ON AN IMPROVED UNIFIED CONVERGENCE ANALYSIS FOR A CERTAIN CLASS OF EULER–HALLEY TYPE METHODS

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ABSTRACT. Using more precise majorizing sequences than before [6], [10], [11], [14] we provide a finer semilocal convergence analysis for a certain class of Euler–Halley type methods for approximating a solution of an equation in a Banach space setting.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution $x^*$ of a nonlinear equation

$$F(x) = 0$$

where $F$ is a twice-Fréchet differentiable operator defined on a convex subset $D$ of a Banach space $X$ with values in a Banach space $Y$.

Let $\lambda \in [0,2]$ be a given parameter. We use the class of Euler–Halley type approximations given by

$$x_{n+1} = T_{F,\lambda}(x_n) = x_n + G_F(x_n) + H_{F,\lambda}(x_n) \ (x_0 \in D), \ (n \geq 0),$$

where

$$G_F(x) = -F'(x)^{-1}F(x)$$

$$Q_{F,\lambda}(x) = \left[I + \frac{\lambda}{2}F'(x)^{-1}F''(x)G_F(x)\right]^{-1},$$

and

$$H_{F,\lambda}(x) = -\frac{1}{2}F'(x)^{-1}F''(x)G_F(x)Q_{F,\lambda}(x)G_F(x)$$

for all $x \in D$.

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This class of approximations includes as special cases many cubically convergent methods: If \( \lambda = 0 \) we obtain the Euler method \([1], [2], [9], [10]\); if \( \lambda = 1 \) we get the Halley method \([2], [8], [10]\) whereas for \( \lambda = 2 \) method (1.2) reduces to the super-Haley method \([7], [10]\). A convergence analysis of method (1.2) has been given by us in [6], Gutierrez and Hernandez in [10], Han in [11] and more recently by Wang and Li [14] under various hypotheses that have certain advantages over each other.

In particular in the elegant paper by Wang and Li a unified convergence analysis was provided for method (1.2) using the concept of the average function. This idea has already been used in [12], [13] and in an improved way in [3]–[5] on Newton’s method.

Here we show that using more precise majorizing sequences than in [14] and under the same computational cost and the same or weaker hypotheses we can provide finer error bounds on the distances \( \|x_{n+1} - x_n\|, \|x_n - x^*\| (n \geq 0) \) and a more precise information on the location of the solution \( x^* \).

2. SEMILOCAL CONVERGENCE ANALYSIS OF METHOD (1.2)

Let \( x_0 \in D \) such that \( F'(x_0)^{-1} \in L(Y, X) \). Let \( R > 0 \). We denote by \( U(x_0, R) \) the ball \( U(x_0, R) = \{ x \in X \mid \|x - x_0\| < R \} \), whereas \( \overline{U}(x_0, R) \) the corresponding closed ball.

We need the definition [5], [12]:

**Definition 2.1.** An operator \( F'(x_0)^{-1}F'(x) \) satisfies the center Lipschitz condition in \( U(x_0, r) \) with \( L_0 \) average if

\[
(2.1) \quad \|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \int_0^{\|x - x_0\|} L_0(u)du
\]

for some positive integrable function \( L_0 \) on \([0, R]\). Usually \( R \) is taken such that

\[
(2.2) \quad \int_0^R (R - u)L_0(u)du = R.
\]

Let \( r_0^* > 0 \) be such that

\[
(2.3) \quad \int_0^{r_0^*} L_0(u)du = 1
\]

and define

\[
(2.4) \quad b_0 = \int_0^{r_0^*} uL_0(u)du.
\]
For $\beta \in [0, b_0]$, define

$$h_0(t) = \beta - t + \int_0^t (t-u)L_0(u)du, \quad t \in [0, R].$$

We need the Lemma [5], [12]:

**Lemma 2.2.** The function $h_0$ nonincreasing in $[0, r_0^*]$ and nondecreasing in $[r_0, R]$. Moreover, if

$$\beta \leq b_0$$

$$h_0(\beta) > 0, h_0(r_0) = \beta - b_0 \leq b, \text{ and } h_0(R) = \beta > 0. \text{ That is, } h_0 \text{ has a unique zero in } [0, r_0] \text{ denoted by } r_1^* \text{ and a unique zero } r_2^* \text{ in } [r_0, R], \text{ satisfying}$$

$$\beta < r_1^* < \frac{r_0^*}{b_0} \beta < r_0^* < r_2^* < R$$

when $\beta < b_0$ and $r_1^* = r_2^*$ when $\beta = b_0$.

We also need the following Lemmas:

**Lemma 2.3** ([5]). If operator $F'(x_0)^{-1}F$ satisfies (2.1) in $U(x_0, r)$ with $L_0$ average and $r > r_0^*$. Then for each $x \in U(x_0, r_0^*)$, $F'(x)^{-1} \in L(Y, X)$ and

$$\|F'(x)^{-1}F'(x_0)\| \leq \left[1 - \int_0^{\|x-x_0\|} L_0(u)du\right]^{-1}$$

**Proof.** In view of (2.1) we get

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \int_0^{\|x-x_0\|} L_0(u)du < 1.$$ 

It follows by (2.9) and the Banach Lemma on invertible operators [2] that $F'(x)^{-1} \in L(Y, X)$ and (2.8) holds true.

That completes the proof of Lemma 2.3.

**Lemma 2.4** ([5], [12]). Let $\beta = \|F'(x_0)^{-1}F(x_0)\| \leq b_0$. Assume that $r_1^* < r < r_2^*$ if $\beta < b$, or $r = r_1^*$ if $\beta = b_0$. If operator $F'(x_0)^{-1}F$ satisfies (2.1) in $U(x_0, r)$ with $L_0$ average, the equation $F(x) = 0$ has a unique solution

$$x^* \in \overline{U}(x_0 - F'(x_0)^{-1}F(x_0), r_1^* - \beta) \subset \overline{U}(x_0, r_1^*).$$

**Lemma 2.5.** Assume that

$$\|F'(x_0)^{-1}F''(x_0)\| = L_0(0)$$

$$\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq L(\|x-x_0\| + \|x-y\|) - L(\|y-x_0\|)$$
for all $y \in \bar{U}(x_0, r), \ x \in \bar{U}(y, r - \|y - x_0\|)$, and some positive integrable function $L$ on $[0, R]$. Then, for each $x \in U(x_0, r_0)$

$$\|F'(x_0)^{-1}F''(x)\| \leq h''(\|x - x_0\|),$$

where,

$$h(t) = \beta - t + \int_0^t (t - u)L(u)du, \quad t \in [0, R]$$

and

$$\|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h'_0(\|x - x_0\|)}.$$

Proof. In view of (2.11) and (2.12) it follows

$$L_0(u) \leq L(u), \quad u \in [0, R]$$

and $\frac{L(u)}{L_0(u)}$ can be arbitrarily large [1], [2]. Using (2.11), (2.12), (2.14) and (2.16) we obtain in turn

$$\|F'(x_0)^{-1}F''(x)\| = \|F'(x_0)^{-1}[F''(x) - F''(x_0)] + F''(x_0)\|$$

$$\leq \|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| + \|F'(x_0)^{-1}F''(x_0)\|$$

$$\leq L_0(0) + L(\|x - x_0\|) - L(0)$$

$$\leq L(0) + L(\|x - x_0\|) - L(0) \leq h''(\|x - x_0\|),$$

which shows (2.13).

Estimate (2.15) follows immediately from (2.5) and (2.8).

That completes the proof of Lemma 2.5. \[\square\]

Remark 2.6. If equality holds in (2.16) then our results reduce to the corresponding ones in [14]. Otherwise they constitute an improvement. Indeed, let

$$b = \int_0^{r_0} uL(u)du$$

where $r_0$ is such that

$$\int_0^{r_0} L(u)du = 1.$$

If

$$\beta \leq b$$
denote the corresponding zeros of $h$ by $r_1$ and $r_2$. It then follows by (2.3)–(2.5), (2.14), (2.18) and (2.19) that under (2.6) and (2.20) the following hold true:

\begin{align*}
(2.21) & \quad r_0 < r_0^* \\
(2.22) & \quad r_1^* < r_1 \\
(2.23) & \quad r_2 < r_2^* \\
\end{align*}

and

\begin{equation}
(2.24) \quad b < b_0.
\end{equation}

That is we obtain a better information on the location of the solution and wider upper bound on $\|F'(x_0)^{-1}F(x_0)\|$.

Let $\{t_n\}$ be the majorizing sequence for $\{x_n\}$ given by

\begin{equation}
(2.25) \quad t_{n+1} = T_{h,\lambda}(t_n) \quad (n \geq 0).
\end{equation}

Define scalar sequence $\{s_n\} \ (n \geq 0)$ as $\{t_n\}$ but with $h(t_n)^{-1}$ replaced by $h_0^{-1}(t_n)$ $(n \geq 0)$. If equality holds in (2.16) then $s_n = t_n \ (n \geq 0)$. Otherwise it can easily be seen using induction on $n \geq 0$

\begin{align*}
(2.26) & \quad s_n < t_n \\
(2.27) & \quad s_{n+1} - s_n < t_{n+1} - t_n \\
(2.28) & \quad s^* = \lim_{n \to \infty} s_n \leq r_1 = \lim_{n \to \infty} t_n \\
\end{align*}

and

\begin{equation}
(2.29) \quad s^* - s_n \leq r_1 - t_n.
\end{equation}

Note that under (2.20) scalar sequence $\{t_n\}$ is nondecreasing and converges to $r_1$ (see Lemma 2.2 in [14]). Therefore under the same condition (2.20) scalar sequence $\{s_n\}$ is also nondecreasing and converges to $s^*$.

Let us also define scalar sequence

\begin{equation}
(2.30) \quad u_{n+1} = T_{h_0,\lambda}(u_n) \quad (n \geq 0).
\end{equation}

It follows that under (2.6) $\{v_n\}$ is nondecreasing and converges to $r_1^*$. By comparing sequences $\{v_n\}$ and $\{s_n\}$ we deduce

\begin{align*}
(2.31) & \quad v_n \leq s_n \\
(2.32) & \quad v_{n+1} - v_n \leq s_{n+1} - s_n \\
(2.33) & \quad r_1^* \leq s^*.
\end{align*}
and

\[(2.34) \quad r_1^* - v_n \leq s^* - s_n,\]

although \(\{v_n\}\) is not a majorizing sequence for \(\{x_n\}\).

We can now state the main unifying result for method (1.2):

**Theorem 2.7.** Assume for \(r > r_1^*\) conditions (2.11), (2.12) hold true on \(\overline{U}(x_0, r)\). If

\[(2.35) \quad \beta \leq b_0\]

then sequence \(\{x_n\}\) generated by method (1.2) is well defined, for all \(\lambda \in [0, 2]\), remains in \(U = \overline{U}(x_0 - F'(x_0)^{-1}F(x_0), r_1^* - \beta)\) and converges to a solution \(x^* \in U\).

Moreover, for \(r \in [r_1^*, r_2^*]\) if \(\beta < b_0\) and \(r = r_1^*\) if \(\beta = b_0\), the equation \(F(x) = 0\) has a unique solution in \(\overline{U}(x_0, r)\). Furthermore the following estimates hold true for all \(n \geq 0:\)

\[(2.36) \quad \|x_{n+1} - x_n\| \leq s_{n+1} - s_n\]

and

\[(2.37) \quad \|x_n - x^*\| \leq s^* - s_n.\]

**Proof.** It follows exactly as Theorem 3.1 in [14] with the only crucial difference that we are using sharper (2.15) instead of

\[(2.38) \quad \|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h'(\|x - x_0\|)}\]

used in [14] and majorizing sequence \(\{s_n\}\) instead of \(\{t_n\}\).

That completes the proof of Theorem 2.7.

**Remark 2.8.** Another way of improving the results in [14] is to consider the approximation (see Lemma 3.1 in [14] or [6] or [10]):

\begin{align*}
F(x_{n+1}) &= \frac{1}{2} F''(x_n) \{(2 - \lambda) G_F(x_n) + H_{F, \lambda}(x_n)\} H_{F, \lambda}(x_n) \\
(2.39) &\quad + \int_0^1 \{F''(x_n + \theta(x_{n+1} - x_n)) - F''(x_n)\}(1 - \theta) d\theta (x_{n+1} - x_n)^2
\end{align*}

and the corresponding iteration \(\{w_n\}\) given by

\[(2.40) \quad w_0 = 0, \quad w_1 - w_0 = \|x_1 - x_0\|, \quad w_{n+2} = w_{n+1} + \overline{T}_{h_0, h, \lambda}(w_{n+1}),\]
where,

\begin{equation}
T_{h_0, h_\lambda}(w_{n+1}) = \frac{\varepsilon_{n+1}}{-h'_0(w_{n+1})},
\end{equation}

and

\begin{equation}
\varepsilon_{n+1} = \frac{1}{2} h''(w_{n+1}) \left\{ (2 - \lambda) G_h(w_{n+1}) + H_{h, \lambda}(w_{n+1}) \right\} H_{h, \lambda}(w_{n+1})
+ \int_0^1 [L(w_n + \theta(w_{n+1} - w_n)) - L(w_n)] (1 - \theta) d\theta (w_{n+1} - w_n)^2.
\end{equation}

It follows from the proof of the theorem that \( \{w_n\} \) is the finer majorizing sequence for \( \{x_n\} \) such that

\begin{align*}
(2.43) & \quad w_n \leq s_n \\
(2.44) & \quad w_{n+1} - w_n \leq s_{n+1} - s_n \\
(2.45) & \quad w^* = \lim_{n \to \infty} w_n \leq s^*
\end{align*}

and

\begin{equation}
(2.46) \quad w^* - w_n \leq s^* - s_n.
\end{equation}

Therefore if one finds sufficient convergence conditions for sequence \( \{w_n\} \) weaker than the ones given in our Theorem 2.7 or Theorem 3.1 in [14] then the results obtained here or in [14] can be improved even further. We note that sufficient convergence conditions for scalar sequences more general than (2.40) have already been given by us in [1], [2]. Therefore we do not write down explicitly those conditions. Instead we refer the motivated reader to [1], [2].

**Remark 2.9.** In view of (2.11) there exists a positive integrable function \( L_1 \) on \([0, R]\) such that

\begin{equation}
(2.47) \quad \|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \leq L_1(\|x - x_0\|) - L_1(0)
\end{equation}

for all \( x \in \overline{U}(x_0, r) \).

Clearly

\begin{equation}
(2.48) \quad L_1(u) \leq L(u)
\end{equation}

holds and \( \frac{L(u)}{L_1(u)} \) can be arbitrarily large [1], [2]. If \( L_1 \) replaces \( L_0 \) in condition (2.5), (2.11) then if we denote the new \( h \) function by \( h_1 \) we again get

\begin{equation}
(2.49) \quad \|F'(x)^{-1}F'(x_0)\| \leq \frac{1}{h'_1(\|x - x_0\|)}.
\end{equation}
Indeed by Taylor’s formula,

\[ F'(x) = F'(x_0) + F''(x_0)(x - x_0) \]

\[ + \int_0^1 \left[ F''(x_0 + t(x - x_0)) - F''(x_0) \right] dt(x - x_0) \]

which leads to the estimate

\[ \|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_1(0)\|x - x_0\| \]

\[ + \int_0^1 \|F''(x_0 + t(x - x_0)) - F''(x_0)\| \|x - x_0\|dt \]

\[ \leq \int_0^{\|x - x_0\|} L_1(u)du < 1. \]

(2.51)

It follows from (2.51) and the Banach Lemma on invertible operators that $F'(x)^{-1}$ exists so that (2.49) holds true. That is all the results obtained here hold with $L_1$ replacing $L_0$. Note however that although $L_1$ connects better than $L_0$ to $L$ it follows from (2.1) and (2.51) that $L_1$ may be chosen at least as small as $L_0$.

We now complete this study with an example to show that (2.16) and (2.48) can hold as strict inequalities:

Example 2.10. Let $X = Y = \mathbb{R}$, $x_0 = 0$, $D = [-1, 1]$ and define function $F$ on $D$ by

\[ F(x) = e^x - 2x + c \]

for some constant $c$. It can easily be seen that (2.1), (2.12), and (2.47) hold for

\[ L_0(u) = L_1(u) = (e - 1)u, \quad \text{and} \quad L(u) = eu. \]

(2.53)

Moreover in view of (2.53) we get

\[ L_0(u) = L_1(u) < L(u) \quad \text{for all} \quad u \in [0, R]. \]

(2.54)

Other choices for the “$L$” functions can be found in [3]–[7], [10], [14] and the references there.

References


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