REMARKS ON THE RADIAL SOLUTIONS OF THE SELF-DUAL ABELIAN CHERN-SIMONS MODEL

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ABSTRACT. We consider the nonrelativistic limit for the radial solutions to the self-dual equations in the self-dual Abelian Chern-Simons model. We achieve the limit by fixing the common maximum value of solutions.

1. INTRODUCTION

Let us consider the following equation which arises from the relativistic self-dual Abelian Chern-Simons model in $\mathbb{R}^2$ [7, 10]:

\begin{equation}
\Delta u = \frac{4q^4}{\kappa^2c^4} e^u(e^u - \sigma^2) + 4\pi \sum_{j=1}^{k} n_j \delta_{p_j},
\end{equation}

\[ u \to -\infty \quad \text{as} \quad |x| \to \infty. \]

Here, $\kappa > 0$ is the Chern-Simons coupling constant, $q$ is the charge of electron, $\sigma > 0$ is the symmetry breaking parameter, and $c$ is the speed of light. The vortex points $p_1, \ldots, p_k$ are distinct in $\mathbb{R}^2$, $n_1, \ldots, n_k$ are positive integers, and $\delta_{p_j}$ denotes the Dirac measure concentrated on the point $p_j$. The existence and properties of solutions to (1.1) have attracted much attention and some results can be found in [1, 3, 4, 5, 12]. See also [13] for the results on the other boundary conditions.

In this paper, we are interested in the limit $c \to \infty$ for the solutions to (1.1), which is called the nonrelativistic limit. Considering the Lagrangian density of the relativistic Abelian Chern-Simons model, we find the mass of the scalar Higgs field is $m = \hbar q^2 \sigma^2 / \kappa c^3$, where $\hbar$ is the Plank constant (see [6, 7, 10] for more information). We will accompany the limit $c \to \infty$ with $m$ fixed. To this end, we set

$\kappa c = \text{constant} =: \mu$. 

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\[ \kappa = \frac{\mu}{c} \quad \text{and} \quad \sigma = c \sqrt{\frac{m \mu}{h q^2}}, \]

which give a difference from the method in [9]. Indeed, in [9], not only \( m \) but also \( \kappa \) is kept fixed and only \( \sigma \) varies in the limit \( c \to \infty \). Then the matter part of the nonrelativistic Lagrangian contains the constant \( c \), which yields difficulties in proving the nonrelativistic limit by mathematical arguments. To overcome this, we vary both \( \kappa \) and \( \sigma \) as (1.2) in the limit \( c \to \infty \) (see [6] for more information).

If we set \( \nu = u + \ln 2m \), then (1.1) becomes

\[ \Delta \nu = \frac{q^4}{m^2 \kappa^2 c^4} e^{\nu v} (e^{2\nu} - 2m \sigma^2) + 4\pi \sum_{j=1}^{k} n_j \delta_{p_j} \]

\[ = \frac{q^4}{m^2 \kappa^2 c^4} e^{2\nu} - \frac{2q^2}{\hbar \mu} e^{\nu} + 4\pi \sum_{j=1}^{k} n_j \delta_{p_j}. \]

Letting \( c \to \infty \), we may derive formally that \( \nu \) converges to a solution of

\[ \Delta w = -\frac{2q^2}{\hbar \mu} e^{w} + 4\pi \sum_{j=1}^{k} n_j \delta_{p_j}, \]

\[ w \to -\infty \quad \text{as} \quad |x| \to \infty. \]

This is the well-known Liouville equation with singular sources and appears in the nonrelativistic self-dual Abelian Chern-Simons model [8, 9].

It is known that there are infinitely many solutions of (1.3). See [1, 3, 4, 12] for example. Therefore it is not surprising that there may be a sequence of solutions to (1.3) which blows up as \( c \to \infty \) instead of converging to a solution to (1.4). In this point of view, it is important to find a sequence of solutions to (1.3) converging to a solution to (1.4). This means we need a kind of conditions for solutions of (1.3) to make sure the convergence in the limit \( c \to \infty \). In [6], the authors consider such a condition for radial solutions when there is only one-vortex point. They fix a shooting constant for radial solutions in the limit \( c \to \infty \) to prove the nonrelativistic limit. In this paper, we consider a different condition for radial solutions to guarantee the limit. In fact, we will show that if we choose a sequence of radial solutions to (1.3) with the common maximum value, then it converges to a solution to (1.4). In the next section, we prove this statement and establish the nonrelativistic limit.
2. Main Theorem

In this section, we assume that there is only one-vortex point. In this case, the equations (1.3) and (1.4) are rewritten as

\[
\Delta v = \frac{q^4}{m^2 \kappa^2 c_4} e^v (e^v - 2m \sigma^2) + 4\pi N \delta_0, \\
v \to -\infty \text{ as } |x| \to \infty, \tag{2.1}
\]

and

\[
\Delta w = -\frac{2q^2}{h\mu} e^w + 4\pi N \delta_0, \\
w \to -\infty \text{ as } |x| \to \infty. \tag{2.2}
\]

We are interested in the radial solutions to (2.1) and (2.2).

To investigate (2.1) and (2.2) further, we first transform the equation (2.2) into

\[
w_{rr} + \frac{1}{r} w_r = -\frac{2q^2}{h\mu} e^w, \quad r = |x| > 0 \tag{2.3}
\]

with the constraint

\[
\lim_{r \to 0} \frac{w(r)}{\ln r} = \lim_{r \to 0} r w_r(r) = 2N, \quad \lim_{r \to \infty} w(r) = -\infty. \tag{2.4}
\]

It follows from the result of [2, 11] that every radial solution to (2.3) with (2.4) is of the form

\[
w(r) = \ln \left\{ \frac{8\lambda(N + 1)^2 r^{2N}}{\lambda + r^{2N+2}} \right\} - \ln \frac{2q^2}{h\mu}, \tag{2.5}
\]

where \(\lambda\) is any positive constant.

Similarly, for (2.1) we get

\[
v_{rr} + \frac{1}{r} v_r = \frac{q^4}{m^2 \kappa^2 c_4} e^v (e^v - 2m \sigma^2) =: g(v) \tag{2.6}
\]

with the constraint

\[
\lim_{r \to 0} \frac{v(r)}{\ln r} = \lim_{r \to 0} r v_r(r) = 2N, \quad \lim_{r \to \infty} v(r) = -\infty. \tag{2.7}
\]

Concerning to (2.6) and (2.7), we have one parameter family of radial solutions by the result of [12] as follows:

**Theorem 1** ([12]). For any number \(\alpha \leq \ln m \sigma^2\), there exist a number \(r_0 = r_0(\alpha)\) and a solution \(v(r) = v(r; \alpha)\) to (2.6) and (2.7) satisfying \(v(r_0) = \alpha, \ v_r(r_0) = 0\). Moreover, \(\alpha\) is the unique maximum value of \(v(r; \alpha)\).
In [12] Theorem 1 was proved under the conditions \( c = q = \sigma = 1 \) and \( m = 1/2 \) for simplicity. In the following, we give a sketch of its proof keeping all the constants for later use.

**Sketch of Proof of Theorem 1.** Let \( \alpha \leq \ln m\sigma^2 \) be fixed. By an elementary ODE argument, one can show that for any \( r_0 \in \mathbb{R}^+ \), there exists a unique global solution \( v(r) = v(r; r_0, \alpha) \) to (2.6) with initial data \( v(r_0) = \alpha \), \( v_r(r_0) = 0 \). Moreover, it holds that \( v(0) = -\infty = v(\infty) \), \( v_r(r) > 0 \) for \( r < r_0 \), and \( v(r) \leq \alpha \) for all \( r > 0 \).

It remains to find a suitable \( r_0 \) such that \( v(r; r_0, \alpha) \) satisfies (2.7). To this end, let us define \( \eta(r_0, \alpha) = \lim_{r \to 0} rv_r(r; r_0, \alpha) \). Multiplying the inequality from (2.6)

\[
(2.8) \quad (rv_r)_r > -\frac{2q^4\sigma^2}{m\kappa^2 c^4} r e^v
\]

by \( rv_r + 2 \) and integrating it over \((r; r_0)\) for \( r < r_0 \), we obtain

\[
0 < rv_r(r; r_0, \alpha) < -2 + 2 \sqrt{1 + \frac{q^4\sigma^2}{m\kappa^2 c^4} e^{\alpha r_0^2} =: R(r_0, \alpha), \quad \forall r < r_0.}
\]

In particular,

\[
(2.10) \quad \eta(r_0, \alpha) < R(r_0, \alpha).
\]

Integrating (2.9) on \((r, r_0)\), we obtain

\[
\alpha + R(\ln r - \ln r_0) < v(r; r_0, \alpha) \leq \alpha \quad \text{for} \quad r < r_0.
\]

Since \( g \) defined in (2.6) is decreasing if \( v \leq \ln(m\sigma^2) \), we are led to

\[
\frac{1}{r} (rv_r)_r = g(v) < g(\alpha + R(\ln r - \ln r_0)).
\]

Since \( \alpha \leq \ln(m\sigma^2) \), the integration of the above inequality over \((0, r_0)\) yields that

\[
\eta(r_0, \alpha) > \frac{q^4}{m\kappa^2 c^4} e^{\alpha r_0^2} \left( \frac{2\sigma^2}{R + 2} - \frac{e^\alpha}{m(2R + 2)} \right)
\]

\[
> \frac{q^4\sigma^2 e^{\alpha r_0^2}}{2m\kappa^2 c^4} \left( 1 + \frac{q^4\sigma^2}{m\kappa^2 c^4} e^{\alpha r_0^2} \right)^{-1/2} =: S(r_0, \alpha).
\]

Now let \( T_1 \) solve \( R(T_1, \alpha) = 2N \), namely,

\[
(2.12) \quad T_1 = T_1(\alpha) = \sqrt{N^2 + 2N} \frac{\sqrt{m\kappa c^2}}{e^{\alpha/2} q^2 \sigma}.
\]

Then by (2.10), \( \eta(T_1, \alpha) < 2N \). Similarly, let \( T_2 \) be a solution to \( S(T_2, \alpha) = 2N \). Then we have

\[
(2.13) \quad T_2 = T_2(\alpha) = \frac{2(2N^2 + N\sqrt{4N^2 + 1})^{1/2}}{q^2 \sigma e^{\alpha/2}} \sqrt{m\kappa c^2}.
\]
Obviously, \( T_1 < T_2 \). By virtue of (2.10) and (2.11), we find
\[
\eta(T_1, \alpha) < 2N < \eta(T_2, \alpha).
\]
Since \( \eta \) is continuous on \( r_0 \), there exists \( r_0 = r_0(\alpha) \in (T_1, T_2) \) such that \( \eta(r_0, \alpha) = 2N \), which completes the proof. \( \Box \)

**Remark 1.** It is easily shown that if \( v \) is a solution to (2.6) and (2.7), then \( v < \ln(2m\sigma^2) \) by the maximum principle. Theorem 1 shows the existence of solutions for \( \alpha = \sup v \leq \ln(m\sigma^2) \). It is still open whether there exists a solution with the property \( \ln(m\sigma^2) < \alpha < \ln(2m\sigma^2) \).

Now we proceed in the proof of nonrelativistic limit for the solutions given by Theorem 1. This theorem implies that for each \( c \) there are infinitely many solutions to (2.6) and (2.7). Thus, as mentioned at the end of the previous section, if we choose an arbitrary sequence of solutions as \( c \to \infty \), then it may diverge. For example, for any sequence \( c_n \to \infty \), if we choose a sequence \( \alpha_n \to -\infty \) and a sequence \( v_n \) of solutions with the maximum value \( \alpha_n \), then \( v_n \to -\infty \). Hence, we need an additional condition to make sure the convergence of solutions in the nonrelativistic limit. Although it seems not to be easy to find such a condition for a sequence of solutions to the general equation (1.1), it is not difficult to get a condition for radial solutions of Theorem 1 as we shall see.

From now on, we adopt the relation (1.2) and let \( \hbar, q, \mu, m > 0 \) be fixed and \( \alpha \in \mathbb{R} \) be given. Set
\[
\alpha_c = \ln m\sigma^2 = \ln \frac{\mu m^2 c^2}{\hbar q^2}.
\]
Since \( \alpha_c \to \infty \) as \( c \to \infty \), if \( c \) is sufficiently large, then there exists a solution to (2.6) and (2.7) satisfying Theorem 1 for \( \alpha \). Let us denote it by \( v(r, c) \). Then it follows from Theorem 1 that for a given large \( c > 0 \) there exists an \( r_0 = r_0(c) \) satisfying
\[
\begin{align*}
\{ & v(r_0, c) = \alpha, & v_r(r_0, c) = 0, \\
& \max_{r \in \mathbb{R}^+} v(r, c) = \alpha, & \lim_{r \to 0} r v_r(r, c) = 2N.
\end{align*}
\]
Furthermore, \( 0 < T_1 \leq r_0 \leq T_2 \), where \( T_1 \) and \( T_2 \) are defined by (2.12) and (2.13). Using (1.2), we can rewrite \( T_1 \) and \( T_2 \) as
\[
\begin{align*}
T_1 &= \frac{\sqrt{N^2 + 2N \sqrt{\hbar \mu}}}{qe^{\alpha/2}}, \\
T_2 &= \frac{2(2N^2 + N\sqrt{4N^2 + 1})^{1/2} \sqrt{\hbar \mu}}{qe^{\alpha/2}}.
\end{align*}
\]
We observe that $T_1$ and $T_2$ are independent of $c$. The following Theorem completely characterizes the nonrelativistic limit concerning (2.3), (2.4), (2.6), and (2.7).

**Theorem 2.** Let $h,q,\mu,m > 0$ be fixed, $N$ be a positive integer, and $\alpha \in \mathbb{R}$ be given. Let $v(r,c)$ be a solution to (2.6) and (2.7) satisfying (2.14) which is constructed by Theorem 1. Then, as $c \to \infty$, $v(r,c)$ converges to $w(r)$ which is a solution to (2.3) and (2.4). The function $w(r)$ is explicitly given by (2.5) with $\lambda = \lambda(\alpha)$ defined by

\begin{equation}
\lambda = \lambda(\alpha) = N^N(N + 2)^{N+2} \left(\frac{h\mu}{q^2}\right)^{N+1} e^{-\alpha(N+1)}.
\end{equation}

Moreover, if we set

\[
\tilde{v}(r,c) = v(r,c) - 2N \ln r, \quad \tilde{w}(r) = w(r) - 2N \ln r,
\]

then for any nonnegative integers $k$

\begin{equation}
\| \tilde{v} - \tilde{w} \|_{C^k(B_R)} = \| v - w \|_{C^k(B_R)} \to 0
\end{equation}
as $c \to \infty$, where $B_R$ is the ball of radius $R$ centered at the origin.

**Proof.** Let $c_n$ be an arbitrary sequence such that $c_n \to \infty$. Set $r_n = r_0(c_n)$ and $v_n(r) = v(r,c_n)$. We split the proof into four steps.

**Step 1. Convergence of $v_n$.**

It follows from (2.6) that

\[
v_n(r) = \alpha + \int_{r_n}^r \frac{1}{s} \int_{r_n}^s \frac{q^4}{m^2\kappa^2 c_n^4} r e^{\nu_n} (e^{\nu_n} - 2m\sigma^2) \, dr \, ds.
\]

Since $0 < T_1 < r_n < T_2$ and $T_1, T_2$ are independent of $c_n$, we may assume that there exists a subsequence of \{r_n\}, still denoted by \{r_n\}, satisfying $r_n \to r_* \in [T_1, T_2]$. We note that

\[
|g(v_n)| \leq \frac{q^4}{m^2\kappa^2 c_n^4} (e^{2\alpha} + 2m\sigma^2 e^\alpha) = \frac{q^4}{m^2\kappa^2 c_n^4} e^{2\alpha} + \frac{2q^2}{h\mu} e^\alpha,
\]

which means that $g$ is uniformly bounded for all $r$ as $c_n \to \infty$. Therefore for any given $R_2 > R_1 > 0$, if $R_1 < r < R_2$,

\[
|v_n(r)| \leq |\alpha| + \left| \int_{r_n}^r \frac{1}{s} \int_{r_n}^s \tau g(v_n) \, dr \, ds \right| \leq C
\]

for some constant $C$ dependent only on $R_1$ and $R_2$. Thus we have

\[
\sup_{[R_1, R_2]} |v_n(r)| \leq C,
\]
Since
\[
\sup_{\mathbb{R}^2} |\Delta v_n| = \sup_{\mathbb{R}^2} |g(v_n)| \leq C,
\]
we conclude that \(\|v_n\|_{W^{2,p}(B_{R_2}\setminus B_{R_1})} \leq C\) for all \(p > 1\). This implies that there exist a subsequence, denoted by the same notation, \(v_n\) and a function \(w \in W^{2,p}(B_{R_2}\setminus B_{R_1})\) such that \(v_n \to w \in C^{1,\beta}(B_{R_2}\setminus B_{R_1})\) for any \(\beta \in (0,1)\) as \(c_n \to \infty\). Moreover, it follows from the bootstrap argument that \(v_n \to w\) in \(C^k(B_{R_2}\setminus B_{R_1})\) for all nonnegative integers \(k\).

**Step 2. Explicit form of \(w\).**

Let us show that \(w\) is a solution to (2.3). Since
\[
g(v_n) = \frac{q^4}{m^2 \kappa^2 c_n^4} e^{2v_n} - \frac{2q^2}{\tilde{h}\mu} e^{v_n} \to -\frac{2q^2}{\tilde{h}\mu} e^w,
\]
\(w\) satisfies
\[
w(r) = \alpha + \int_{r_*}^r \frac{1}{s} \int_{r_*}^s \tau \left(\frac{-2q^2}{\tilde{h}\mu}\right) e^{w(\tau)} d\tau ds,
\]
which yields
\[
w_{rr} + \frac{1}{r} w_r = - \frac{2q^2}{\tilde{h}\mu} e^w, \quad r > 0
\]
with \(w_r(r_*) = 0, w(r_*) = \alpha\).

Next, we verify that \(w\) satisfies (2.4). Integrating (2.6) on \((0, r_n)\), we get
\[
-2N = \int_0^{r_n} \frac{q^4}{m^2 \kappa^2 c^4} re^{v_n}(e^{v_n} - 2m\sigma^2) dr.
\]
Taking the limit, we obtain
\[
2N = \int_0^{r_*} \frac{2q^2}{\tilde{h}\mu} re^w dr,
\]
which implies \(\lim_{r \to 0} rw_r = 2N\). Since \(w_r < 0\) for \(r > r_*\), there exists
\[
\gamma = \inf_{r > r_*} w(r) = \lim_{r \to \infty} w(r) \geq -\infty.
\]
If \(\gamma > -\infty\), we arrive at a contradiction:
\[
\infty > \lim_{r \to \infty} |w(r)| \geq -|\alpha| + \frac{2q^2 e^\gamma}{\tilde{h}\mu} \lim_{r \to \infty} \int_{r_*}^r \frac{1}{s} \int_{r_*}^s \tau d\tau ds = \infty.
\]
Hence \(w(\infty) = -\infty\) and (2.4) is proved.

As a consequence, \(w\) is a radial solution to (2.3) and (2.4). By (2.5), there exists \(\lambda > 0\) such that \(w\) is of the form
\[
w(r) = 2N \ln r - 2 \ln(\lambda + r^{2N+2}) + \ln 8\lambda(N + 1)^2 - \ln(2q^2/\tilde{h}\mu).
\]
It remains to show (2.16). Since $w_r(r_*) = 0$, we have

\begin{equation}
\lambda = \frac{N + 2}{N} r_*^{2N+2}. \tag{2.18}
\end{equation}

In addition, from $w(r_*) = \alpha$, we obtain

\[ r_* = q^{-1} e^{-\alpha/2} \sqrt{N(N + 2) \hbar \mu}. \]

Now (2.16) is a consequence of substitution of this identity into (2.18).

**Step 3. Convergence of $\tilde{v}_n$.**

It is not difficult to show that $\tilde{v}_n$ and $\tilde{w}$ are smooth functions on $\mathbb{R}^2$. For instance, see Lemma 3.2 of [12]. We know from Step 1 that for any $0 < R_1 < R_2$

\[ \| \tilde{v}_n - \tilde{w} \|_{C^k(B_{R_2} \setminus B_{R_1})} \to 0. \]

We will extend the convergence on any ball $B_R$ for $R > 0$. From $(r(v_n)_r)_r = \tau g(v_n)$, we get

\[ \tilde{v}_n(r) = \alpha - 2N \ln r_n + \int_{r_n}^r \frac{1}{s} \int_0^s \tau g(v_n) d\tau ds, \quad r > 0. \]

Since $|g(v_n)| \leq C$, we are led that for any $r \leq R$

\[ \left| \int_{r_n}^r \frac{1}{s} \int_0^s \tau g(v_n) d\tau ds \right| \leq C_R, \]

which implies that $\sup_{B_R} |\tilde{v}_n| \leq C_R$. Since $\sup_{\mathbb{R}^2} |\Delta \tilde{v}_n| \leq C$, we conclude that $\| \tilde{v}_n \|_{W^{2,p}(B_R)} \leq C_R$ for all $p > 1$. Then as in the Step 1, we can show by the bootstrap argument that $\tilde{v}_n \rightarrow \tilde{w}$ in $C^k(B_R)$ for any nonnegative integers $k$.

**Step 4. Convergence of the whole sequence.**

Finally, the uniqueness of $w$ implies that the convergence holds true for the whole sequence $c_n$. Since $\{c_n\}$ was an arbitrary sequence, we conclude that $\tilde{v}(r, c) \rightarrow \tilde{w}(r)$ as $c \rightarrow \infty$ in $C^k(B_R)$ for any $R > 0$.

**Remark 2.** We observe that $\lambda$ is a decreasing function of $\alpha$ in (2.16), which implies that Theorem 2 completely characterizes the nonrelativistic limit for radial solutions of a one-vortex case. In other words, for each solution $w(r)$ to the nonrelativistic equations (2.3) and (2.4), we can find one parameter family of solutions $v(r, c)$ to the relativistic equations (2.6) and (2.7) such that $v(r, c) \rightarrow w(r)$ as $c \rightarrow \infty$. Indeed, $w(r)$ is determined by $\lambda$ via (2.5) and the corresponding $v(r, c)$ converging to $w(r)$ can be realized by the common maximum value $\alpha$ given by (2.16).
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