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A NOTE ON PARTIAL SIGN-SOLVABILITY

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ABSTRACT. In this paper we prove that if $A\mathbf{x} = \mathbf{b}$ is a partial sign-solvable linear system with A being sign non-singular matrix and if $\alpha = \{j : x_j \text{ is sign-determined by } A\mathbf{x} = \mathbf{b}\}$, then $A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha$ is a sign-solvable linear system, where A_α denotes the submatrix of A occupying rows and columns in α and $\mathbf{x}_\alpha$ and $\mathbf{b}_\alpha$ are subvectors of $\mathbf{x}$ and $\mathbf{b}$ whose components lie in α.

For a sign non-singular matrix A, let $A_1, \ldots, A_k$ be the fully indecomposable components of A and let α_i denote the set of row numbers of A_r, $r = 1, \ldots, k$. We also show that if $A\mathbf{x} = \mathbf{b}$ is a partial sign-solvable linear system, then, for $r = 1, \ldots, k$, if one of the components of $\mathbf{x}_{\alpha_r}$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$, then so are all the components of $\mathbf{x}_{\alpha_r}$.

1. Introduction

For a real number a, the sign of a, sign(a), is defined by

$$\operatorname{sign}(a) = \begin{cases} 1, & \text{if } a > 0, \\ -1, & \text{if } a < 0, \\ 0, & \text{if } a = 0. \end{cases}$$

For a real matrix A, let $\mathcal{Q}(A)$ denote the set of all real matrices $B = [b_{ij}]$ with the same size as A such that $\operatorname{sign}(b_{ij}) = \operatorname{sign}(a_{ij})$ for all i, j. For a matrix A, the matrix obtained from A by replacing each of the entries by its sign is a $(1, -1, 0)$-matrix. In that sense, a $(1, -1, 0)$-matrix is called a *sign-pattern matrix.*

Consider a linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = (x_1, \ldots, x_n)^T$. A component x_j of $\mathbf{x}$ is said to be *sign-determined* by $A\mathbf{x} = \mathbf{b}$ if, for every $\widetilde{A} \in \mathcal{Q}(A)$ and every $\widetilde{\mathbf{b}} \in \mathcal{Q}(\mathbf{b})$, $\widetilde{x}_j$ is determined by $\widetilde{A}\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}}$ and all

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the elements of the set $\{\widetilde{x}_j : \widetilde{A}\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}},\ \widetilde{A} \in \mathcal{Q}(A),\ \widetilde{\mathbf{b}} \in \mathcal{Q}(\mathbf{b})\}$ have the same sign. Such a common sign will be denoted by $\text{sign}(x_j : A\mathbf{x} = \mathbf{b})$ in the sequel. In particular, if $\text{sign}(x_j : A\mathbf{x} = \mathbf{b}) = 0$, then we call x_j a *fixed zero solution.* $A\mathbf{x} = \mathbf{b}$ is called *partial sign-solvable* if at least one of the components of $\mathbf{x}$ is sign-determined by $A\mathbf{x} = \mathbf{b}$ [3]. The system is *sign-solvable* if all of the components of $\mathbf{x}$ are sign-determined [5]. A matrix A is called an *L-matrix* if every $\widetilde{A} \in \mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is called a *sign non-singular matrix* (abb. SNS-matrix) [2]. It is well known that A is an SNS-matrix if and only if at least one of the $n!$ terms in the expansion of $\det A$ is not zero and all the nonzero terms have the same sign [1, 2, 5]. It is noted in [3] that, for an $m \times n$ matrix A having no $p \times q$ zero submatrix, where $p + q \geq n$, if $A\mathbf{x} = \mathbf{b}$ is partial sign-solvable then A^T is an L-matrix. Thus A is a SNS-matrix if $m = n$. An $n \times n$ matrix is called *fully indecomposable* if it contains no $p \times q$ zero submatrix, where $p + q = n$.

Suppose that $A\mathbf{x} = \mathbf{b}$ is a partial sign-solvable linear system, where A is a sign non-singular matrix. Let $\alpha = \{j : x_j \text{ is sign-determined by } A\mathbf{x} = \mathbf{b}\}$. Let A_α denote the principal submatrix of A occupying rows and columns in α and let $\mathbf{x}_\alpha, \mathbf{b}_\alpha$ denote the subvectors of $\mathbf{x}$ and $\mathbf{b}$ respectively whose components lie in α.

In this paper we prove that $A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha$ is a sign-solvable linear system and that, for $j \in \alpha$,

$$\text{sign}(x_j : A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha) = \text{sign}(x_j : A\mathbf{x} = \mathbf{b})$$

unless $\text{sign}(x_j : A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha) = 0$ while $\text{sign}(x_j : A\mathbf{x} = \mathbf{b}) \neq 0$.

By a theorem of Frobenius [4], A can be transformed into the form

$$\begin{bmatrix} A_1 & O & \cdots & O & O \\ A_{21} & A_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{k-1,1} & A_{k-1,2} & \cdots & A_{k-1} & O \\ A_{k1} & A_{k2} & \cdots & A_{k,k-1} & A_k \end{bmatrix}, \tag{1.1}$$

where $A_r, r = 1, \ldots, k$, are fully indecomposable matrices by permutations of rows and columns if necessary. The matrices $A_1, \ldots, A_k$ are called the *fully indecomposable components* of A. For each $r = 1, \ldots, k$, let α_r denote the set of row numbers of A_r. Let $\mathbf{x}^{(r)}, \mathbf{b}^{(r)}$ denote the vectors $\mathbf{x}_{\alpha_r}$ and $\mathbf{b}_{\alpha_r}$ for brevity.

In this paper we show that if $A\mathbf{x} = \mathbf{b}$ is partial sign-solvable, then, for each $r = 1, \ldots, k$, either none of the components of $\mathbf{x}^{(r)}$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$ or all of the components of $\mathbf{x}^{(r)}$ are fixed zero solutions.

2. Preliminaries

For an $n \times n$ matrix $A = [a_{ij}]$, the digraph $\mathcal{D}(A)$ associated with A is defined as the one whose vertices are $1, 2, \ldots, n$ and there is an arc $i \to j$ in $\mathcal{D}(A)$ if and only if $a_{ij} \neq 0$. A *1-factor* of $\mathcal{D}(A)$ is a set of cycles or loops $\gamma_1, \ldots, \gamma_k$ such that each of the vertices $1, 2, \ldots, n$ belongs to exactly one of the γ_i's. A 1-factor of $\mathcal{D}(A)$ gives rise to a nonzero term in the expansion of $\det A$. We call a 1-factor a *positive* (*negative* resp.) *1-factor* if the product of signs of the arcs in the 1-factor is positive (negative resp.), where the sign of an arc $i \to j$ is defined to be the same as $\operatorname{sign}(a_{ij})$.

A square matrix A is called *irreducible* if there does not exist a permutation matrix P such that

$$PAP^T = \begin{bmatrix} X & O \\ * & Y \end{bmatrix},$$

where X and Y are nonvacuous square matrices. Clearly a fully indecomposable matrix is an irreducible matrix. A digraph $\mathcal{D}$ is called *strongly connected* if for any two vertices u, v of $\mathcal{D}$ there is a path from u to v. It is well known that, for a square matrix A, A is irreducible if and only if $\mathcal{D}(A)$ is strongly connected [4].

Given a linear system $A\mathbf{x} = \mathbf{b}$, let $B\mathbf{x} = \mathbf{c}$ be the linear system obtained from $A\mathbf{x} = \mathbf{b}$ by applying one or more of the following operations.

- Permuting rows of $(A, \mathbf{b})$.
- Simultaneously permuting columns of A and components of $\mathbf{x}$.
- Multiplying a row of $(A, \mathbf{b})$ by -1.
- Multiplying a column of A and the corresponding component of $\mathbf{x}$ by -1.

Then $B\mathbf{x} = \mathbf{c}$ has the 'same' solution as $A\mathbf{x} = \mathbf{b}$.

If A is fully indecomposable, then there are permutation matrices P, Q, R and diagonal matrices D, E with diagonal entries 1 or -1 such that $RDPAQER^T$ has the form (1.1) and has -1's on its main diagonal. Assume that A is an $n \times n$ SNS-matrix. Suppose that $A\mathbf{x} = \mathbf{b}$ is partially sign-solvable linear system. Since, by the Cramer's rule,

$$x_j = \frac{\det A(j \leftarrow \mathbf{b})}{\det A}, \ (j = 1, 2, \ldots, n),$$

where and in the sequel $A(j \leftarrow \mathbf{b})$ denotes the matrix obtained from A by replacing the column j by the vector $\mathbf{b}$, it is clear that x_j is sign-determined if and only if either $A(j \leftarrow \mathbf{b})$ has identically zero determinant or $A(j \leftarrow \mathbf{b})$ is an SNS-matrix.

In the sequel, for a submatrix B of A, we assume that B uses the same row number and column number as those in A. From example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ B = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

then row 2 and row 3 of B are $(a_{22}, a_{23}), (a_{32}, a_{33})$ and the column 2 and column 3 of B are $(a_{22}, a_{32})^T, (a_{23}, a_{33})^T$. By the same way, the indices of components of subvectors of $\mathbf{x}$ and $\mathbf{b}$ will be used.

We close this section with the following lemma.

LEMMA 2.1. *Let A be an $n \times n$ SNS-matrix with negative main diagonal entries and let $\alpha \subset \{1, 2, \ldots, n\}$. Then the following hold.*

(1) *A_α is an SNS-matrix.*
(2) *If, for some $j \in \alpha$, x_j is sign-determined by $A\mathbf{x} = \mathbf{b}$, then x_j is sign-determined by $A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha$.*

Proof. (1) Since A_α is a principal submatrix of A, $\mathcal{D}(A_\alpha)$ has a 1-factor. Since every 1-factor of $\mathcal{D}(A_\alpha)$ gives rise to a 1-factor of A, the sign non-singularity of A_α follows from that of A.

(2) Suppose that $j \in \alpha$ and that x_j is not sign-determined by $A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha$. Then $A_\alpha(j \leftarrow \mathbf{b}_\alpha)$ has both a positive 1-factor and a negative 1-factor, which yield a positive 1-factor and a negative 1-factor of $A(j \leftarrow \mathbf{b})$ since A has negative main diagonal entries, contradicting that x_j is sign-determined by $A\mathbf{x} = \mathbf{b}$. □

3. Main results

We first show that a partial sign-solvable linear system has a sign-solvable subsystem.

THEOREM 3.1. *Let A be an $n \times n$ SNS-matrix with negative main diagonal entries. Let $A\mathbf{x} = \mathbf{b}$ be a partial sign-solvable linear system and let $\alpha = \{j : x_j$ is sign-determined by $A\mathbf{x} = \mathbf{b}\}$. Then the following hold.*

(1) *A_α is an SNS-matrix and $A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha$ is sign-solvable.*
(2) *For $j \in \alpha$,*

(*i*) *if* $x_j = 0$ *is a fixed zero solution of* $A\mathbf{x} = \mathbf{b}$, *then* $x_j = 0$ *is a fixed zero solution of* $A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha$, *and*

(*ii*) *if* x_j *is not a fixed zero solution of* $A\mathbf{x} = \mathbf{b}$, *then* $\operatorname{sign}(x_j : A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha) = \operatorname{sign}(x_j : A\mathbf{x} = \mathbf{b})$ *unless* $\operatorname{sign}(x_j : A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha) = 0$.

Proof. (1) follows directly from Lemma 2.1.

(2) (*i*) Every 1-factor of $\mathcal{D}(A_\alpha(j \leftarrow \mathbf{b}_\alpha))$ yields a 1-factor of $\mathcal{D}(A(j \leftarrow \mathbf{b}))$. Since $x_j = 0$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$, $\mathcal{D}(A(j \leftarrow \mathbf{b}))$ has no 1-factor. Therefore $\mathcal{D}(A_\alpha(j \leftarrow \mathbf{b}_\alpha))$ has no 1-factor, telling us that $A_\alpha(j \leftarrow \mathbf{b}_\alpha)$ has identically zero determinant.

(*ii*) Suppose that x_j is not a fixed zero solution of $A\mathbf{x} = \mathbf{b}$. Let $|\alpha| = m$, where $|\alpha|$ stands for the number of elements of α. Then $\operatorname{sign}(\det A(j \leftarrow \mathbf{b})) = (-1)^{n-m}\operatorname{sign}(\det A_\alpha(j \leftarrow \mathbf{b}_\alpha))$ unless $A_\alpha(j \leftarrow \mathbf{b}_\alpha)$ has identically zero determinant. Since $\operatorname{sign}(\det A) = (-1)^{n-m}$ $\operatorname{sign}(\det A_\alpha)$ we see that

$$\frac{\det A_\alpha(j \leftarrow \mathbf{b}_\alpha)}{\det A_\alpha}, \quad \frac{\det A(j \leftarrow \mathbf{b})}{\det A}$$

have the same sign unless $\det A_\alpha(j \leftarrow \mathbf{b}_\alpha) = 0$. □

It may well be possible that $\operatorname{sign}(x_j : A_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha) = 0$ even though $\operatorname{sign}(x_j : A\mathbf{x} = \mathbf{b}) \neq 0$ as we see in the following example.

EXAMPLE 3.2. Let

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

and let $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ be a solution of $A\mathbf{x} = \mathbf{b}$. Then $\operatorname{sign}(x_3 : A\mathbf{x} = \mathbf{b}) = -\operatorname{sign}(x_4 : A\mathbf{x} = \mathbf{b}) = -1$, but x_3 and x_4 are fixed zero solutions in the sign-solvable subsystem

$$\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In the rest of this section we investigate the distribution of fixed zero solutions of a partial sign-solvable linear system.

LEMMA 3.3. *Let* A *be a fully indecomposable SNS-matrix and let* $A\mathbf{x} = \mathbf{b}$ *be a partial sign-solvable linear system. Then the following are equivalent.*

(1) $A\mathbf{x} = \mathbf{b}$ *has a fixed zero solution.*

(2) $\mathbf{b} = \mathbf{0}$.
(3) *Every component of* $\mathbf{x}$ *is a fixed zero solution of* $A\mathbf{x} = \mathbf{b}$.

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathbf{b} = (b_1, \ldots, b_n)^T \neq \mathbf{0}$. Then $b_i \neq 0$ for some i. Since $x_j = 0$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$, it follows that $A(j \leftarrow \mathbf{b})$ has identically zero determinant. Since the (i,j)-entry of $A(j \leftarrow \mathbf{b})$, which is b_i, is not zero, it follows that $A(i|j)$, the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row i and the column j, has identically zero determinant so that there exist a $p \times q$ zero submatrix of $A(i|j)$, where $p + q = (n-1) + 1$, contradicting the fully indecomposability of A.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear. □

LEMMA 3.4. *Let* A *be a square matrix of the form*

$$A = \begin{bmatrix} A_1 & O & \cdots & O & B_1 \\ B_2 & A_2 & \cdots & O & O \\ * & B_3 & \ddots & O & O \\ \vdots & \vdots & \ddots & A_{k-1} & O \\ * & * & \cdots & B_k & A_k \end{bmatrix},$$

where A_r *is fully indecomposable and* $B_r \neq O$ *for each* $r = 1, 2, \ldots, k$. *Then* $\mathcal{D}(A)$ *has a 1-factor.*

Proof. If, for some distinct $p, q \in \{1, \ldots, k\}$, there is an arc in $\mathcal{D}(A)$ from a vertex of $\mathcal{D}(A_p)$ to a vertex of $\mathcal{D}(A_q)$, we simply say that there is an arc from $\mathcal{D}(A_p)$ to $\mathcal{D}(A_q)$. By the structure of A, we see that there is an arc e_r from $\mathcal{D}(A_r)$ to $\mathcal{D}(A_{r-1})$ for each $r = 1, 2, \ldots, k$, where we assume that A_0 equals A_k. Since $\mathcal{D}(A_r)$ is strongly connected for each $r = 1, \ldots, k$, there occurs a cycle in $\mathcal{D}(A)$ containing all e_r's as its part, and the lemma is proved. □

Now we are ready to prove our last main theorem.

THEOREM 3.5. *Let* A *be an SNS-matrix of the form* (1.1) *with negative main diagonal entries. Then for each* $r = 1, \ldots, k$, *either non of the components of* $\mathbf{x}_{\alpha_r}$ *is a fixed zero solution of* $A\mathbf{x} = \mathbf{b}$ *or all of the components of* $\mathbf{x}_{\alpha_r}$ *are fixed zero solutions of* $A\mathbf{x} = \mathbf{b}$. *Moreover, in the latter case,* $\mathbf{b}_{\alpha_r} = \mathbf{0}$.

Proof. We write $\mathbf{x}^{(r)}$ and $\mathbf{b}^{(r)}$ instead of $\mathbf{x}_{\alpha_r}$ and $\mathbf{b}_{\alpha_r}$ for the sake of simplicity. We prove the theorem by induction on k.

The theorem for $k = 1$ follows directly from Lemma 3.3.

Let $k \geq 2$. In the proof of the theorem, we shall call a diagonal block A_r an *isolated block* if all the blocks in the block row r are O's except for A_r and if $\mathbf{b}_r = \mathbf{0}$.

Let $x_j = 0$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$, where $j \in \alpha_p$.

Suppose that non of $A_1, \ldots, A_p$ is an isolated block. Then, first of all, $\mathbf{b}_1 \neq \mathbf{0}$ because A_1 is not an isolated block. If $\mathbf{b}_p \neq \mathbf{0}$, then there is an $i \in \alpha_p$ such that $b_i \neq 0$. $A(j \leftarrow \mathbf{b})_{\alpha_p} = A_p(j \leftarrow \mathbf{b}_p)$. Note that $b_i \neq 0$ is the (i,j)-entry of $A_p(j \leftarrow \mathbf{b}_p)$ so that there is an arc e in $\mathcal{D}(A_p(j \leftarrow \mathbf{b}_p))$. Remember that $A_p(j \leftarrow \mathbf{b}_p)$ uses the same row numbers and column numbers. Since A_p is strongly connected there is a path γ from j to i in $\mathcal{D}(A_p)$ and hence in $\mathcal{D}(A_p(j \leftarrow \mathbf{b}_p))$. Now, the path γ together with the preceded arc e gives rise to a 1-factor of $A_p(j \leftarrow \mathbf{b}_p)$ which yields a 1-factor of $A(j \leftarrow \mathbf{b})$, contradicting that $x_j = 0$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$. Thus $\mathbf{b}_p = \mathbf{0}$. Since $\mathbf{b}_1 \neq \mathbf{0}$, it must be that $p \geq 2$. We claim that there exists a sequence $r_1, \ldots, r_m$ such that $p = r_1 > r_2 > \cdots > r_m \geq 1$, $A_{r_1 r_2} \neq O$, $A_{r_2 r_3} \neq O, \ldots, A_{r_{m-1} r_m} \neq O$ and $\mathbf{b}_m \neq \mathbf{0}$. Since A_p is not isolated and $\mathbf{b}_p = \mathbf{0}$, $[A_{p1}, \ldots, A_{p,p-1}] \neq O$ and hence there is an $r_2 < p$ such that $A_{pr_2} \neq O$. If $\mathbf{b}_{r_2} \neq \mathbf{0}$, then we end up with the sequence p, r_2. If $\mathbf{b}_{r_2} = \mathbf{0}$, then since $[A_{r_2 1}, \ldots, A_{r_2, r_2 - 1}] \neq O$, there is an $r_3 < r_2$ such that $A_{r_2 r_3} \neq O$. If $\mathbf{b}_{r_3} \neq \mathbf{0}$, then we are done. Otherwise we repeat this process. Eventually we will end up with a sequence with the required property because $\mathbf{b}_1 \neq \mathbf{0}$. Thus there occurs a 1-factor of $A(j \leftarrow \mathbf{b})$ by Lemma 3.4 contradicting that $x_j = 0$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$. Therefore it is shown that A_q is an isolated block for some $q \in \{1, \ldots, p\}$.

Now the system $A\mathbf{x} = \mathbf{b}$ can be written as

$$\begin{cases} A_q \mathbf{x}_q = \mathbf{0} \\ A_\beta \mathbf{x}_\beta = \mathbf{b}_\beta, \end{cases}$$

where $\beta = \{1, \ldots, n\} - \alpha_q$, and it follows, by induction, that every component of $\mathbf{x}_p$ is a fixed zero solution of $A\mathbf{x} = \mathbf{b}$. □

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