J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 13, Number 3 (August 2006), Pages 197–206

MULTIVALUED MIXED QUASI-VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. This paper introduces a class of multivalued mixed quasi-variational-like inequalities and shows the existence of solutions to the class of quasi-variational-like inequalities in reflexive Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The importance of applications of variational inequalities to many areas, for examples, optimization problems, differential equations, equilibrium problems in nonlinear analysis is known in many researches (see [1, 5, 10, 11] and references therein). Parida and Sen [7] firstly posed variational-like inequalities and Aubin and Ekeland [1] also firstly introduced quasi-variational inequalities. Recently, Verma [9] intorduced a class of monotone nonlinear variational inequalities and considered the existence of solutions. Very recently, Cho et al. [2], Fang et al. [4] and Huang et al. [6] generalized and improved the results of Verma [9] to a class of nonlinear quasi-variational-like inequalities.

This paper introduces a new class of generalized quasi-variational-like inequalities and, generalizes and improves the results of Cho et al. [2]. In the proof, some wrong part of the proof in [2] is corrected.

Let X be a real Banach space with dual space $X^\star$ and K a nonempty convex closed subset of X. Denote $\langle \ell, x\rangle = \ell(x)$, for all $\ell \in X^\star$ and $x \in X$. Let $S, T : K \to 2^{X^\star}$ be two multivalued mappings, $N : X^\star \times X^\star \to X^\star$ and $g : K \to X^\star$ be mappings.

Recently, the following nonlinear mixed quasi-variational-like inequality was studied by Cho et al. [2].

Received by the editors March 24, 2006 and, in revised form, July 21, 2006.
2000 *Mathematics Subject Classification.* 49J40, 90C33.
Key words and phrases. multivalued quasi-variational-like inequalities, monotone type mapping, Lipschitz type mapping, KKM-mapping, Fan-KKM Theorem.

For any $\ell \in X^\star$, find $u \in K$ such that

$$\sup_{x\in S(u),\, y\in T(u)} \langle N(x,y) - \ell,\ v-u\rangle + f(v) - f(u) \ \geq\ 0, \text{ for all } v \in K. \tag{1.1}$$

Also, recently the following quasi-variational-like inequality was considered by Fang et al. [4].

For any $\ell \in X^\star$, find $u \in K$ such that

$$\sup_{x\in S(u),\, y\in T(u)} \langle x-y-\ell, \eta(v,u)\rangle + f(v) - f(u) \ \geq\ 0, \text{ for all } v \in K. \tag{1.2}$$

For single-valued mappings S and T, the following quasi-variational-like inequality was studied by Huang et al. [6].

For any $\ell \in X^\star$, find $u \in K$ such that

$$\langle N(S(u), T(u)) - \ell, \eta(v,u)\rangle + f(v) - f(u) \ \geq\ 0, \text{ for all } v \in K. \tag{1.3}$$

In this paper, we consider the following generalized quasi-variational-like inequality;

For any $\ell \in X^\star$, find $u \in K$ such that

$$\sup_{x\in S(u),\, y\in T(u)} \langle (g(u) + N(x,y)) - \ell,\ G(v) - G(u)\rangle + f(v) - f(u) \ \geq\ 0, \tag{1.4}$$

for all $v \in K$, where $G : K \to K$ is a mapping.

Definition 1.1. A mapping $S : K(\subset X) \to 2^{X^\star}$ is said to be G-φ-p-monotone with respect to the first argument of a mapping $N : X^\star \times X^\star \to X^\star$ if there exist a function $\varphi : [0,+\infty) \to [0,+\infty)$, a mapping $G : K \to K$ and a constant $p > 1$ such that

$$\langle N(x,\cdot) - N(y,\cdot), G(u) - G(v)\rangle \ \geq\ \varphi(\|G(u) - G(v)\|)\, \|G(u) - G(v)\|^p, \tag{1.5}$$

for all $u, v \in K$, $x \in S(u)$ and $y \in S(v)$.

Definition 1.2. A mapping $T : K(\subset X) \to 2^{X^\star}$ is said to be G-ψ-p-monotone with respect to the second argument of a mapping $N : X^\star \times X^\star \to X^\star$ if there exist a function $\psi : [0,+\infty) \to [0,+\infty)$, a mapping $G : K \to K$ and a constant $p > 1$ such that

$$\langle N(\cdot,x) - N(\cdot,y), G(u) - G(v)\rangle \ \geq\ -\psi(\|G(u) - G(v)\|)\, \|G(u) - G(v)\|^p, \tag{1.6}$$

for all $u, v \in K$, $x \in T(u)$ and $y \in T(v)$.

Definition 1.3. A mapping $g : K(\subset X) \to X^\star$ is said to be G-ϕ-p-relaxed Lipschitzian if there exist a function $\phi : [0,+\infty) \to [0,+\infty)$, a mapping $G : K \to K$ and a constant $p > 1$ such that

(1.7)

$$\langle g(v) - g(u), G(v) - G(u)\rangle \ \geq \phi(\|G(v) - G(u)\|)\, \|G(v) - G(u)\|^p, \text{ for all } u, v \in K.$$

Definition 1.4. Let X and Y be topological spaces. A mapping $F : X \to 2^Y$ is said to be lower semi-continuous at $x \in X$ if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there exists a subnet $\{x_\beta\} \subset \{x_\alpha\}$ and $y_\beta \in F(x_\beta)$ for each β such that $\{y_\beta\}$ converging to y.

Definition 1.5 ([8]). A mapping $g : K \to X^\star$ is said to be hemi-continuous if for all $u, v, z \in K$, the mapping $t \to \langle g(u + tv), z\rangle$ is continuous on [0,1]. A mapping $T : K \to 2^{X^\star}$ is said to be lower hemi-continuous if for all $u, v, z \in K$, the multivalued mapping

$$t \to \langle T(u + tv), z\rangle$$

is lower semi-continuous on [0,1].

2. Main Results

Now, we consider two kinds of variational inequalities, whose solution sets are the same.

Theorem 2.1. *Let X be a reflexive Banach space, $X^\star$ be its dual and K be a nonempty convex closed subset of X, let $g : K \to X^\star$ be a hemi-continuous mapping satisfying (1.7) and also let S and $T : K \to 2^{X^\star}$ be lower semi-continuous multivalued mappings satisfying (1.5) and (1.6), respectively, where for functions $\varphi, \psi, \phi : [0, +\infty) \to [0, +\infty)$ satisfying $\varphi(t) + \phi(t) > \psi(t)$ for all $t > 0$, $\varphi + \phi - \psi$ is bounded in $[0, \delta]$ for some $\delta > 0$. In addition, suppose that $G : K \to K$ is an affine mapping, $f : K \to \mathbb{R} \cup \{+\infty\}$ is a proper convex functional and $N : X^\star \times X^\star \to X^\star$ is continuous with respect to the weak* topology of X. Let a multivalued mapping $v \mapsto \{N(z, w) \in X^\star : z \in S(v), w \in T(v)\}$ be lower hemi-continuous. Then for any $\ell \in X^\star$, $u \in K$ is a solution of problem (1.4) if and only if $u \in K$ is a solution of the following problem: Find $u \in K$ such that*

$$\begin{aligned} &\langle (g(v) + N(z, w)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \\ &\geq (\varphi(\|G(v) - G(u)\|) - \psi(\|G(v) - G(u)\|) \\ &\quad + \phi(\|G(v) - G(u)\|))\|G(v) - G(u)\|^p \end{aligned} \tag{2.1}$$

for all $v \in K$, $z \in S(v)$ and $w \in T(v)$.

Proof. Suppose that the problem (1.4) holds. Since mappings S, T and g satisfy (1.5) (1.6) and (1.7), respectively, then for all $u, v \in K$, $x \in S(u)$, $z \in S(v)$, $y \in T(u)$ and $w \in T(v)$, we have

$$\begin{aligned}
&\langle (g(v) + N(z,w)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \\
&= \langle -\ell, G(v) - G(u)\rangle + \langle g(v), G(v) - G(u)\rangle + \langle N(z,w), G(v) - G(u)\rangle \\
&\quad + f(v) - f(u) \\
&= \langle -\ell, G(v) - G(u)\rangle - \langle N(x,w) - N(z,w), G(v) - G(u)\rangle \\
&\quad - \langle N(x,y) - N(x,w), G(v) - G(u)\rangle + \langle N(x,y), G(v) - G(u)\rangle \\
&\quad + \langle g(v) - g(u), G(v) - G(u)\rangle + \langle g(u), G(v) - G(u)\rangle + f(v) - f(u) \\
&= \langle -\ell, G(v) - G(u)\rangle + \langle N(x,w) - N(z,w), G(u) - G(v)\rangle \\
&\quad + \langle N(x,y) - N(x,w), G(u) - G(v)\rangle + \langle N(x,y), G(v) - G(u)\rangle \\
&\quad + \langle g(v) - g(u), G(v) - G(u)\rangle + \langle g(u), G(v) - G(u)\rangle + f(v) - f(u) \\
&\geq \langle (g(u) + N(x,y)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) + (\varphi(\|G(v) - G(u)\|) \\
&\quad - \psi(\|G(v) - G(u)\|) + \phi(\|G(v) - G(u)\|))\|G(v) - G(u)\|^p.
\end{aligned}$$

Put $A = \langle (g(v) + N(z,w)) - \ell, G(v) - G(u)\rangle + f(v) - f(u)$

$B = \langle (g(u) + N(x,y)) - \ell, G(v) - G(u)\rangle + f(v) - f(u)$

$C = (\varphi - \psi + \phi)(\|G(v) - G(u)\|)\|G(v) - G(u)\|^p.$

Taking suprema on both sides of the following inequality;

$$A \geq B + C,$$

we have

$$A = \sup_{x\in S(u),\, y\in T(u)} A \geq \sup_{x\in S(u),\, y\in T(u)} (B + C) = \sup_{x\in S(u),\, y\in T(u)} B + C.$$

Since $\sup_{x\in S(u),\, y\in T(u)} B \geq 0$ for all $v \in K$ from (1.4), $A \geq C$. Hence

$$\begin{aligned}
&\langle (g(v) + N(z,w)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \\
&\geq (\varphi(\|G(v) - G(u)\|) - \psi(\|G(v) - G(u)\|) + \phi(\|G(v) - G(u)\|))\, \|G(v) - G(u)\|^p
\end{aligned}$$

for all $v \in K$, $z \in S(v)$ and $w \in T(v)$.

Conversely, suppose that (2.1) holds, without loss of generality, choose a point $v \in K$ such that $f(v) < +\infty$ and so $f(u) < +\infty$. Letting $v_n = \left(1 - \frac{1}{n}\right) u + \frac{1}{n} v$ for $n \in \mathbb{N}$, we have $v_n \in K$.

For any $x \in S(u)$ and $y \in T(u)$, since the mapping $v \mapsto \{N(z,w) \in X^\star : z \in S(v), w \in T(v)\}$ is lower hemi-continuous, the mapping $v \mapsto g(v)$ is hemi-continuous and $v_n \to u$ as $n \to \infty$, there exists a subsequence $\{v_{n_j}\} \subset \{v_n\}$ and there are $z_{n_j} \in S(v_{n_j})$, $w_{n_j} \in T(v_{n_j})$ such that for any $\tau \in X$

$$z_{n_j} \to x,\ w_{n_j} \to y,\ \langle g(v_{n_j}) + N(z_{n_j}, w_{n_j}), \tau\rangle \to \langle g(u) + N(x,y), \tau\rangle \tag{2.2}$$

as $j \to \infty$. By the affinity of G, $G(v_n) = (1 - \frac{1}{n})G(u) + \frac{1}{n}G(v_n)$ for all $n \in \mathbb{N}$, hence it follows from (2.1) that

$$\begin{aligned}
&\langle g(v_{n_j}) + N(z_{n_j}, w_{n_j}) - \ell, G(v_{n_j}) - G(u)\rangle + f(v_{n_j}) - f(u) \\
&\geq (\varphi(\|G(v_{n_j}) - G(u)\|) - \psi(\|G(v_{n_j}) - G(u)\|) \\
&\quad - \phi(\|G(v_{n_j}) - G(u)\|))\, \|G(v_{n_j}) - G(u)\|^p \\
&= \left(\frac{1}{n_j}\right)^p \left(\varphi\left(\frac{1}{n_j}\|G(v) - G(u)\|\right) - \psi\left(\frac{1}{n_j}\|G(v) - G(u)\|\right)\right. \\
&\quad \left. -\phi\left(\frac{1}{n_j}\|G(v) - G(u)\|\right)\right) \|G(v) - G(u)\|^p.
\end{aligned} \tag{2.3}$$

Since f is convex and $v_{n_j} = (1 - \frac{1}{n_j})u + \frac{1}{n_j}v$,

$$\begin{aligned}
f(v) - f(u) &= n_j \left(\left(1 - \frac{1}{n_j}\right) f(u) + \frac{1}{n_j} f(v) - f(u)\right) \\
&\geq n_j f\left(\left(1 - \frac{1}{n_j}\right) u + \frac{1}{n_j} v\right) - n_j f(u) \\
&= n_j f(v_{n_j}) - n_j f(u) \\
&= n_j (f(v_{n_j}) - f(u)),
\end{aligned}$$

from (2.3), it follows that

$$\begin{aligned}
&\langle (g(v_{n_j}) + N(z_{n_j}, w_{n_j})) - \ell, G(v_{n_j}) - G(u)\rangle + f(v) - f(u) \\
&\geq \left(\frac{1}{n_j}\right)^{p-1} \left(\varphi\left(\frac{1}{n_j}\|G(v) - G(u)\|\right) - \psi\left(\frac{1}{n_j}\|G(v) - G(u)\|\right)\right. \\
&\quad \left. +\phi\left(\frac{1}{n_j}\|\eta(v,u)\|\right)\right) \|G(v) - G(u)\|^p.
\end{aligned} \tag{2.4}$$

It follows from (2.2) and (2.4) that

$$\langle (g(u) + N(x,y)) - \ell,\ G(v) - G(u)\rangle + f(v) - f(u) \geq 0$$

for all $v \in K$, $x \in S(u)$, $y \in T(u)$. □

From Theorem 2.1, we have the following theorem, the main result of Cho et al. [2] as a corollary.

Corollary 2.2 ([2]). *Let G be the identity mapping, $g \equiv 0$, $\phi \equiv 0$ and $N(x,y) = x-y$ for x, $y \in X^\star$ in Theorem 2.1. Then for any $\ell \in X^\star$, $u \in K$ is a solution of*

$$\sup_{x\in S(u), y\in T(u)} \langle N(x,y)-\ell, v-u\rangle + f(v) - f(u) \geq 0 \quad \text{for all } v \in K$$

if and only if $u \in K$ is a solution of

$$\langle N(z,w)-\ell, v-u\rangle + f(v) - f(u) \geq (\varphi\|v-u\|) - \psi(\|v-u\|)\|v-u\|^p$$

for all $v \in K$, $z \in S(v)$ and $w \in T(v)$.

For the next result, we need the following KKM mapping and Fan-KKM Theorem.

Definition 2.1 ([10]). Let X be a topological vector space. A mapping $F : X \to 2^X$ is called a KKM mapping if for any $\{x_1, x_2, \quad , x_n\} \subset X$,

$$co\{x_1, x_2, \quad , x_n\} \subset \bigcup_{i=1}^{n} F(x_i).$$

Fan-KKM Theorem 2.3 ([3]). *Let K be a nonempty subset of a topological vector space X and $F : K \to 2^X$ be a KKM-mapping. If $F(x)$ is closed in Y for every x in K and there exists at least a point $x_0 \in K$ such that $F(x_0)$ is compact, then*

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

Theorem 2.4. *Let X be a real reflexive Banach space, $X^\star$ be its dual space and K be a nonempty bounded closed convex subset of X. Let S, T, g, N, G, φ, ψ and ϕ be the same as those in Theorem 2.1. Suppose that the mappings $\varphi - \psi + \phi$, η are continuous and $f : K \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous. Then the problem (1.4) has a solution. Moreover if G is injective, then the solution is unique.*

Proof. We first prove the existence of a solution of the problem (1.4). Define two multivalued mappings F, $H : K \to 2^K$ by, for any $\ell \in X^\star$,

$$F(v) = \{u \in K : \langle (g(u) + N(x,y)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \geq 0, \\ \text{for some } x \in S(u), \text{ some } y \in T(u)\}$$

for all $v \in K$, and

$$\begin{aligned} H(v) =\{u \in K : &\langle (g(v) + N(z,w)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \\ &\geq (\varphi(\|G(v) - G(u)\|) - \psi(\|G(v) - G(u)\|) \\ &\quad + \phi(\|G(v) - G(u)\|))\, \|G(v) - G(u)\|^p, \text{ for all } z \in S(v),\ w \in T(v)\} \end{aligned}$$

for all $v \in K$. □

We show that F is a KKM-mapping. Assume that F is not a KKM-mapping. Then there exists $\{v_1, v_2, \quad, v_n\} \subset K$ and $t_i > 0$, $i = 1, 2, \quad, n$, such that

$$\sum_{i=1}^{n} t_i = 1,\ v = \sum_{i=1}^{n} t_i v_i \notin \bigcup_{i=1}^{n} F(v_i).$$

For any $z \in S(u)$ and $w \in T(u)$, by the definition of F, we have

$$\langle (g(v) + N(z,w)) - \ell, G(v_i) - G(v)\rangle + f(v_i) - f(v) < 0$$

for $i = 1, 2, \quad, n$. It follows that

$$\begin{aligned} 0 &= \langle (g(v) + N(z,w)) - \ell, G(v) - G(v)\rangle \\ &= \langle (g(v) + N(z,w)) - \ell, G\Big(\sum_{i=1}^{n} t_i v_i\Big) - G(v)\rangle \\ &= \sum_{i=1}^{n} t_i \langle (g(v) + N(z,w)) - \ell, G(v_i) - G(v)\rangle < \sum_{i=1}^{n} t_i (f(v) - f(v_i)) \\ &= f(v) - \sum_{i=1}^{n} t_i f(v_i) \\ &\leq f(v) - f(v) = 0, \end{aligned}$$

which is a contradiction. This implies that F is a KKM-mapping. Now we prove that $F(v) \subset H(v)$ for all $v \in K$. Let $u \in F(v)$, then there exist $x \in S(u)$, $y \in T(u)$ such that

$$\langle (g(u) + N(x,y)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \geq 0.$$

Since the mappings g, S and T satisfy (1.7), (1.5) and (1.6) respectively, we have

$$\begin{aligned} &\langle (g(v) + N(z,w)) - \ell, G(v) - G(u)\rangle + f(v) - f(u) \\ &= \langle -\ell, G(v) - G(u)\rangle + \langle N(x,w) - N(z,w), G(u) - G(v)\rangle \\ &\quad + \langle N(x,y) - N(x,w), G(u) - G(v)\rangle + \langle N(x,y), G(v) - G(u)\rangle \end{aligned}$$

$$+ \langle g(v) - g(u), G(v) - G(u)\rangle + \langle g(u), G(v) - G(u)\rangle + f(v) - f(u)$$
$$\geq (\varphi(\|G(v) - G(u)\|) - \psi(\|G(v) - G(u)\|) + \phi(\|G(v) - G(u)\|))\| G(v) - G(u)\|^p$$
$$+ \langle (g(u) + N(x,y) - \ell, G(v) - G(u)\rangle + f(v) - f(u)$$
$$\geq (\varphi(\|G(v) - G(u)\|) - \psi(\|G(v) - G(u)\|) + \phi(\|G(v) - G(u)\|)) \|G(v) - G(u)\|^p,$$

for all $v \in K$, $z \in S(v)$ and $w \in T(v)$. This implies that $u \in H(v)$ and so H is also a KKM-mapping.

On the other hand, from the assumption, it follows that $H(v)$ is weakly closed for all $v \in K$. Since K is bounded closed convex, we know that K is weakly compact and so $H(v)$ is weakly compact in K for all $v \in K$. It follows from Fan-KKM Theorem that

$$\bigcap_{v \in K} H(v) \neq \emptyset.$$

Hence for any $\ell \in X^\star$ there exists a point $u_0 \in K$ such that

$$\begin{aligned}&\langle (g(v) + N(z,w)) - \ell, G(v) - G(u_0)\rangle + f(v) - f(u_0)\\ &\geq (\varphi(\|G(v) - G(u_0)\|) - \psi(\|G(v) - G(u_0)\|)\\ &\quad + \phi(\|G(v) - G(u_0)\|)) \|G(v) - G(u_0)\|^p,\end{aligned}$$

for all $z \in S(v)$, $w \in T(v)$ for all $v \in K$. Thus

$$\langle (g(u_0) - N(x,y)) - \ell, G(v) - G(u_0)\rangle + f(v) - f(u_0) \geq 0,$$

for all $v \in K$, for some $x \in S(v)$ and for some $y \in T(v)$, which shows that u_0 is a solution of (1.4).

Let u_1 and $u_2 \in K$ be solutions of the problem (1.4). Since $k_1(v) := \sup_{\substack{x \in S(u_1)\\ y \in T(u_1)}} \langle (g(u_1) + N(x,y)) - \ell, G(v) - G(u_1)\rangle + f(v) - f(u_1) \geq 0$, for each fixed $v \in K$ and $k_2(v) := \sup_{\substack{x \in S(u_2)\\ y \in T(u_2)}} \langle (g(u_2) + N(x,y)) - \ell, G(v) - G(u_2)\rangle + f(v) - f(u_2) \geq 0$ for each fixed $v \in K$, by the definition of supremum, for any positive number ε, there exist $x_1 \in S(u_1)$ and $y_1 \in T(u_1)$ such that

$$k_1(v) - \varepsilon < \langle (g(u_1) + N(x_1, y_1)) - \ell, G(v) - G(u_1)\rangle + f(v) - f(u_1) \leq k_1(v) \tag{2.5}$$

and, there exist $x_2 \in S(u_2)$ and $y_2 \in T(u_2)$ such that

$$k_2(v) - \varepsilon < \langle (g(u_2) + N(x_2, y_2)) - \ell, G(v) - G(u_2)\rangle + f(v) - f(u_2) \leq k_2(v) \tag{2.6}$$

Setting $v = u_2$ in (2.5) and $v = u_1$ in (2.6) and adding, we have

$$k_1(u_2) + k_2(u_1) - 2\varepsilon < \langle g(u_1) - g(u_2) + N(x_1, y_1) - N(x_2, y_2), G(u_2) - G(u_1)\rangle \\ \leq k_1(u_2) + k_2(u_1).$$

Since ε is arbitrary,

$$\langle g(u_1)-g(u_2)+N(x_1,y_1)-N(x_2,y_2), G(u_2)-G(u_1)\rangle = k_1(u_2)+k_2(u_1) \geq 0. \tag{2.7}$$

By (1.5), (1.6) and (1.7), we obtain

$$\begin{aligned}
&\langle g(u_1) - g(u_2) + N(x_1, y_1) - N(x_2, y_2), G(u_2) - G(u_1)\rangle \\
&= \langle g(u_1) - g(u_2), G(u_2) - G(u_1)\rangle + \langle N(x_1, y_1) - N(x_2, y_1), G(u_2) - G(u_1)\rangle \\
&\quad + \langle N(x_2, y_1) - N(x_2, y_2), G(u_2) - G(u_1)\rangle \\
&= -\langle g(u_1) - g(u_2), G(u_1) - G(u_2)\rangle - \langle N(x_1, y_1) - N(x_2, y_1), G(u_1) - G(u_2)\rangle \\
&\quad - \langle N(x_2, y_1) - N(x_2, y_2), G(u_1) - G(u_2)\rangle \\
&\leq (-\phi(\|G(u_1) - G(u_2)\|) - \varphi(\|G(u_1) - G(u_2)\|) \\
&\quad + \psi(\|G(u_1) - G(u_2)\|))\| G(u_1) - G(u_2)\|^p \\
&= -(\phi(\|G(u_1) - G(u_2)\|) + \varphi(\|G(u_1) - G(u_2)\|) \\
&\quad - \psi(\|G(u_1) - G(u_2)\|)) \|G(u_1) - G(u_2)\|^p. \qquad (2.8)
\end{aligned}$$

Due to the inequality $\phi(t) + \varphi(t) > \psi(t)$ for all $t > 0$, it follows from (2.7) and (2.8) that $\|G(u_1) - G(u_2)\|^p = 0$.

Since G is injective, $u_1 = u_2$. Hence (1.4) has a unique solution. □

Remark 2.1. Theorem 2.4 also improves and extends Theorem 2.4 of [2].

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