\right| \geq \lambda\right\} \leq P(2||\tilde{X}_n|| \geq \lambda) \leq P(2\xi \geq \lambda).$$

Thus, by Rohatgi's SLLN[11] for weighted sums of real-valued random variables,

$$\sum_{i=1}^{n} |\lambda_{ni}| \left(\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}\right| - E\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k})\right|\right) \to 0 \;\; a.s.$$

Thus by (4),

$$\begin{aligned}
\text{(I)} &= \sum_{i=1}^{n} |\lambda_{ni}| \left(\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}\right| - E\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}\right|\right) \\
&\quad + \sum_{i=1}^{n} |\lambda_{ni}| E\left|X^l_{i,\alpha^+_{k-1}} - X^l_{i,\alpha_k}\right| \\
&< \epsilon a.s. \text{ for large } n.
\end{aligned}$$

For (II), since

$$P\left\{\left|X^l_{n,\alpha_k}\right| \geq \lambda\right\} \leq P(||\tilde{X}_n|| \geq \lambda) \leq P(\xi \geq \lambda).$$

by Rohatgi's result[11], (II) $\to 0$ $a.s.$

Finally, it is trivial that (III) $< \epsilon$ by (4).

Hence we obtain

$$\begin{aligned}
&\sup_{0\leq\alpha\leq1} \left|\sum_{i=1}^{n} \lambda_{ni}(X^l_{i,\alpha} - EX^l_{i,\alpha})\right| \\
&= \max_{1\leq k\leq m} \sup_{\alpha_{k-1}<\alpha\leq\alpha_k} \left|\sum_{i=1}^{n} \lambda_{ni}(X^l_{i,\alpha} - EX^l_{i,\alpha})\right| \\
&< 2\epsilon \;\; a.s. \text{ for large } n.
\end{aligned}$$

Similarly, it can be proved that

$$\sup_{0\leq\alpha\leq1} \left|\sum_{i=1}^{n} \lambda_{ni}(X^r_{i,\alpha} - EX^r_{i,\alpha})\right| < 2\epsilon \;\; a.s. \text{ for large } n.$$

Therefore,

$$\begin{aligned} & d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\right) \\ = & \max\left(\sup_{0\le\alpha\le1}\left|\sum_{i=1}^n \lambda_{ni}(X_{i,\alpha}^l - EX_{i,\alpha}^l)\right|, \sup_{0\le\alpha\le1}\left|\sum_{i=1}^n \lambda_{ni}(X_{i,\alpha}^r - EX_{i,\alpha}^r)\right|\right) \\ < & \ 2\epsilon \ \ a.s. \text{ for large } n. \end{aligned}$$

Since ϵ is arbitrary, this completes the proof. □

The following corollary is an analogue of strong law of large numbers obtained by Joo and Kim [6].

Corollary 3.4. *Let* $\{\tilde{X}_n\}$ *be a sequence of independent and identically distributed fuzzy random variables with* $E||\tilde{X}_1||^{1+\frac{1}{\gamma}} < \infty$ *for some* $\gamma > 0$. *If* $\{\lambda_{ni}\}$ *is a Toeplitz sequence satisfying* $\max_{1\le i\le n}|\lambda_{ni}| = O(n^{-\gamma})$, *then*

$$\lim_{n\to\infty} d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\right) = 0 \ a.s.$$

Corollary 3.5. *Let* $\{\tilde{X}_n\}$ *be a sequence of independent fuzzy random variables satisfying* (3.1). *If*

$$\sup_n E||\tilde{X}_n||^p = M < \infty \ for\ some\ p > 1,$$

then for any Toeplitz sequence $\{\lambda_{ni}\}$ *satisfying* $\max_{1\le i\le n}|\lambda_{ni}| = O(n^{-\gamma})$, *for* $\gamma > \frac{1}{p-1}$,

$$\lim_{n\to\infty} d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i\right) = 0 \ a.s.$$

Recall that we can define the concept of convexity on $F(R)$ as in the case of a vector space. That is, $A \subset F(R)$ is said to be convex if $\lambda\tilde{u} \oplus (1-\lambda)\tilde{v} \in A$ whenever $\tilde{u}, \tilde{v} \in A$ and $0 \le \lambda \le 1$. Using this concept of convexity on $F(R)$, we can introduce the following concepts.

Definition 3.6. Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables.

(a) $\{\tilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$ there exists a compact subset K of $(F(R), d_s)$ such that

$$P(\tilde{X}_n \notin K) < \epsilon \text{ for all } n.$$

If we can choose a convex and compact subset K, then $\{\tilde{X}_n\}$ is said to be convexly tight.

(b) $\{\tilde{X}_n\}$ is said to be compactly uniformly integrable if for each $\epsilon > 0$ there exists a compact subset K of $(F(R), d_s)$ such that

$$\int_{\{\tilde{X}_n \notin K\}} ||\tilde{X}_n|| dP < \epsilon \text{ for all } n.$$

If we can choose a convex and compact subset K, then $\{\tilde{X}_n\}$ is said to be convex-compactly uniformly integrable.

If $\{\tilde{X}_n\}$ is convexly tight, then it is tight. But the converse is not true (for a counter-example, see Joo [4]). This problem arises from the fact that there exists a compact subset K of $(F(R), d_s)$ such that its convex hull $co(K)$ is not compact(For details, see Kim [8]). Similar statements can be applied for compact uniform integrability.

Corollary 3.7. *Let $\{\tilde{X}_n\}$ be a sequence of independent and convex-compactly uniformly integrable fuzzy random variables. If*

$$\sup_n E||\tilde{X}_n||^p = M < \infty \; for \; some \; p > 1,$$

then for any Toeplitz sequence $\{\lambda_{ni}\}$ satisfying $\max_{1\le i\le n} |\lambda_{ni}| = O(n^{-\gamma})$, for $\gamma > \frac{1}{p-1}$,

$$\lim_{n\to\infty} d_\infty \left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i \right) = 0 \; a.s.$$

Corollary 3.8. *Let $\{\tilde{X}_n\}$ be a sequence of independent and convexly tight fuzzy random variables. If*

$$\sup_n E||\tilde{X}_n||^p = M < \infty \; for \; some \; p > 1,$$

then for any Toeplitz sequence $\{\lambda_{ni}\}$ satisfying $\max_{1\le i\le n} |\lambda_{ni}| = O(n^{-\gamma})$, for $\gamma > \frac{1}{p-1}$,

$$\lim_{n\to\infty} d_\infty \left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni} E\tilde{X}_i \right) = 0 \; a.s.$$

Corollary 3.9. *Let $\{\tilde{X}_n\}$ be a sequence of independent and p-th order convex-compactly uniformly integrable $(p > 1)$, i.e. for each $\epsilon > 0$ there exists a compact convex subset K of $(F(R), d_s)$ such that*

$$\int_{\{\tilde{X}_n \notin K\}} ||\tilde{X}_n||^p dP < \epsilon \; for \; all \; n.$$

Then for any Toeplitz sequence $\{\lambda_{ni}\}$ *satisfying* $\max_{1\le i\le n}|\lambda_{ni}| = O(n^{-\gamma})$, *for* $\gamma > \frac{1}{p-1}$,

$$\lim_{n\to\infty} d_\infty\left(\oplus_{i=1}^n \lambda_{ni}\tilde{X}_i, \oplus_{i=1}^n \lambda_{ni}E\tilde{X}_i\right) = 0 \ a.s.$$

References

[1] A. Colubi, J. S. Dominguez-Menchero, M. L ópez-Diáz, and M. A. Gil, *A generalized strong law of large numbers*, Probab. Theory Rel. Fields **114** (1999), 401-417.

[2] R. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems **18** (1986),31-43.

[3] H. Inoue, *A strong law of large numbers for fuzzy random sets*, Fuzzy Sets and Systems **41** (1991), 285-291.

[4] S. Y. Joo, *Strong law of large numbers for tight fuzzy random variables*, Jour. Korean Statist. Soc. **31** (2002), 129-140.

[5] S. Y. Joo and Y. K. Kim, The Skorokhod topology on the space of fuzzy numbers, Fuzzy Sets and Systems **246** (2000), 576-590.

[6] S. Y. Joo and Y. K. Kim, Kolmogorovs strong law of large numbers for fuzzy random variables, Fuzzy Sets and Systems **120** (2001), 499-503.

[7] Y. K. Kim, Measurability for fuzzy valued functions, Fuzzy Sets and Systems **129** (2002), 105-109.

[8] Y. K. Kim, Compactness and convexity on the space of fuzzy sets, J. Math. Anal. Appl. **264** (2001), 122-132.

[9] E. P. Klement, M. L. Puri and D. A. Ralescu, Limit theorems for fuzzy random variables, Proc. Roy. Soc. Lond. Ser. A **407** (1986), 171-182.

[10] I. Molchanov, On strong law of large numbers for random upper semicontinuous functions, J. Math. Anal. Appl. **235** (1999), 349-355.

[11] V. K. Rohatgi, Convergence of weighted sums of independent random variables, Proc. Cambridge Philos. Soc. **69** (1971), 305-307.

Young Nam Hyun
Department of Statistics
Kangwon National University
Chunchon 200–701, Korea
E-mail: `hyn0117@gwd.go.kr`

Sang Yeol Joo
Department of Statistics
Kangwon National University
Chunchon 200–701, Korea
E-mail: `syjoo@kangwon.ac.kr`

Seung Soo Lee
Department of Statistics
Kangwon National University
Chunchon 200–701, Korea
E-mail: sslee@kangwon.ac.kr