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ON THE SOLUTION OF A BI-JENSEN FUNCTIONAL EQUATION AND ITS STABILITY

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ABSTRACT. In this paper, we obtain the general solution and the stability of the bi-Jensen functional equation

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w).$$

1. Introduction

In 1940, Ulam proposed the general Ulam stability problem (see [8]): *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d\big(h(xy), h(x)h(y)\big) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d\big(h(x), H(x)\big) < \varepsilon$ for all $x \in G_1$?*

In 1941, this problem was solved by Hyers [3] in the case of Banach space. Thereafter, we call that type the Hyers-Ulam stability. In 1978, Th. M. Rassias [7] extended the Hyers-Ulam stability by considering variables. It also has been generalized to the function case by Găvruta [2].

Throughout this paper, let X and Y be vector spaces.

A mapping $g : X \to Y$ is called a *Jensen mapping* if g satisfies the functional equation

$$2g\left(\frac{x+y}{2}\right) = g(x) + g(y).$$

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DEFINITION. A mapping $f : X \times X \to Y$ is called a *bi-Jensen mapping* if f satisfies the system of equations

$$\begin{aligned} 2f\left(\frac{x+y}{2}, z\right) &= f(x,z) + f(y,z), \\ 2f\left(x, \frac{y+z}{2}\right) &= f(x,y) + f(x,z). \end{aligned} \tag{1.1}$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy + bx + cy + d$ is a solution of (1.1). In particular, letting $y = x$, we get a function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) := f(x,x) = ax^2 + (b+c)x + d$.

For a mapping $f : X \times X \to Y$, consider the functional equation:

$$4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w). \tag{1.2}$$

For a mapping $g : X \to Y$, consider the functional equation:

$$\begin{aligned} &9g\left(\frac{x+y+z}{3}\right) + g(x) + g(y) + g(z) \\ &= 4\left[g\left(\frac{x+y}{2}\right) + g\left(\frac{y+z}{2}\right) + g\left(\frac{z+x}{2}\right)\right]. \end{aligned} \tag{1.3}$$

In [5], Y.-W. Lee solved the solution and proved the stability of the equation (1.3). The equation (1.3) generalized by S.-H. Lee [4] and Y.-W. Lee [6].

In this paper, we investigate the relation between (1.2) and (1.3). And we find out the general solution and the generalized Hyers-Ulam stability of (1.1).

2. The relation between (1.2) and (1.3)

THEOREM 1. *Let $g : X \to Y$ be a mapping satisfying (1.3) and let $f : X \times X \to Y$ be the mapping given by*

$$f(x,y) := \frac{1}{2}[5g(x+y) - g(2x+2y) - g(x) - g(y)] \tag{2.1}$$

for all $x, y \in X$. Then f satisfies (1.2) and

$$g(x) = f(x,x) \tag{2.2}$$

for all $x \in X$.

Proof. By Theorem 2.1 in [5], there exist a quadratic mapping $Q : X \to Y$ and an additive mapping $A : X \to Y$ such that

$$g(x) = Q(x) + A(x) + g(0) \tag{2.3}$$

for all $x \in X$. Putting $y = x$ in (2.1) and then using (2.3), the equality (2.2) holds.

By (2.3), we get

$$\begin{aligned} & 2\left[5g\left(\frac{x+y+z+w}{2}\right) - g(x+y+z+w)\right. \\ & \left.-g\left(\frac{x+y}{2}\right) - g\left(\frac{z+w}{2}\right)\right] \\ = \; & \frac{1}{2}\big[5g(x+z) + 5g(x+w) + 5g(y+z) + 5g(y+w) \\ & -g(2x+2z) - g(2x+2w) - g(2y+2z) - g(2y+2w)\big] \\ & -g(x) - g(y) - g(z) - g(w) \end{aligned} \tag{2.4}$$

for all $x, y, z, w \in X$. By (2.1) and the above equality, f satisfies (1.2). □

THEOREM 2. *Let $f : X \times X \to Y$ be a mapping satisfying* (1.2) *and $g : X \to Y$ the mapping given by* (2.2). *If f satisfies* (2.1), *then g satisfies* (1.3).

Proof. By (1.2) and (2.1), g satisfies (2.4). Setting $y = x$ in (2.1) and using (2.2), we have

$$g(4x) = 4g(x) - 5g(2x) \tag{2.5}$$

for all $x \in X$. Taking $w = 0$ in (2.4), we obtain that

$$\begin{aligned} & 2\left[5g\left(\frac{x+y+z}{2}\right) - g(x+y+z) - g\left(\frac{x+y}{2}\right) - g\left(\frac{z}{2}\right)\right] \\ = \; & \frac{1}{2}\big[5g(x+z) + 5g(x) + 5g(y+z) + 5g(y) - g(2x+2z) - g(2x) \\ & -g(2y+2z) - g(2y)\big] - g(x) - g(y) - g(z) - g(0) \end{aligned} \tag{2.6}$$

for all $x, y, z \in X$. Letting $y = z = 0$ and replacing x by $2x$ in (2.6), we get

$$g(4x) = 8g(x) - 6g(2x) - 3g(0)$$

for all $x \in X$. By (2.5) and the above equation, we have

$$4g(x) = g(2x) + 3g(0) \tag{2.7}$$

for all $x \in X$. Putting $y = z = x$ in (2.6), we see that

$$g(4x) - 2g(3x) - 4g(2x) + 10g\left(\frac{3x}{2}\right) - 4g(x) - 2g\left(\frac{x}{2}\right) + g(0) = 0$$

for all $x \in X$. Replacing x by $2x$ in the above equation, we get

$$g(8x) - 2g(6x) - 4g(4x) + 10g(3x) - 4g(2x) - 2g(x) + g(0) = 0$$

for all $x \in X$. By the above equation and using (2.7), we have

$$9g(x) = g(3x) + 8g(0) \tag{2.8}$$

for all $x \in X$. By (2.7) and (2.8), we get

$$16g(3x) = 9g(4x) - 7g(0) \tag{2.9}$$

for all $x \in X$.

By (2.5) and (2.6), we obtain that

$$\begin{aligned} & 8g\left(\frac{x+y+z}{4}\right) + g(x) + g(y) + g(z) + g(0) \\ = \; & 2\left[g\left(\frac{x+y}{2}\right) + g\left(\frac{y+z}{2}\right) + g\left(\frac{z+x}{2}\right)\right. \\ & \left. + g\left(\frac{x}{2}\right) + g\left(\frac{y}{2}\right) + g\left(\frac{z}{2}\right)\right] \end{aligned}$$

for all $x, y, z \in X$. By (2.7), (2.9) and the last equation, we see that g satisfies the equation (1.3). □

3. Solutions of (1.1) and (1.2)

THEOREM 3. *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if there exist a bi-additive mapping $B : X \times X \to Y$ and two additive mappings $A, A' : X \to Y$ such that $f(x,y) = B(x,y) + A(x) + A'(y) + f(0,0)$ for all $x, y \in X$.*

Proof. We first assume that f is a solution of (1.1). Define $g_x, g'_y : X \to Y$ by $g_x(y) = g'_y(x) := f(x,y)$ for all $x, y \in X$. Then g_x, g'_y are Jensen mappings for all $x, y \in X$. By [1], there exists additive mappings $A_x, A'_y : X \to Y$ such that $g_x(y) = A_x(y) + g_x(0)$ and $g'_y(x) = A'_y(x) + g'_y(0)$ for all $x, y \in X$. Define $A, A' : X \to Y$ by

$$A(x) := f(x,0) - f(0,0) \quad \text{and} \quad A'(y) := f(0,y) - f(0,0)$$

for all $x, y \in X$. Then $A(x) = g_0'(x) - g_0'(0) = A_0'(x)$ and $A'(y) = g_0(y) - g_0(0) = A_0(y)$ for all $x, y \in X$. So A and A' are additive. Define $B : X \times X \to Y$ by

$$B(x,y) := f(x,y) - f(x,0) - f(0,y) + f(0,0)$$

for all $x, y \in X$.

Note that

$$\begin{aligned}
B(x+y,z) &= f(x+y,z) - A(x+y) - A'(z) - f(0,0) \\
&= g_z'(x+y) - A(x) - A(y) - g_0(z) \\
&= A_z'(x+y) + g_z'(0) - A(x) - A(y) - g_0(z) \\
&= A_z'(x) + A_z'(y) - A(x) - A(y) \\
&= f(x,z) - f(0,z) + f(y,z) - f(0,z) - A(x) - A(y) \\
&= f(x,z) - A'(z) - f(0,0) + f(y,z) - A'(z) - f(0,0) \\
&\quad -A(x) - A(y) \\
&= B(x,z) + B(y,z)
\end{aligned}$$

for all $x, y, z \in X$. By the same method as above, one can obtain that

$$B(x, y+z) = B(x,y) + B(x,z)$$

for all $x, y, z \in X$. Hence B is bi-additive.

Conversely, we assume that there exist a bi-additive mapping $B : X \times X \to Y$ and additive mappings $A, A' : X \to Y$ such that $f(x,y) = B(x,y) + A(x) + A'(y) + f(0,0)$ for all $x, y \in X$. Since B is additive in the first variable, $2B\left(\frac{x}{2}, y\right) = B(x,y)$ and so

$$\begin{aligned}
&2f\left(\frac{x+y}{2}, z\right) \\
&= 2B\left(\frac{x+y}{2}, z\right) + 2A\left(\frac{x+y}{2}\right) + 2A'(z) + 2f(0,0) \\
&= B(x,z) + B(y,z) + A(x) + A(y) + 2A'(z) + 2f(0,0) \\
&= f(x,z) + f(y,z)
\end{aligned}$$

for all $x, y, z \in X$. Similarly, we get

$$2f\left(x, \frac{y+z}{2}\right) = f(x,y) + f(x,z)$$

for all $x, y, z \in X$. □

THEOREM 4. *A mapping* $f : X \times X \to Y$ *satisfies* (1.1) *if and only if it satisfies* (1.2).

Proof. If f satisfies (1.1), then

$$\begin{aligned} 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) &= 2f\left(x, \frac{z+w}{2}\right) + 2f\left(y, \frac{z+w}{2}\right) \\ &= f(x,z) + f(x,w) + f(y,z) + f(y,w) \end{aligned}$$

for all $x, y, z, w \in X$.

Conversely, assume that f satisfies (1.2). Putting $w = z$ in (1.2), we have

$$2f\left(\frac{x+y}{2}, z\right) = f(x,z) + f(y,z)$$

for all $x, y, z \in X$. Similarly, we get

$$2f\left(x, \frac{y+z}{2}\right) = f(x,y) + f(x,z)$$

for all $x, y, z \in X$. □

COROLLARY 5. *A function $g : X \to Y$ satisfies* (1.3) *if and only if there exists a symmetric bi-additive function $S : X \times X \to Y$ and an additive mapping $A : X \to Y$ such that $g(x) = S(x,x) + A(x) + g(0)$ for all $x \in X$.*

Proof. Define $f : X \times X \to Y$ by (2.1) for all $x, y \in X$. By Theorem 1, f satisfies (1.2) and (2.2). Using Theorem 4, f also satisfies (1.1). By Theorem 3, there exist a bi-additive mapping $B : X \times X \to Y$ and two additive mappings $A_0, A_0' : X \to Y$ such that

$$f(x,y) = B(x,y) + A_0(x) + A_0'(y) + f(0,0)$$

for all $x, y \in X$. By (2.2), we have

$$g(x) = B(x,x) + A_0(x) + A_0'(x) + g(0) \tag{3.1}$$

for all $x \in X$. Define $S : X \times X \to Y$ and $A : X \to Y$ by

$$S(x,y) := \frac{1}{2}\big[B(x,y) + B(y,x)\big] \quad \text{and} \quad A(x) := A_0(x) + A_0'(x)$$

for all $x, y \in X$. Then S is symmetric bi-additive, A is additive and

$$g(x) = S(x,x) + A(x) + g(0)$$

for all $x \in X$.

The converse is obviously true. □

4. Stability of (1.1)

Let Y be complete and let $\varphi : X \times X \times X \to [0, \infty)$ and $\psi : X \times X \times X \to [0, \infty)$ be two functions such that

$$\tilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} \left[\varphi(3^j x, 3^j y, z) + \varphi(x, y, 3^j z)\right] < \infty \tag{4.1}$$

and

$$\tilde{\psi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} \left[\psi(x, 3^j y, 3^j z) + \psi(3^j x, y, z)\right] < \infty \tag{4.2}$$

for all $x, y, z \in X$.

THEOREM 6. *Let $f : X \times X \to Y$ be a mapping such that*

$$\left\|2f\left(\frac{x+y}{2}, z\right) - f(x, z) - f(y, z)\right\| \leq \varphi(x, y, z) \tag{4.3}$$

$$\left\|2f\left(x, \frac{y+z}{2}\right) - f(x, y) - f(x, z)\right\| \leq \psi(x, y, z) \tag{4.4}$$

for all $x, y, z \in X$. Then there exist two bi-Jensen mappings F, $F' : X \times X \to Y$ such that

$$\|f(x, y) - f(0, y) - F(x, y)\| \leq \tilde{\varphi}(x, -x, y) + \tilde{\varphi}(-x, 3x, y), \tag{4.5}$$

$$\|f(x, y) - f(x, 0) - F'(x, y)\| \leq \tilde{\psi}(x, y, -y) + \tilde{\psi}(x, -y, 3y) \tag{4.6}$$

for all $x, y \in X$. The mappings F, $F' : X \times X \to Y$ are given by

$$F(x, y) := \lim_{j\to\infty} \frac{1}{3^j} f(3^j x, y) \quad \text{and} \quad F'(x, y) := \lim_{j\to\infty} \frac{1}{3^j} f(x, 3^j y)$$

for all $x, y \in X$.

Proof. Letting $y = -x$ in (4.3) and replacing x by $-x$ and y by $3x$ in (4.3), one can obtain that

$$\|2f(0, z) - f(x, z) - f(-x, z)\| \leq \varphi(x, -x, z),$$

$$\|2f(x, z) - f(-x, z) - f(3x, z)\| \leq \varphi(-x, 3x, z),$$

respectively, for all $x, z \in X$. By the above two inequalities and replacing z by y, we get

$$\|3f(x, y) - 2f(0, y) - f(3x, y)\| \leq \varphi(x, -x, y) + \varphi(-x, 3x, y)$$

for all $x, y \in X$. Thus we have

$$\begin{aligned} &\left\| \frac{1}{3^j} f(3^j x, y) - \frac{2}{3^{j+1}} f(0, y) - \frac{1}{3^{j+1}} f(3^{j+1} x, y) \right\| \\ \leq \;& \frac{1}{3^{j+1}} [\varphi(3^j x, -3^j x, y) + \varphi(-3^j x, 3^{j+1} x, y)] \end{aligned}$$

for all $x, y \in X$ and all j. For given integers $l, m (0 \leq l < m)$, we obtain that

$$\begin{aligned} &\left\| \frac{1}{3^l} f(3^l x, y) - \sum_{j=l}^{m-1} \frac{2}{3^{j+1}} f(0, y) - \frac{1}{3^m} f(3^m x, y) \right\| \\ \leq \;& \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[\varphi(3^j x, -3^j x, y) + \varphi(-3^j x, 3^{j+1} x, y)\right] \end{aligned} \tag{4.7}$$

for all $x, y \in X$. By (4.1), the sequence $\{\frac{1}{3^j} f(3^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{3^j} f(3^j x, y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(3^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (4.7), one can obtain the inequality (4.5). By (4.3), we get

$$\begin{aligned} &\left\| \frac{2}{3^j} f\left(\frac{3^j(x+y)}{2}, y\right) - \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(3^j y, z) \right\| \\ \leq \;& \frac{1}{3^j} \varphi(3^j x, 3^j y, y) \end{aligned}$$

for all $x, y, z \in X$ and all j. By (4.4), we have

$$\left\| \frac{2}{3^j} f\left(3^j x, \frac{y+z}{2}\right) + \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(3^j x, z) \right\| \leq \frac{1}{3^j} \psi(3^j x, y, z)$$

for all $x, y, z \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using (4.1) and (4.2), F is a bi-Jensen mapping.

Define $F' : X \times X \to Y$ by $F'(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(x, 3^j y)$ for all $x, y \in X$. By the same method in the above argument, F' is a bi-Jensen mapping satisfying (4.6). □

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