A Proposal of Simplified Eigenvalue Equation for an Analysis of Dielectric Slab Waveguide

Young-Kyu Choi*

Abstract - In dielectric waveguide analysis and synthesis, we often encounter an awkward task of solving the eigenvalue equation to find the value of propagation constant. Since the dispersion equation is an irrational equation, we cannot solve it directly. Taking advantage of approximated calculation, we attempt here to solve this irrational dispersion equation. A new type of eigenvalue equation, in which guide index is expressed as a function of frequency, has been developed. In practical optical waveguide designing and in calculating the propagation mode, this equation will be used more conveniently than the previous one. To expedite the design of the waveguide, we then solve the eigenvalue equation of a slab waveguide, which is sufficiently accurate for practical purpose.

Keywords: Waveguide, Eigenvalue equation, Dispersion equation, Effective index method

1. Introduction

In dielectric waveguide analysis, we often encounter a difficult task of solving the eigenvalue equation to find the propagation constant in the guided region. Since an analytical closed-form solution is not available, many authors have reported different solution methods [1, 2, 3]. Goell solved the problem with a rigorous circular harmonic computer analysis [1]. Marcartili derived a closed-form solution for the well guided field by mode matching approximation [2]. And Hocker and Bruns used the well-known effective index method [3]. Among these works, Goell's result seems to be the exact one to date even though the solution technique is complex. Parallel to the above works, we have also tried to solve the two-dimensional transcendental eigenvalue equation approximately.

In this paper, we first describe the approximate solution technique of the 2-D eigenvalue equation to get a closed-form solution. Then, a model example is presented to replace a 3-D waveguide by a 2-D one using effective index method, so that this closed-form solution can also be applied. The obtained results of this conversion are compared with that of [1, 2] and [4].

2. Two-dimensional Eigenvalue Equation

The basic structure of slab waveguide is shown in Fig. 1. In this waveguide, the light confinement is on x-direction, the propagation of light is on z-direction and the length of the waveguide is considered infinite in y-direction. The refractive index of the film region n_f is considered higher than that of the cover n_c and substrate n_s regions.

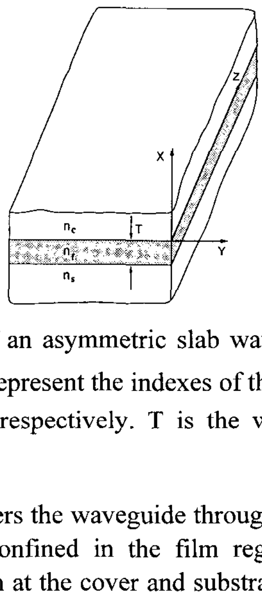

Fig. 1. Sketch of an asymmetric slab waveguide; n_c, n_f and n_s represent the indexes of the cover, film and substrate respectively. T is the width of the film region.

Once light enters the waveguide through the film region, it will remain confined in the film region by the total internal reflection at the cover and substrate boundaries. In the figure, the width of the film of guided region is denoted by T. A mode in the waveguide must satisfy Maxwell's equation as well as wave equation. Let us consider that the time varying electric and magnetic fields are propagating in the z-direction as the form,

$$\overline{E} = E(x,y)\cdot e^{j(\omega t-\beta z)}$$
$$\overline{H} = H(x,y)\cdot e^{j(\omega t-\beta z)} \quad (1)$$

where ω is the angular frequency and β is the propagating

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Received: July 9, 2005 ; Accepted: July 9, 2006

constant in the z-direction. If incident light free space waveguide and velocity are designated by λ and C_0 respectively, the angular frequency $\omega=2\pi C_0/\lambda$., considering uniform field in the y-direction (i.e., $\partial/\partial y = 0$), we can then write the 2-D wave equation as,

$$\frac{\partial^2 E_y}{\partial x^2} + (k_o^2 - \beta^2) = 0 \tag{2}$$

where k_o is the free space wave number equal to $2\pi/\lambda$ and n is index of the medium. From Maxwell's equation we find that the TE mode field components are E_y, H_x and H_z, and the TM mode field components are H_y, H_x and E_z. Since the treatment for TE and TM modes are the same, from now on we shall consider the TE modes only. Thus for TE modes, the transverse and longitudinal magnetic field components can be expressed in terms of the transverse electric field components as,

$$\begin{aligned} H_x &= -\frac{\beta}{\omega\mu_o} E_y \\ H_z &= -\frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial x} \end{aligned} \tag{3}$$

where μ_o is the vacuum permeability. The solutions of the wave equation for transverse electric field in three layers of the slab waveguide are found in [5, 6, 7]

$$\begin{aligned} E_y &= E_c \exp(-\gamma_c x) && x > 0 \\ E_y &= E_f \cos(k_x x + \phi_c) && -T < x < 0 \\ E_y &= E_s \exp[\lambda_s (x+T)] && x < -T \end{aligned} \tag{4}$$

where E_c, E_f and E_s are the complex field amplitudes of the respective layers. Their value will be determined by matching the field components at the interface of discontinuity. The parameters k_x , γ_s , γ_c and ϕ_c are defined by,

$$\begin{aligned} k_x &= k_o\sqrt{n_f^2 - N^2} \\ \gamma_c &= k_o\sqrt{N^2 - n_c^2} \\ \gamma_s &= k_o\sqrt{N^2 - n_s^2} \\ \tan\phi &= \gamma_c/\gamma_s \end{aligned} \tag{5}$$

In the above equation, N is the effective index of the guide. At the boundaries $x = 0.0$ and $x = -T$, the transverse electric and magnetic field should be continuous. Using these boundary conditions, we get two sets of equations from which the 2-D eigenvalue equation is found [7],

$$k_x T = (m'+1)\pi - \tan^{-1}\left(\frac{k_x}{\gamma_c}\right) - \tan^{-1}\left(\frac{k_x}{\lambda_s}\right) \tag{6}$$

where m' denotes the mode number, $m' = 0,1,2,\cdots$.

Equation (6) is a transcendental equation and cannot be solved without approximation or rigorous computer simulation. Defining the normalized frequency by V , normalized guide index by b and asymmetry measure by a as given below.

$$\begin{aligned} V &= k_o T\sqrt{n_f^2 - n_s^2} \\ b &= (N^2 - n_s^2)/(n_f^2 - n_s^2) \\ a &= (n_s^2 - n_c^2)/(n_f^2/N_s^2) \end{aligned} \tag{7}$$

Eq. (6) can be written as,

$$V\sqrt{1-b} = (m-1)\pi + \tan^{-1}\frac{\sqrt{b}}{\sqrt{1-b}} + \tan^{-1}\frac{\sqrt{b+a}}{1-b} \tag{8}$$

Here, m is the mode number, $m = 1,2,3,\cdots$. A graphical representation of Eq. (8) is rather easy in comparison to Eq. (6), but the objective to know the value of b which is a function of effective index N or the propagation constant β, is not yet fulfilled. As a matter of fact, a nearly exact value of b cannot be obtained without rigorous computer simulation. Therefore, to find a simple but accurate solution of Eq. (8), we then followed an approximate solution technique. First, we approximated this equation for two extreme conditions, which are used to derive a comprehensive closed-form solution of the eigenvalue equation, Eq. (8).

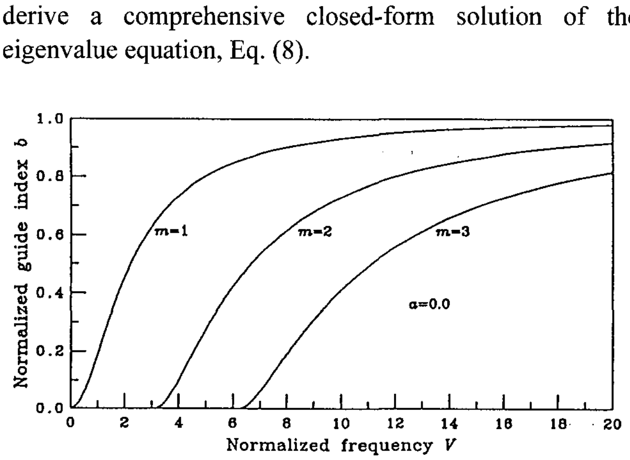


Fig. 2. Two-dimensional dispersion curves for the eigenvalue equation

3. The closed-form Solution of the Eigenvalue Equation

The characteristic curve of Eq. (8) is shown in Fig. 2. We first approximate one of these curves, say the curve for $m = 1$ at $b \cong 0$. That means in the vicinity of cutoff, Eq. (8) can be approximately solved.

$$\begin{aligned} V &\cong (m-1)\pi + \tan^{-1}\sqrt{a} + \tan^{-1}\sqrt{b} \\ &\cong (m-1)\pi + \tan^{-1}\sqrt{a} + \sqrt{b} \end{aligned} \tag{9}$$

Therefore, we get,

$$b = \left[V - (m-1)\pi - \tan^{-1}\sqrt{a}\right]^2 \tag{10}$$

Now, Eq. (8) is rearranged as,

$$\begin{aligned} V\sqrt{1-b} &= (m-1)\pi + \frac{\pi}{2} - \tan^{-1}\frac{\sqrt{1-b}}{\sqrt{b}} + \frac{\pi}{2} \\ &\quad - \tan^{-1}\frac{\sqrt{1-b}}{b+a} \\ &= m\pi - \tan^{-1}\frac{\sqrt{1-b}}{\sqrt{b}} - \tan^{-1}\frac{\sqrt{1-b}}{b+a} \end{aligned} \tag{11}$$

For the well-guided condition i.e., $b \cong 1$, Eq. (11) is approximately solved [4].

$$V\sqrt{1-b} \cong m\pi - \frac{\sqrt{1-b}}{\sqrt{b}} - \frac{\sqrt{1-b}}{\sqrt{b+a}} \tag{12}$$

From the above equation we get,

$$b = 1 - \left(\frac{m\pi}{V + (1 + 1/\sqrt{a+1})}\right)^2 \tag{13}$$

Equations (10) and (13) are accurate only in two extreme regions ($b \cong 0$ and $b \cong 1$). To fit these equations with the value region of the $b - V$ curve of Fig. 2, a combined equation is derived given by,

$$b = B^{1+\Delta}, \quad B \equiv 1 - \left(\frac{m\pi}{V + \pi - \tan^{-1}\sqrt{a}}\right)^2 \tag{14}$$

This combined equation reduces to Eq. (10) when $b \cong 1.0$. The parameter Δ is small and for better accuracy, we have added this parameter in the power of the equation. Even when we do consider Δ in Eq. (14), we find that the combined equation is very close to the original curve (see Fig. 3).

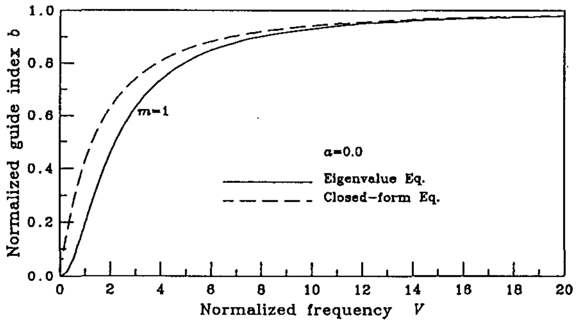


Fig. 3. The versus V plot for the eigenvalue Eq. (8) and also for the combined approximated Eq. (14) without considering Δ.

We defined Δ as,

$$\Delta = \frac{n(a,m)}{d(V)} \tag{15}$$

From the above equation, we can see that Δ is actually a function of three variables, a, m and V. For convenience of calculation, we have expressed Δ as a function of two variables, $n(a,m)$, and $d(V)$. Taking log on both sides of Eq. (14), we get,

$$\Delta = \frac{\ln b - \ln B}{\ln B} \tag{16}$$

From Eq. (16) we can write,

$$d(V) = \frac{n(a,m)}{(\ln b/\ln B) - 1} \tag{17}$$

In the above equation again, all three variables are present and it is difficult to get an analytical solution. So we followed a graphical approximation method.

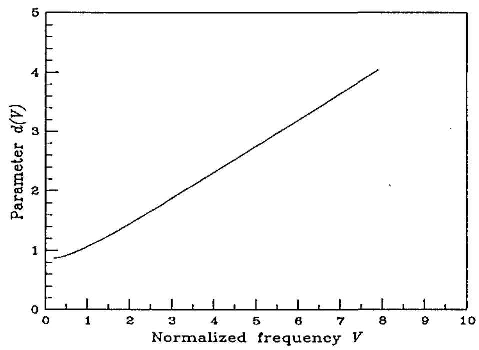


Fig. 4. Plot of the parameter $d(V)$ as a function of the normalized frequency V in Eq. (17)

Let us consider $a \cong 0$ and $m \cong 1$. Changing the value of the normalized guide index b, we calculated the corresponding value of V from Eq. (8). Substituting these values of V in Fig Eq. (17), we can plot $d(V)$ as a function of V as shown in Fig. 4. Upon close inspection of this figure we find that the upper part of the curve can be approximated by a straight line and the lower part by an exponential function. This leads to an approximate expression for $d(V)$ as,

$$d(V) = 0.44V + 0.55 + 0.29^{-1.4V} \tag{18}$$

and from Eqs. (14) and (15) we also get,

$$n(a,m) = d(V)(\ln b/\ln B - 1) \tag{19}$$

In order to find a simple approximate equation, we consider two separate cases;

Case 1: $m = 1$, $a =$ variable. From the graphical solution $d(V)$, Eq. (19) may be approximated as,

$$n(a,\, 1) \cong 0.43^{-2.7\sqrt{a}} - 0.15e^{-a/10} + 0.72 \tag{20}$$

Case 2: $a = 0$, $m =$ variable. From the graphical solution $d(V)$, Eq. (19) may be approximated as,

$$n(0,\, m) \cong 1 + 0.46(1 - e^{-\sqrt{m-1/2}}) \tag{21}$$

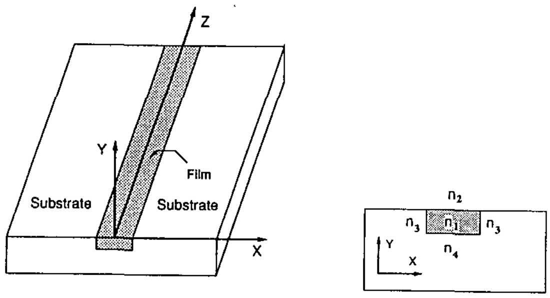
From Eq. (20) and Eq. (21) we can derive a general solution for $n(a,\, m)$ as,

$$n(a,\, m) \cong n(a,\, 1) + \frac{n(0,\, m) - 1}{a+1} \tag{22}$$

Eqs. (14), (15), (18), (20), (21) and (22), collectively give a closed-form solution to the eigenvalue Eq. (8). It is clear from the above description, that if we fix the normalized frequency V, for any mode number, m and the asymmetry measure a, we can calculate the normalized guide index b with a simple calculator. Whereas, conventionally, we must first fix the value of b to get value of V, which is a lengthy process. Thus the closed-form solution expedites the design of an optical waveguide.

4. Three-Dimensional Eigenvalue Equation

The dielectric slab waveguide explained in the earlier section is a useful model for more complicated waveguide structures. However, in most practical applications, more complicated waveguides are used. The waveguides used in integrated optics are usually rectangular strips of dielectric material that are embedded in order dielectrics.



(a) (b)

Fig. 5. (a) A rectangular dielectric waveguide in an integrated optics application. (b) A cross-section view of the rectangular guide showing the index of different layers.

Fig. 5(a) shows the geometry of an embedded type rectangular waveguide. An exact analytical treatment of such a 3-D guide is not possible. Approximate solutions by numerical methods have been obtained that can be made as accurate as desired [1].

In a 3-D guide, there are two types of modes that the waveguide can support. One type, which we will call E^x_{pq} mode, is polarized predominately in the x-direction. The other mode, E^y_{pq} is polarized predominately in the y-direction. The integers, p and q, are the mode numbers, and indicate the number of maxima of the field of distribution in x and y direction. We can adjust the amplitude coefficients of longitudinal field components E_z and H_z so that one of the transverse field components vanishes. Thus, each of the modes of E^x_{pq} and E^y_{pq} have two eigenvalue equations.

By matching the field components of E^x_{pq} at regions 1, 2 and 4 (see Fig. 5(b)), we get the obtained eigenvalue equation, which corresponds to the eigenvalue equation of the TM modes of the infinite slab [8]. On the other hand, by matching the field components of E^x_{pq} at regions 1, 3 and 4, we get another eigenvalue equation which corresponds to the TM modes of the infinite slab [5]. Solving of these eigenvalue equations is now even tougher than the 2-D eigenvalue equation. Since our closed-form solution is derived for a 2-D guide, we cannot apply it in this case directly. One way of attacking the problem is to employ the effective index method. But in applying this method we must consider field polarization of both the TE and TM modes. When the index difference $(n_1 - n_2)$ is large and $(n_1 - n_3)$ and $(n_1 - n_4)$ are small, the normalized dispersion curve for TE modes can be used in both x- and y-direction [4] with an approximate change in the asymmetry measure a. However, if the index difference $(n_1 - n_3)$ is also large, the polarization characteristics of the mode must be retained, and the normalized mode dispersion curves for both TE and TM modes must be employed [3].

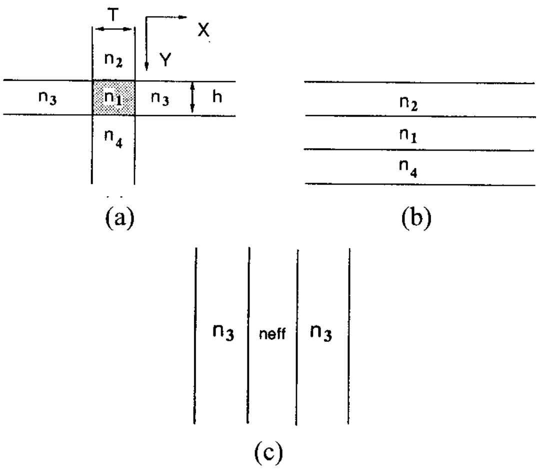


Fig. 6. (a) Cross-sectional view of a rectangular dielectric waveguide, (b) 2-D equivalent guide considering y directional confinement of light, (c) 2-D equivalent guide considering x directional confinement of light.

5. Effective Index Method

In order to use the closed-form solution described earlier, we reduce the 3-D structure into a 2-D one by using this method. In this method the propagating constant of the 3-D guide is first calculated letting the width of the guide approach infinity Fig. 6(b). This propagation constant is then used to define an effective dielectric constant. A second slab guide, filled with material of the previously calculated dielectric constant, is obtained by allowing the short dimension, or the height of the rectangular guide, to approach infinity (Fig. 6(c)). The propagation constant of this second equivalent slab (Fig. 6(c)) describes the modes of the original rectangular waveguide. We shall follow reference [1] and extend the theory for the effective index method as described in [2].

Let us consider $n_{2,3,4} = n_s = 2.2$ and $n_1 = n_f = 1.002 \times n_s$, where the width of the guide is T and the height is h. Firstly, let us consider the light confinement in the y-direction with T approaching infinity. From Eq. (7), we get the normalized height of the guide of Fig. 6(b) as,

$$V_y = \frac{2\pi}{\lambda} \cdot h\sqrt{(n_f^2 - n_s^2)} = \frac{h}{\lambda} \cdot 2\pi\sqrt{(n_f^2 - n_s^2)} \tag{23}$$

By varying h/λ, we can calculate the value of V and then the closed-form solution. Using Eq. (7), the effective index denoted by n_{eff} is calculated for the guide Fig. 6(b). The addition of confinement of light in the x-direction is now represented by the 2-D guide of Fig. 6(c) with the previously calculated effective index. The normalized guide width of Fig. 6(c) is given by,

$$V_x = \frac{2\pi}{\lambda} T\sqrt{(n_{eff}^2 - n_f^2)} = V_y b^{1/2} T / h \tag{24}$$

As before, the normalized guide index, b' of guide, of Fig. 6(c) is then calculated using the closed-form solution. The calculated effective index n_{eff} describes the mode of the original 3-D structure in this time.

6. Numerical Results

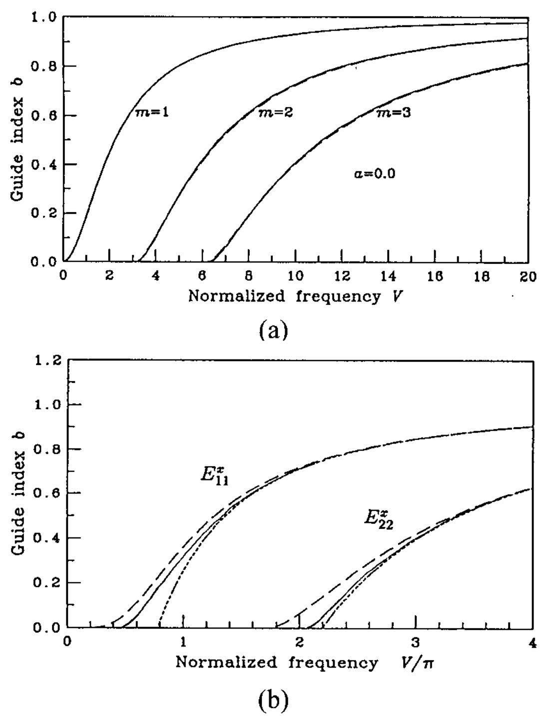


Fig. 7. A comparison of the mode dispersion curves for (a) two-dimensional waveguide, (b) three dimensional rectangular waveguide; $T = h$, $n_1 = 1.002$; Goell's computer solution (solid curve), Marcatili's analysis (dash and dot curve), Closed-form solution (broken line curve).

In Fig. 7 we have shown the b versus V curves of the original eigenvalue equation and the closed-form solution. The exact value denoted by b is compared with the closed-form value denoted by b'. For two-dimensional cases, we compared our result with that of reference [6] in Fig. 7(a). The solid line curves are approximate solutions.

There is no appreciable difference between the approximate and the exact curves in the fundamental mode, but for higher order modes ($m = 2, 3, \cdots$), a small deviation is observed for b, within 0.3 to 0.8.

We then applied this closed-form equation to a square dielectric guide for 2-D light confinement using effective index method. For small index difference between core and cladding ($n_1 = 1.002 \times n_{2,3,4}$), and with unity aspect ration, our curves are closer to Goell's curves [3] than Marcatili's curves [4] as shown in Fig. 7(b). As b approaches to unity, our closed-form solution exactly fits Goell's curve, but for $b \cong 0.0$, there is a small discrepancy. Even though near cutoff, our results were more favorable than Marcatili's.

Table 1. A comparison between b (exact) and b' (closed-form) for the 2-D waveguide.

Asymm. parameter a	Mode no. m	Exact value b	Closed-form b'	Error (b-b')
0.00	1	0.2000	0.2009	-0.0009
0.00	1	0.4000	0.4008	-0.0008
0.00	1	0.6000	0.6005	-0.0005
0.00	1	0.8000	0.8002	-0.0002
0.00	2	0.2000	0.2002	-0.0002
0.00	2	0.4000	0.3942	0.0058
0.00	2	0.6000	0.5932	0.0068
0.00	2	0.8000	0.7962	0.0038
0.00	3	0.2000	0.2021	-0.0021
0.00	3	0.4000	0.3952	0.0058
0.00	3	0.6000	0.5939	0.0068
0.00	3	0.8000	0.7966	0.0038
100.00	1	0.2000	0.1982	0.0018
100.00	1	0.4000	0.3887	0.0113
100.00	1	0.6000	0.5870	0.0130
100.00	1	0.8000	0.7923	0.0077
100.00	2	0.2000	0.1993	0.0007
100.00	2	0.4000	0.3921	0.0079
100.00	2	0.6000	0.5915	0.0085
100.00	2	0.8000	0.9754	0.0046
100.00	3	0.2000	0.1998	0.0002
100.00	3	0.4000	0.3942	0.0058
100.00	3	0.6000	0.5938	0.0062
100.00	3	0.8000	0.7967	0.0033

In Table 1, we have presented the calculated values of the normalized guide index b (exact) and b' (the closed-form solution), for the first few modes of the 2-D guide. From Table 1 we find that the maximum error is about 0.011 for higher order modes.

7. Conclusions

The approximate solution of the 2-D Eigenvalue equation expedites the design of the waveguide. The results for a 2-D waveguide show positive support of reference [6]. For the fundamental mode we found no difference between our result and those of reference [6]. For higher order modes we observed a small error around half the value of the normalized guide index b. On the other hand, this error becomes dominant near cutoff for the 3-D waveguide. Our curves exactly coincide with Goell's curve as b approaches unity, and are better for the whole range of b versus V curves. Although our closed-form equation is derived for TE modes, we can also apply this equation for TM mode [6]. If we adopt the equivalent index method, we can utilize this closed-form solution for design of the 3-D rectangular waveguide.

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