Bull. Korean Math. Soc. **43** (2006), No. 3, pp. 479–486

THE SEQUENTIAL UNIFORM LAW OF LARGE NUMBERS

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ABSTRACT. Let $Z_n(s,f) = n^{-1}\sum_{i=1}^{[ns]}(f(X_i) - Pf)$ be the sequential empirical process based on the independent and identically distributed random variables. We prove that convergence problems of $\sup_{(s,f)} |Z_n(s,f)|$ to zero boil down to those of $\sup_f |Z_n(1,f)|$. We employ Ottaviani's inequality and the complete convergence to establish, under bracketing entropy with the second moment, the almost sure convergence of $\sup_{(s,f)} |Z_n(s,f)|$ to zero.

1. Introduction and the main result

Let $X_1, \ldots, X_n$ be independent and identically distributed random variables. Let $\{D_n\}$ be the sequential process defined by

$$D_n(s) = n^{-1/2}\sum_{i=1}^{[ns]} X_i \text{ for } 0 \le s \le 1, \tag{1.1}$$

where $[x]$ denotes the integral part of x. It is well known that, under the topic of Donsker's invariance principle, the process D_n in (1.1) converges weakly to a Gaussian process. See, for example, Billingsley [1].

We have encountered the sequential process $\{G_n\}$ defined by

$$G_n(s) = n^{-1}\sum_{i=1}^{[ns]} X_i \text{ for } 0 \le s \le 1 \tag{1.2}$$

in mathematical finance literatures. See Shreve [6].

The fact that $G_n(s)$ converges almost surely to sEX_1 for each fixed $s \in [0,1]$ is well known as a strong law of large numbers.

It is surprising that the following question is not yet settled down.

Received April 7, 2005.

2000 Mathematics Subject Classification: Primary 60F15; Secondary 62G20.

Key words and phrases: sequential Glivenko-Cantelli class, Ottaviani's inequality, complete convergence, almost sure convergence, uniform law of large numbers.

QUESTION. *What are the asymptotic behaviors of*

$$\sup_{0\le s\le 1}|G_n(s)|?$$

In this paper we provide a solution to the question by Ottaviani's inequality and complete convergence.

We begin by formulating the above question.

Let $\{X_i : i \geq 1\}$ be a sequence of independent and identically distributed random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. Given a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$, we see that $\{f(X_i) : i \geq 1\}$ forms a sequence of independent and identically distributed random variables that are more flexible in applications than $\{X_i : i \geq 1\}$.

We state the following sequential strong law of large numbers.

THEOREM 1.1. *Suppose that* $\int f^2(x)P(dx) < \infty$. *Then,*

$$\sup_{0\le s\le 1}\left|\frac{1}{n}\sum_{i=1}^{[ns]}(f(X_i)-Pf)\right| \to 0 \quad \textit{almost surely.}$$

The proof of Theorem 1.1 will be given at the end of the paper.

2. Sequential Glivenko-Cantelli classes

In addition to the setting in previous section, we consider a class $\mathcal{F}$ of real-valued Borel measurable functions defined on $\mathbb{R}$. Define a sequential empirical process Z_n by

$$Z_n(s,f) = \frac{1}{n}\sum_{i=1}^{[ns]}(f(X_i)-Pf) \text{ for } (s,f) \in [0,1]\otimes\mathcal{F}.$$

Define the empirical measure by

$$\mathbf{P}_n(f) := \frac{1}{n}\sum_{i=1}^{n} f(X_i) \text{ for } f \in \mathcal{F}.$$

Observe that

$$Z_n(s,f) = \frac{[ns]}{n}(\mathbf{P}_{[ns]} - P)(f)$$

and

$$Z_n(1,f) = (\mathbf{P}_n - P)(f).$$

For a process $\{Y_t\}$ indexed by an arbitrary set, we denote $||Y||$ to mean $\sup_t |Y_t|$. Let P^* denote the outer probability with respect to the underlying probability P.

We need the following Ottaviani's inequality.

LEMMA 2.1. *Let $X_1, \dots, X_n$ be independent stochastic processes indexed by an arbitrary set. Let $S_n := X_1 + \cdots + X_n$. Then for $\lambda, \mu > 0$,*

$$P^* \left(\max_{1 \le k \le n} ||S_k|| > \lambda + \mu \right) \le \frac{P^* \left(||S_n|| > \lambda \right)}{1 - \max_{1 \le k \le n} P^* \left(||S_n - S_k|| > \mu \right)}.$$

Proof. See Proposition A.1.1 in Van der Vaart and Wellner [8]. □

We obtain the following maximal inequality that illuminates the role of $s \in [0,1]$ is negligible in the problem of law of large numbers. Compare with the role of $s \in [0,1]$ in the problem of central limit theorem in section 2.12 in Van der Vaart and Wellner [8]. Let $\mathcal{S} := [0,1] \otimes \mathcal{F}$.

THEOREM 2.2. *There exists a universal constant C such that*

$$P^* \left(||Z_n||_{\mathcal{S}} > 2\epsilon \right) \le CP^* \left(||Z_n(1, \cdot)||_{\mathcal{F}} > \epsilon \right) \text{ for every } \epsilon > 0.$$

Proof. For $s \in [0,1]$ and $f \in \mathcal{F}$, we see that $|Z_n(s,f)| \le ||Z_n(s,\cdot)||_{\mathcal{F}}$. Take the sup over $\mathcal{S}$ on both sides to obtain

$$||Z_n||_{\mathcal{S}} \le \sup_{0 \le s \le 1} ||Z_n(s,\cdot)||_{\mathcal{F}}. \tag{2.1}$$

In the right hand side of (2.1) the parameter s may be restricted to the points k/n with k ranging over $1, 2, \dots, n$. Since $Z_n(s,f) = \frac{[ns]}{n}(\mathbf{P}_{[ns]} - P)(f)$, we have

$$||Z_n||_{\mathcal{S}} \le \max_{1 \le k \le n} \frac{k}{n} ||\mathbf{P}_k - P||_{\mathcal{F}}.$$

Ottaviani's inequality in Lemma (2.1) gives

$$P^* \left(\max_{1 \le k \le n} \frac{k}{n} ||\mathbf{P}_k - P||_{\mathcal{F}} > 2\epsilon \right)$$
$$\le \frac{P^*(||\mathbf{P}_n - P||_{\mathcal{F}} > \epsilon)}{1 - \max_{1 \le k \le n} P^* \left(\frac{k}{n} ||\mathbf{P}_k - P||_{\mathcal{F}} > \epsilon \right)}.$$

The term $\max_{1 \le k \le n} P^* \left(\frac{k}{n} ||\mathbf{P}_k - P||_{\mathcal{F}} > \epsilon \right)$ indexed by $k \le n_0$ can be controlled with the help of the inequality

$$k||\mathbf{P}_k - P||_{\mathcal{F}} \le 2 \sum_{i=1}^{n_0} F(X_i) + 2n_0 P^* F$$

for an envelope function F. For sufficiently large n_0 the term indexed by $k > n_0$ are bounded away from 1 by the uniform weak law of large numbers for $\mathbf{P}_n$. Conclude that the denominator is bounded away from zero. The proof is completed. $\square$

Let T^* denote the measurable cover function for any mapping $T : \Omega \to \mathbb{R}$.

DEFINITION 2.3. A class $\mathcal{F}$ of measurable functions is a sequential weak Glivenko-Cantelli if

$$||Z_n||_{\mathcal{S}}^* \to 0 \text{ in probability.}$$

The class $\mathcal{F}$ is a weak Glivenko-Cantelli if

$$Z_n(1,\cdot)^* \to 0 \text{ in probability.}$$

Sequential strong Glivenko-Cantelli class and strong Glivenko-Cantelli class are defined in a similar fashion by using almost sure convergence.

COROLLARY 2.4. *A class $\mathcal{F}$ is a sequential weak Glivenko-Cantelli if and only if it is a weak Glivenko-Cantelli.*

Proof. Observe that $||Z_n(1,\cdot)||_{\mathcal{F}} \le ||Z_n||_{\mathcal{S}}$. By Theorem 2.2, we get

$$P^*\left(||Z_n(1,\cdot)||_{\mathcal{F}} > 2\epsilon\right) \le P^*\left(||Z_n||_{\mathcal{S}} > 2\epsilon\right) \le CP^*\left(||Z_n(1,\cdot)||_{\mathcal{F}} > \epsilon\right).$$

The proof is completed. $\square$

COROLLARY 2.5. *Let f be such that $\int |f(x)|P(dx) < \infty$. Then, singleton set $\{f\}$ is a sequential weak Glivenko-Cantelli class.*

Proof. For a singleton set $\{f\}$, since $\int |f(x)|P(dx) < \infty$, we have by weak law of large numbers

$$||Z_n(1,\cdot)||_{\{f\}} = (\mathbf{P}_n - P)(f) \to 0 \text{ in probability.}$$

Apply Corollary 2.4 to finish the proof. $\square$

DEFINITION 2.6. A sequence of random variables $\{Y_n\}$ converges completely to constant c if the series $\sum_{n=1}^{\infty} P(|Y_n - c| > 1)$ converges. A class $\mathcal{F}$ is a sequential complete Glivenko-Cantelli if

$$||Z_n||_{\mathcal{S}}^* \to 0 \text{ completely.}$$

It is a complete Glivenko-Cantelli class if

$$||Z_n(1,\cdot)||_{\mathcal{F}}^* \to 0 \text{ completely.}$$

REMARK 2.7. It is clear that if $\sum_{n=1}^{\infty} P(|Y_n - c| > 1)$ converges then $\sum_{n=1}^{\infty} P(|Y_n - c| > \epsilon)$ converges for every $\epsilon > 0$. See Erdos [3].

We will use the following result on complete convergence in proving the results on almost sure convergence.

COROLLARY 2.8. *A class of measurable functions is a sequential complete Glivenko-Cantelli if and only if it is complete Glivenko-Cantelli.*

Proof. Recall that

$$P^*\left(\|Z_n(1,\cdot)\|_{\mathcal{F}} > 2\epsilon\right) \le P^*\left(\|Z_n\|_{\mathcal{S}} > 2\epsilon\right) \le CP^*\left(\|Z_n(1,\cdot)\|_{\mathcal{F}} > \epsilon\right).$$

The proof is completed by taking the summation on each side. □

3. A sequential strong Glivenko-Cantelli class

Let $\mathcal{F} \subset \mathcal{L}_2(P) := \{f : \int f^2(x)P(dx) < \infty\}$ be a class of real-valued measurable functions defined on $\mathbb{R}$. In this paper, we will use the $\mathcal{L}^2$ metric.

In order to measure the size of the function space, we define the following version of metric entropy with bracketing. See, for example, Van der Vaart and Wellner [8] and Van der Geer [7] for recent references.

DEFINITION 3.1. Given two functions l and u, the bracket $[l,u]$ is the set of all functions f with $l \le f \le u$. An ϵ-bracket is a bracket $[l,u]$ with $\left[\int (u-l)^2(x)P(dx)\right]^{1/2} < \epsilon$. The bracketing number $N_{[\,]}(\epsilon) := N_{[\,]}(\epsilon, \mathcal{F}, d)$ is the minimum number of ϵ-brackets needed to cover $\mathcal{F}$. We say that $\mathcal{F}$ has a bracketing entropy if $\int_0^\infty [\ln N_{[\,]}(\epsilon, \mathcal{F}, d)]^{1/2} d\epsilon < \infty$.

Our goal is to find a condition on $\mathcal{F}$ that suffices to be a sequential strong Glivenko-Cantelli class.

We are ready to state the following.

THEOREM 3.2. *Suppose that $\mathcal{F}$ has a bracketing entropy. Then, it is a sequential strong Glivenko-Cantelli class. That is,*

$$\sup_{\mathcal{S}} \left| \frac{1}{n} \sum_{i=1}^{[ns]} (f(X_i) - Pf) \right|^* \to 0 \quad \textit{almost surely.}$$

We will use the following complete law of large numbers that appears in Hsu and Robbins [4].

PROPOSITION 3.3. *Suppose that $\int f^2(x)P(dx) < \infty$. Then,*

$$\int f(x)(\mathbf{P}_n - P)(dx) \to 0$$

completely. That is, the series

$$\sum_{n=1}^{\infty} P\left(\left| \int f(x)(\mathbf{P}_n - P)(dx) \right| > 1 \right)$$

converges.

REMARK 3.4. It is known that the assumption $\int f^2(x)P(dx) < \infty$ is essential. See Theorem 2 and their conjecture in Hsu and Robbins [4]. See also Erdos [3] for the affirmative answer to the conjecture.

We are ready to perform the proof of Theorem 3.2. We find that the idea of DeHardt [2] is still worked in getting Theorem 3.2.

Proof. In order to get the almost sure convergence, it suffices to show the complete convergence. See Proposition 5.7 in [5]. In view of Corollary 2.8, to obtain complete convergence of $\sup_{\mathcal{S}} |\frac{1}{n}\sum_{i=1}^{[ns]}(f(X_i) - Pf)|^*$ to zero, it is enough to prove $\sup_{f\in\mathcal{F}} |\frac{1}{n}\sum_{i=1}^{n}(f(X_i) - Pf)|^* \to 0$ completely. Represent $\frac{1}{n}\sum_{i=1}^{n}(f(X_i) - Pf)$ as an integral form $\int f(x)(\mathbf{P}_n - P)(dx)$. We name the integral process as $U_n(f)$. Fix $\epsilon > 0$. Choose finitely many ϵ-brackets $[l_i, u_i]$ whose union contains $\mathcal{F}$ and such that $\int (u_i - l_i)^2(x)P(dx) < \epsilon^2$ for every $i = 1, \ldots, N_{[\,]}(\epsilon)$. Then, for every $f \in \mathcal{F}$, there is a bracket such that

$$\begin{aligned} U_n(f) &= \int f(x)\mathbf{P}_n(dx) - \int f(x)P(dx) \\ &= \int f(x)\mathbf{P}_n(dx) - \int u_i(x)P(dx) \\ &\quad + \int u_i(x)P(dx) - \int f(x)P(dx) \\ &\le \int u_i(x)(\mathbf{P}_n - P)(dx) + \int (u_i - l_i)(x)P(dx). \end{aligned}$$

Now, observe that

$$\int (u_i - l_i)(x)P(dx) \le \left[\int (u_i - l_i)^2(x)P(dx)\right]^{1/2} < \epsilon.$$

Consequently,

$$\sup_{f\in\mathcal{F}} U_n(f) \le \max_{1\le i\le N_{[\,]}(\epsilon)} \int u_i(x)(\mathbf{P}_n - P)(dx) + \epsilon.$$

The right hand side converges completely to ϵ by Proposition 3.3. Combination with a similar argument for $\inf_{f\in\mathcal{F}} U_n(f)$ yields that

$$\limsup_{n\to\infty} \sup_{f\in\mathcal{F}} |U_n(f)|^* \le \epsilon,$$

completely, for every $\epsilon > 0$. Take a sequence $\epsilon_m \downarrow 0$ to see that the limsup must actually be zero completely. The proof of Theorem 3.2 is completed. □

COROLLARY 3.5. *Suppose that $\mathcal{F}$ has a bracketing entropy. Then, it is a sequential complete Glivenko-Cantelli class. That is,*

$$\sup_{S} \left| \frac{1}{n} \sum_{i=1}^{[ns]} (f(X_i) - Pf) \right|^* \to 0 \quad \textit{completely.}$$

Proof. This is not a corollary to Theorem 3.2 itself but a corollary to the proof of Theorem 3.2. □

We are ready to finish the proof of Theorem 1.1.

Proof. The singleton set $\mathcal{F} = \{f\}$ certainly satisfies the bracketing entropy condition. The result follows from Theorem 3.2. □

REMARK 3.6.

1. To the best of our knowledge, we cannot weaken the second moment assumption of $\int f^2(x)P(dx) < \infty$. See Remark 3.4.
2. Convergence of $\sup_{0 \le s \le 1} \left| n^{-1} \sum_{i=1}^{[ns]} (f(X_i) - Pf) \right|$ to zero in probability is valid under the first moment assumption. See Corollary 2.5.
3. It is our opinion that considering G_n in (1.2) without accompanying D_n in (1.1) is not natural. When we consider G_n together with D_n, the second moment condition is presumably given. See Shreve [6].
4. Under a second moment condition, the weak laws can be obtained as a result of central limit theorem by an application of Slutsky's theorem. However, strong laws have to be dealt with separately.

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