ON THE SOLUTION OF A BI-JENSEN FUNCTIONAL EQUATION AND ITS STABILITY

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ABSTRACT. In this paper, we obtain the general solution and the stability of the bi-Jensen functional equation

\[ 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w). \]

1. Introduction

In 1940, Ulam proposed the general Ulam stability problem (see [8]):

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \) then there is a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \)?

In 1941, this problem was solved by Hyers [3] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability. In 1978, Th. M. Rassias [7] extended the Hyers-Ulam stability by considering variables. It also has been generalized to the function case by Găvruta [2].

Throughout this paper, let \( X \) and \( Y \) be vector spaces.

A mapping \( g : X \to Y \) is called a Jensen mapping if \( g \) satisfies the functional equation

\[ 2g\left(\frac{x+y}{2}\right) = g(x) + g(y). \]

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DEFINITION. A mapping \( f : X \times X \rightarrow Y \) is called a \textit{bi-Jensen mapping} if \( f \) satisfies the system of equations

\[
2f \left( \frac{x + y}{2}, z \right) = f(x, z) + f(y, z),
\]

\[
2f \left( \frac{x}{2}, \frac{y + z}{2} \right) = f(x, y) + f(x, z).
\]

(1.1)

When \( X = Y = \mathbb{R} \), the function \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x, y) := axy + bx + cy + d \) is a solution of (1.1). In particular, letting \( y = x \), we get a function \( g : \mathbb{R} \rightarrow \mathbb{R} \) given by \( g(x) := f(x, x) = ax^2 + (b + c)x + d \).

For a mapping \( f : X \times X \rightarrow Y \), consider the functional equation:

\[
4f \left( \frac{x + y + z}{2} \right) = f(x, z) + f(x, w) + f(y, z) + f(y, w).
\]

(1.2)

For a mapping \( g : X \rightarrow Y \), consider the functional equation:

\[
9g \left( \frac{x + y + z}{3} \right) = g(x, z) + g(y) + g(z)
\]

\[
= 4 \left[ g \left( \frac{x + y}{2} \right) + g \left( \frac{y + z}{2} \right) + g \left( \frac{z + x}{2} \right) \right].
\]

(1.3)

In [5], Y.-W. Lee solved the solution and proved the stability of the equation (1.3). The equation (1.3) generalized by S.-H. Lee [4] and Y.-W. Lee [6].

In this paper, we investigate the relation between (1.2) and (1.3). And we find out the general solution and the generalized Hyers-Ulam stability of (1.1).

2. The relation between (1.2) and (1.3)

THEOREM 1. Let \( g : X \rightarrow Y \) be a mapping satisfying (1.3) and let \( f : X \times X \rightarrow Y \) be the mapping given by

\[
f(x, y) := \frac{1}{2} \left[ 5g(x + y) - g(2x + 2y) - g(x) - g(y) \right]
\]

for all \( x, y \in X \). Then \( f \) satisfies (1.2) and

\[
g(x) = f(x, x)
\]

(2.2)

for all \( x \in X \).
Proof. By Theorem 2.1 in [5], there exist a quadratic mapping \( Q : X \to Y \) and an additive mapping \( A : X \to Y \) such that
\[
(2.3) \quad g(x) = Q(x) + A(x) + g(0)
\]
for all \( x \in X \). Putting \( y = x \) in (2.1) and then using (2.3), the equality (2.2) holds.

By (2.3), we get
\[
2 \left[ 5g \left( \frac{x + y + z + w}{2} \right) - g(x + y + z + w) \right.
\left. - g \left( \frac{x + y}{2} \right) - g \left( \frac{z + w}{2} \right) \right]
\]
\[
= \frac{1}{2} \left[ 5g(x + z) + 5g(x + w) + 5g(y + z) + 5g(y + w)
- g(2x + 2z) - g(2x + 2w) - g(2y + 2z) - g(2y + 2w) \right]
\]
\[
(2.4) \quad -g(x) - g(y) - g(z) - g(w)
\]
for all \( x, y, z, w \in X \). By (2.1) and the above equality, \( f \) satisfies (1.2).

\[ \square \]

**Theorem 2.** Let \( f : X \times X \to Y \) be a mapping satisfying (1.2) and \( g : X \to Y \) the mapping given by (2.2). If \( f \) satisfies (2.1), then \( g \) satisfies (1.3).

**Proof.** By (1.2) and (2.1), \( g \) satisfies (2.4). Setting \( y = x \) in (2.1) and using (2.2), we have
\[
(2.5) \quad g(4x) = 4g(x) - 5g(2x)
\]
for all \( x \in X \). Taking \( w = 0 \) in (2.4), we obtain that
\[
2 \left[ 5g \left( \frac{x + y + z}{2} \right) - g(x + y + z) - g \left( \frac{x + y}{2} \right) - g \left( \frac{z}{2} \right) \right]
\]
\[
= \frac{1}{2} \left[ 5g(x + z) + 5g(x) + 5g(y + z) + 5g(y) - g(2x + 2z) - g(2x) \right.
\left. - g(2y + 2z) - g(2y) \right] - g(x) - g(y) - g(z) - g(0)
\]
\[
(2.6) \quad -g(2y + 2z) - g(2y) \right] - g(x) - g(y) - g(z) - g(0)
\]
for all \( x, y, z \in X \). Letting \( y = z = 0 \) and replacing \( x \) by \( 2x \) in (2.6), we get
\[
g(4x) = 8g(x) - 6g(2x) - 3g(0)
\]
for all \( x \in X \). By (2.5) and the above equation, we have
\[
(2.7) \quad 4g(x) = 2g(2x) + 3g(0)
\]
for all \( x \in X \). Putting \( y = z = x \) in (2.6), we see that
\[
g(4x) - 2g(3x) - 4g(2x) + 10g\left(\frac{3x}{2}\right) - 4g(x) - 2g\left(\frac{x}{2}\right) + g(0) = 0
\]
for all \( x \in X \). Replacing \( x \) by \( 2x \) in the above equation, we get
\[
g(8x) - 2g(6x) - 4g(4x) + 10g(3x) - 4g(2x) - 2g(x) + g(0) = 0
\]
for all \( x \in X \). By the above equation and using (2.7), we have
\[
(2.8) \quad 9g(x) = g(3x) + 8g(0)
\]
for all \( x \in X \). By (2.7) and (2.8), we get
\[
(2.9) \quad 16g(3x) = 9g(4x) - 7g(0)
\]
for all \( x \in X \).

By (2.5) and (2.6), we obtain that
\[
8g\left(\frac{x + y + z}{4}\right) + g(x) + g(y) + g(z) + g(0)
\]
\[
= \left[2g\left(\frac{x + y}{2}\right) + g\left(\frac{y + z}{2}\right) + g\left(\frac{z + x}{2}\right)\right] + g\left(\frac{x}{2}\right) + g\left(\frac{y}{2}\right) + g\left(\frac{z}{2}\right)
\]
for all \( x, y, z \in X \). By (2.7), (2.9) and the last equation, we see that \( g \) satisfies the equation (1.3). \( \square \)

3. Solutions of (1.1) and (1.2)

THEOREM 3. A mapping \( f : X \times X \rightarrow Y \) satisfies (1.1) if and only if there exist a bi-additive mapping \( B : X \times X \rightarrow Y \) and two additive mappings \( A, A' : X \rightarrow Y \) such that \( f(x, y) = B(x, y) + A(x) + A'(y) + f(0, 0) \) for all \( x, y \in X \).

Proof. We first assume that \( f \) is a solution of (1.1). Define \( g_x, g'_y : X \rightarrow Y \) by \( g_x(y) = g'_y(x) := f(x, y) \) for all \( x, y \in X \). Then \( g_x, g'_y \) are Jensen mappings for all \( x, y \in X \). By [1], there exists additive mappings \( A_x, A'_y : X \rightarrow Y \) such that \( g_x(y) = A_x(y) + g(x) \) and \( g'_y(x) = A'_y(x) + g'_y(0) \) for all \( x, y \in X \). Define \( A, A' : X \rightarrow Y \) by
\[
A(x) := f(x, 0) - f(0, 0) \quad \text{and} \quad A'(y) := f(0, y) - f(0, 0)
\]
for all \(x, y \in X\). Then \(A(x) = g'_0(x) - g'_0(0) = A'_0(x)\) and \(A'(y) = g_0(y) - g_0(0) = A_0(y)\) for all \(x, y \in X\). So \(A\) and \(A'\) are additive. Define \(B : X \times X \to Y\) by

\[
B(x, y) := f(x, y) - f(x, 0) - f(0, y) + f(0, 0)
\]

for all \(x, y \in X\).

Note that

\[
\begin{align*}
B(x + y, z) &= f(x + y, z) - A(x + y) - A'(z) - f(0, 0) \\
&= g'_z(x + y) - A(x) - A(y) - g_0(z) \\
&= A'_z(x + y) + g'_z(0) - A(x) - A(y) - g_0(z) \\
&= A'_z(x) + A'_z(y) - A(x) - A(y) - f(x, z) - f(0, 0) + f(y, z) - A'(z) - f(0, 0) \\
&= B(x, z) + B(y, z)
\end{align*}
\]

for all \(x, y, z \in X\). By the same method as above, one can obtain that

\[
B(x, y + z) = B(x, y) + B(x, z)
\]

for all \(x, y, z \in X\). Hence \(B\) is bi-additive.

Conversely, we assume that there exist a bi-additive mapping \(B : X \times X \to Y\) and additive mappings \(A, A' : X \to Y\) such that \(f(x, y) = B(x, y) + A(x) + A'(y) + f(0, 0)\) for all \(x, y \in X\). Since \(B\) is additive in the first variable, \(2B \left( \frac{x}{2}, y \right) = B(x, y)\) and so

\[
\begin{align*}
2f \left( \frac{x + y}{2}, z \right) &= 2B \left( \frac{x + y}{2}, z \right) + 2A \left( \frac{x + y}{2} \right) + 2A'(z) + 2f(0, 0) \\
&= B(x, z) + B(y, z) + A(x) + A(y) + 2A'(z) + 2f(0, 0) \\
&= f(x, z) + f(y, z)
\end{align*}
\]

for all \(x, y, z \in X\). Similarly, we get

\[
2f \left( \frac{x + y}{2}, z \right) = f(x, y) + f(x, z)
\]

for all \(x, y, z \in X\).

\(\square\)

**Theorem 4.** A mapping \(f : X \times X \to Y\) satisfies (1.1) if and only if it satisfies (1.2).
Proof. If \( f \) satisfies (1.1), then
\[
4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = 2f\left(\frac{x}{2}, \frac{z+w}{2}\right) + 2f\left(\frac{y}{2}, \frac{z+w}{2}\right)
\]
\[
= f(x, z) + f(x, w) + f(y, z) + f(y, w)
\]
for all \( x, y, z, w \in X \).

Conversely, assume that \( f \) satisfies (1.2). Putting \( w = z \) in (1.2), we have
\[
2f\left(\frac{x+y}{2}, z\right) = f(x, z) + f(y, z)
\]
for all \( x, y, z \in X \). Similarly, we get
\[
2f\left(\frac{x}{2}, \frac{y+z}{2}\right) = f(x, y) + f(x, z)
\]
for all \( x, y, z \in X \).

\[\square\]

Corollary 5. A function \( g : X \to Y \) satisfies (1.3) if and only if there exists a symmetric bi-additive function \( S : X \times X \to Y \) and an additive mapping \( A : X \to Y \) such that \( g(x) = S(x, x) + A(x) + g(0) \) for all \( x \in X \).

Proof. Define \( f : X \times X \to Y \) by (2.1) for all \( x, y \in X \). By Theorem 1, \( f \) satisfies (1.2) and (2.2). Using Theorem 4, \( f \) also satisfies (1.1). By Theorem 3, there exist a bi-additive mapping \( B : X \times X \to Y \) and two additive mappings \( A_0, A'_0 : X \to Y \) such that
\[
f(x, y) = B(x, y) + A_0(x) + A'_0(y) + f(0, 0)
\]
for all \( x, y \in X \). By (2.2), we have
\[
g(x) = B(x, x) + A_0(x) + A'_0(x) + g(0)
\]
for all \( x \in X \). Define \( S : X \times X \to Y \) and \( A : X \to Y \) by
\[
S(x, y) := \frac{1}{2}[B(x, y) + B(y, x)] \quad \text{and} \quad A(x) := A_0(x) + A'_0(x)
\]
for all \( x, y \in X \). Then \( S \) is symmetric bi-additive, \( A \) is additive and
\[
g(x) = S(x, x) + A(x) + g(0)
\]
for all \( x \in X \).

The converse is obviously true. \[\square\]
4. Stability of (1.1)

Let $Y$ be complete and let $\varphi : X \times X \times X \to [0, \infty)$ and $\psi : X \times X \times X \to [0, \infty)$ be two functions such that

\begin{equation}
\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} \left[ \varphi(3^j x, 3^j y, z) + \varphi(x, y, 3^j z) \right] < \infty
\end{equation}

and

\begin{equation}
\tilde{\psi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} \left[ \psi(x, 3^j y, 3^j z) + \psi(3^j x, y, z) \right] < \infty
\end{equation}

for all $x, y, z \in X$.

**Theorem 6.** Let $f : X \times X \to Y$ be a mapping such that

\begin{equation}
\left\| 2f \left( \frac{x+y}{2}, z \right) - f(x, z) - f(y, z) \right\| \leq \varphi(x, y, z)
\end{equation}

\begin{equation}
\left\| 2f \left( x, \frac{y+z}{2} \right) - f(x, y) - f(x, z) \right\| \leq \psi(x, y, z)
\end{equation}

for all $x, y, z \in X$. Then there exist two bi-Jensen mappings $F, F' : X \times X \to Y$ such that

\begin{equation}
\left\| f(x, y) - f(0, y) - F(x, y) \right\| \leq \tilde{\varphi}(x, -x, y) + \tilde{\varphi}(-x, 3x, y),
\end{equation}

\begin{equation}
\left\| f(x, y) - f(x, 0) - F'(x, y) \right\| \leq \tilde{\psi}(x, y, -y) + \tilde{\psi}(x, -y, 3y)
\end{equation}

for all $x, y \in X$. The mappings $F, F' : X \times X \to Y$ are given by

\[ F(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(3^j x, y) \quad \text{and} \quad F'(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(x, 3^j y) \]

for all $x, y \in X$.

**Proof.** Letting $y = -x$ in (4.3) and replacing $x$ by $-x$ and $y$ by $3x$ in (4.3), one can obtain that

\[ \left\| 2f(0, z) - f(x, z) - f(-x, z) \right\| \leq \varphi(x, -x, z), \]

\[ \left\| 2f(x, z) - f(-x, z) - f(3x, z) \right\| \leq \varphi(-x, 3x, z), \]

respectively, for all $x, z \in X$. By the above two inequalities and replacing $z$ by $y$, we get

\[ \left\| 3f(x, y) - 2f(0, y) - f(3x, y) \right\| \leq \varphi(x, -x, y) + \varphi(-x, 3x, y) \]
for all \( x, y \in X \). Thus we have

\[
\left\| \frac{1}{3^j} f(3^j x, y) - \frac{2}{3^{j+1}} f(0, y) - \frac{1}{3^{j+1}} f(3^{j+1} x, y) \right\|
\leq \frac{1}{3^{j+1}} \left[ \varphi(3^j x, -3^j x, y) + \varphi(-3^j x, 3^{j+1} x, y) \right]
\]

for all \( x, y \in X \) and all \( j \). For given integers \( l, m (0 \leq l < m) \), we obtain that

\[
(4.7) \quad \left\| \frac{1}{3^j} f(3^j x, y) - \sum_{j=l}^{m-1} \frac{2}{3^{j+1}} f(0, y) - \frac{1}{3^{m}} f(3^m x, y) \right\|
\leq \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[ \varphi(3^j x, -3^j x, y) + \varphi(-3^j x, 3^{j+1} x, y) \right]
\]

for all \( x, y \in X \). By (4.1), the sequence \( \{ \frac{1}{3^j} f(3^j x, y) \} \) is a Cauchy sequence for all \( x, y \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{3^j} f(3^j x, y) \} \) converges for all \( x, y \in X \). Define \( F : X \times X \to Y \) by

\[
F(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(3^j x, y)
\]

for all \( x, y \in X \). Putting \( l = 0 \) and taking \( m \to \infty \) in (4.7), one can obtain the inequality (4.5). By (4.3), we get

\[
\left\| \frac{2}{3^j} f \left( \frac{3^j (x + y)}{2}, y \right) - \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(3^j y, z) \right\|
\leq \frac{1}{3^j} \varphi(3^j x, 3^j y, y)
\]

for all \( x, y, z \in X \) and all \( j \). By (4.4), we have

\[
\left\| \frac{2}{3^j} f \left( 3^j x, \frac{y + z}{2} \right) + \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(3^j x, z) \right\| \leq \frac{1}{3^j} \psi(3^j x, y, z)
\]

for all \( x, y, z \in X \) and all \( j \). Letting \( j \to \infty \) in the above two inequalities and using (4.1) and (4.2), \( F \) is a bi-Jensen mapping.

Define \( F' : X \times X \to Y \) by \( F'(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(x, 3^j y) \) for all \( x, y \in X \). By the same method in the above argument, \( F' \) is a bi-Jensen mapping satisfying (4.6). \( \square \)
References


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