OPERATORS WITH THE SINGLE VALUED EXTENSION PROPERTY

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ABSTRACT. In this paper we study some operators with the single valued extension property. In particular, we investigate the Helton class of an operator and an $n \times n$ triangular operator matrix T.

Let H be a complex (separable) Hilbert space and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H. If $T \in \mathcal{L}(H)$, we write $\sigma(T)$ for the spectrum of T. An operator $T \in \mathcal{L}(H)$ is said to have the *single valued extension property* if for any analytic function $f : D \to H$, $D \subset \mathbf{C}$ open, with $(\lambda - T)f(\lambda) \equiv 0$, it results $f(\lambda) \equiv 0$. For an operator $T \in \mathcal{L}(H)$ having the single valued extension property and for $x \in H$ we can consider the set $\rho_T(x)$ of elements $\lambda_0 \in \mathbf{C}$ such that there exists an analytic function $f(\lambda)$ defined in a neighborhood of λ_0, with values in H, which verifies $(\lambda - T)f(\lambda) \equiv x$. Throughout this paper, we denote $\sigma_T(x) = \mathbf{C}\backslash\rho_T(x)$ and $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$, where $F \subset \mathbf{C}$.

In [2] J. W. Helton initiated the study of operators T which satisfy an identity of the form

$$(1) \qquad T^{*m} - \binom{m}{1} T^{*m-1}T + \cdots + (-1)^m T^m = 0.$$

We need further study for this class of operators based on (1). Let R and S be in $\mathcal{L}(H)$ and let $C(R,S) : \mathcal{L}(H) \to \mathcal{L}(H)$ be defined by $C(R,S)(A) = RA - AS$. Then

$$(2) \qquad C(R,S)^k(I) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^j S^{k-j}.$$

DEFINITION 1. Let $R \in \mathcal{L}(H)$. If there is an integer $k \geq 1$ such that an operator S satisfies $C(R, S)^k(I) = 0$, we say that S belongs to Helton class of R with order k. We denote this by $S \in \text{Helton}_k(R)$.

Next we study the Helton class of an operator which has the single valued extension property.

THEOREM 2. Let $R \in \mathcal{L}(H)$ have the single valued extension property. If $S \in \text{Helton}_k(R)$, then S has the single valued extension property.

Proof. Let $f : D \to H$ be an analytic function such that $(\lambda - S)f(\lambda) \equiv 0$. Since the terms of the below equation are equal to zero when $j + s \neq r$, it suffices to consider only the case of $j + s = r$. Then we have the following equations:

$$\sum_{j=0}^{k} \binom{k}{j} (R - \lambda)^j (\lambda - S)^{k-j}$$

$$= \sum_{j=0}^{k} \sum_{r=0}^{j} \sum_{s=0}^{k-j} (-1)^{k-(s+r)} \binom{k}{j} \binom{j}{r} \binom{k-j}{s} R^r \lambda^{j+s-r} S^{k-(j+s)}$$

$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^j S^{k-j}.$$

Hence we have

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^j S^{k-j} f(\lambda) - (R - \lambda)^k f(\lambda)$$

$$= \sum_{j=0}^{k} \binom{k}{j} (R - \lambda)^j (\lambda - S)^{k-j} f(\lambda) - (R - \lambda)^k f(\lambda)$$

$$= \sum_{j=0}^{k-1} \binom{k}{j} (R - \lambda)^j (\lambda - S)^{k-j} f(\lambda)$$

$$= [\sum_{j=0}^{k-1} \binom{k}{j} (R - \lambda)^j (\lambda - S)^{k-j-1}](\lambda - S)f(\lambda) = 0.$$

Since $\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} R^j S^{k-j} = 0$, we get that $(R - \lambda)^k f(\lambda) = 0$. Since R has the single valued extension property, $(R - \lambda)^{k-1} f(\lambda) = 0$. By induction, we have $f(\lambda) \equiv 0$. So we conclude that S has the single valued extension property. $\qquad \square$

If S_1 and S_2 have the single valued extension property, does $S_1 + S_2$ have the single valued extension property? So far we do not know the answer about this question. But we consider the special cases of this question. First, we start with the case of Helton class.

COROLLARY 3. *If S_1 in $\mathcal{L}(H)$ has the single valued extension property, $S_2 \in \mathrm{Helton}_k(S_1)$, and $S_1 S_2 = S_2 S_1$, then $S = S_1 + S_2$ has the single valued extension property.*

Proof. It is easy to calculate that $C(2S_1, S)^k(I) = C(S_1, S_2)^k(I) = 0$. Hence $S = S_1 + S_2 \in \mathrm{Helton}_k(2S_1)$. Since $2S_1$ has the single valued extension property, by Theorem 2 an operator S has the single valued extension property. $\qquad\square$

Next we consider the case of nilpotent perturbation.

THEOREM 4. *Let $T = S + N$ be in $\mathcal{L}(H)$, where $SN = NS$ and $N^k = 0$. Then S has the single valued extension property if and only if T has.*

Proof. ($\Rightarrow$) Assume that S has the single valued extension property. Let $f : D \to H$ be an analytic function such that $(\lambda - T)f(\lambda) \equiv 0$, where D is an open set in $\mathbf{C}$. Then $(\lambda - S)f(\lambda) = Nf(\lambda)$. Since $N^k = 0$ and $SN = NS$, $(\lambda - S)N^{k-1}f(\lambda) = N^k f(\lambda) = 0$. Since S has the single valued extension property, $N^{k-1}f(\lambda) = 0$. Since $(\lambda - S)N^{k-2}f(\lambda) = N^{k-1}f(\lambda) = 0$ and S has the single valued extension property, $N^{k-2}f(\lambda) = 0$. By induction, we can show that $f(\lambda) \equiv 0$. Hence T has the single valued extension property.

($\Leftarrow$) The converse implication is similar. $\qquad\square$

Recall that if $T = S + N$ is in $\mathcal{L}(H)$, where S is similar to a hyponormal operator, $SN = NS$, and $N^k = 0$, then T is called a hypo-Jordan operator of order k. Since every hyponormal operator has the single valued extension property, from Theorem 4 we get the following corollary.

COROLLARY 5. *Every hypo-Jordan operator of order k has the single valued extension property.*

Next we consider another special case of the above question.

LEMMA 6. *Let S_1 and S_2 in $\mathcal{L}(H)$ have the single valued extension property. If $T = S_1 + S_2$, where $S_2 S_1 = 0$, then T has the single valued extension property.*

Proof. If $f : D \to H$ is an analytic function such that $(\lambda - T)f(\lambda) \equiv 0$, where D is an open set in $\mathbf{C}$, then $(\lambda - S_1 - S_2)f(\lambda) = 0$. Since $S_2 S_1 = 0$, we get $(\lambda - S_2)S_2 f(\lambda) = 0$. Since S_2 has the single valued

extension property, $S_2 f(\lambda) = 0$. Since $(\lambda - S_1 - S_2)f(\lambda) = 0$, we have $(\lambda - S_1)f(\lambda) = 0$. Since S_1 has the single valued extension property, $f(\lambda) \equiv 0$. Thus T has the single valued extension property. $\qquad\square$

THEOREM 7. *Let $T \in \mathcal{L}(H \oplus H)$ be the following operator matrix*

$$T = \begin{pmatrix} R & S \\ 0 & N \end{pmatrix},$$

where $N^k = 0$ for some nonnegative integer k. Then R has the single valued extension property if and only if T has.

Proof. ($\Rightarrow$) Assume that R has the single valued extension property. Set $S_1 = R \oplus 0$ and $S_2 = \begin{pmatrix} 0 & S \\ 0 & N \end{pmatrix}$. Then $T = S_1 + S_2$ and $S_2 S_1 = 0$. Since R has the single valued extension property, so has S_1. If $f = f_1 \oplus f_2 : D \to H \oplus H$ is an analytic function such that $(\lambda - S_2)f(\lambda) \equiv 0$, where D is an open set in $\mathbf{C}$, then we have

$$(3) \qquad \begin{cases} \lambda f_1(\lambda) - S f_2(\lambda) = 0 \\ (\lambda - N)f_2(\lambda) = 0. \end{cases}$$

Since $N^k = 0$, $\lambda N^{k-1} f_2(\lambda) = N^k f_2(\lambda) = 0$. Since $N^{k-1}f_2(\lambda) = 0$, if $\lambda \neq 0$ and $N^{k-1}f_2(\lambda)$ is analytic, $N^{k-1}f_2(\lambda) \equiv 0$ from the behavior of an analytic function. Since $(\lambda - N)f_2(\lambda) = 0$ from (3), $\lambda N^{k-2}f_2(\lambda) = N^{k-1}f_2(\lambda) \equiv 0$. By the same reason, $N^{k-2}f_2(\lambda) \equiv 0$. Hence by induction we can show that $f_2(\lambda) = 0$. Hence $\lambda f_1(\lambda) = 0$ from (3). Also we can apply the behavior of an analytic function. Hence we get $f_1(\lambda) = 0$. Thus S_2 has the single valued extension property. By Lemma 6, T has the single valued extension property.

($\Leftarrow$) Assume that T has the single valued extension property. If $f : D \to H$ is an analytic function such that $(\lambda - R)f(\lambda) \equiv 0$, where D is an open set in $\mathbf{C}$, then $(\lambda - T)(f(\lambda) \oplus 0) = (\lambda - R)f(\lambda) \oplus 0 = 0$. Since T has the single valued extension property, $f(\lambda) \equiv 0$. So we complete the proof. $\qquad\square$

Next we investigate an $n \times n$ triangular operator matrix.

LEMMA 8. *Let $T \in \mathcal{L}(\bigoplus_{k=1}^{n} H)$ be the following operator matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & \cdots & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & \cdots & \cdots & T_{2n} \\ 0 & 0 & T_{33} & \cdots & \cdots & T_{3n} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & T_{nn} \end{pmatrix}.$$

Assume that T has the single valued extension property. If $T_{11}T_{ij} = T_{ij}T_{jj}$ for $i = 1, \ldots, j$ and $j = 1, \ldots, n$, then T_{jj} has the single valued extension property for $j = 1, \ldots, n$.

Proof. Assume that $f_j : D \to H$ is an analytic function such that $(\lambda - T_{jj})f_j(\lambda) \equiv 0$ for $j = 1, \ldots, n$, where D is an open set in **C**. Then for $j = 1, \ldots, n$

$$
\begin{aligned}
(\lambda - T)(T_{ij}f_j(\lambda) \oplus \cdots \oplus 0) &= ((\lambda - T_{11})T_{ij}f_j(\lambda)) \oplus 0 \cdots \oplus 0 \\
&= (T_{ij}(\lambda - T_{jj})f_j(\lambda)) \oplus 0 \cdots \oplus 0 \\
&\equiv 0,
\end{aligned}
$$

where $i = 1, \ldots, j$. Since T has the single valued extension property, for $j = 1, \ldots, n$ we get that $T_{ij}f_j(\lambda) \equiv 0$ for $i = 1, 2, \ldots, j$. Hence for $j = 1$

$$
\begin{aligned}
(\lambda - T)(f_1(\lambda) \oplus \cdots \oplus 0) &= ((\lambda - T_{11})f_1(\lambda)) \oplus 0 \cdots \oplus 0 \\
&\equiv 0
\end{aligned}
$$

and for $j = 2, \ldots, n$

$$
\begin{aligned}
&(\lambda - T)(0 \oplus \cdots \oplus f_j(\lambda) \oplus 0 \cdots \oplus 0) \\
=~ &(-T_{1j}f_j(\lambda)) \oplus \cdots \oplus (-T_{j-1,j}f_j(\lambda)) \oplus ((\lambda - T_{jj})f_j(\lambda)) \\
&\oplus 0 \oplus \cdots \oplus 0 \\
\equiv~ &0.
\end{aligned}
$$

Since T has the single valued extension property, we conclude that $f_j(\lambda) \equiv 0$ for $j = 1, \ldots, n$. Hence T_{jj} has the single valued extension property. $\square$

We remark that if we set $S = T_{11} \oplus \cdots \oplus T_{nn}$ and $N = T - S$ in Lemma 8, then $T = S + N$, where $N^n = 0$. But we observe that $SN \neq NS$.

LEMMA 9. *Let $T \in \mathcal{L}(\bigoplus_{k=1}^{n} H)$ be the following operator matrix*

$$
T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & \cdots & \cdots & T_{1n} \\
0 & T_{22} & \cdots & \cdots & \cdots & T_{2n} \\
0 & 0 & T_{33} & \cdots & \cdots & T_{3n} \\
0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & T_{nn}
\end{pmatrix}.
$$

If T_{jj} has the single valued extension property for $j = 1, 2 \ldots, n$, then T has the single valued extension property.

Proof. Let $f = \bigoplus_{k=1}^{n} f_k$ be an analytic $\bigoplus_{k=1}^{n} H$-valued function defined on an open set D, where $f_k : D \to H$ are analytic functions for $k = 1, 2, \ldots, n$. If $(\lambda - T)f(\lambda) \equiv 0$, then we have

$$
0 \equiv (\lambda - T)f(\lambda)
$$

$$= \begin{pmatrix} (\lambda - T_{11})f_1(\lambda) - T_{12}f_2(\lambda) - T_{13}f_3(\lambda)... - T_{1n}f_n(\lambda) \\ (\lambda - T_{22})f_2(\lambda) - T_{23}f_3(\lambda) - T_{24}f_4(\lambda)... - T_{2n}f_n(\lambda) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ (\lambda - T_{n-1,n-1})f_{n-1}(\lambda) - T_{n-1,n}f_n(\lambda) \\ (\lambda - T_{nn})f_n(\lambda) \end{pmatrix}.$$

Hence $(\lambda - T_{nn})f_n(\lambda) \equiv 0$. Since T_{nn} has the single valued extension property, $f_n(\lambda) \equiv 0$. Then we obtain the following equation: $(\lambda - T_{n-1,n-1})f_{n-1}(\lambda) \equiv 0$. Since $T_{n-1,n-1}$ has also the single valued extension property, $f_{n-1}(\lambda) \equiv 0$. By induction, $f_j(\lambda) \equiv 0$ for $j = 1, 2, \ldots, n$. Thus $f(\lambda) \equiv 0$. Hence T has the single valued extension property. $\square$

From Lemmas 8 and 9, we get the following theorem.

THEOREM 10. *Let* $T \in \mathcal{L}(\oplus_{k=1}^n H)$ *be the following operator matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & \cdots & \cdots & T_{1n} \\ 0 & T_{22} & \cdots & \cdots & \cdots & T_{2n} \\ 0 & 0 & T_{33} & \cdots & \cdots & T_{3n} \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & T_{nn} \end{pmatrix},$$

where $T_{11}T_{ij} = T_{ij}T_{jj}$ *for* $i = 1, \ldots, j$ *and* $j = 1, \ldots, n$. *Then* T_{kk} *has the single valued extension property for* $k = 1, \ldots, n$ *if and only if* T *has.*

COROLLARY 11. *Let* $T \in \mathcal{L}(H \oplus H)$ *be the following operator matrix;*

$$T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix},$$

where $T_1 S = S T_2$. *Then* T_1 *and* T_2 *have the single valued extension property if and only if* T *has.*

Proof. It is the special case of Theorem 10. $\square$

COROLLARY 12. *If* T_1 *and* T_2 *have the single valued extension property, then*

$$\sigma_{T_2}(x_2) \subset \sigma_T(x_1 \oplus x_2),$$

where $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix} \in \mathcal{L}(H \oplus H)$.

Proof. Let $\lambda_0 \notin \sigma_T(x_1 \oplus x_2)$. Then $\lambda_0 \in \rho_T(x_1 \oplus x_2)$. There exist a neighborhood D of λ_0 and an analytic function $f = f_1 \oplus f_2 : D \to H \oplus H$

with f_1, f_2 analytic functions such that $(\lambda - T)f(\lambda) \equiv x_1 \oplus x_2$. Then we get

$$\begin{pmatrix} \lambda - T_1 & -S \\ 0 & \lambda - T_2 \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $(\lambda - T_2)f_2(\lambda) \equiv x_2$. Hence $\lambda_0 \in \rho_{T_2}(x_2)$. $\qquad\square$

COROLLARY 13. *If T_1 and T_2 have the single valued extension property, then*

$$\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0),$$

where $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix} \in \mathcal{L}(H \oplus H).$

Proof. Let $\lambda_0 \in \rho_T(x_1 \oplus 0)$. Then there exist a neighborhood D of λ_0 and an analytic function $f = f_1 \oplus f_2 : D \to H \oplus H$ with f_1, f_2 analytic functions such that

$$(\lambda - T) \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Hence we get that

$$\begin{pmatrix} \lambda - T_1 & -S \\ 0 & \lambda - T_2 \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Thus $(\lambda - T_1)f_1(\lambda) - Sf_2(\lambda) \equiv x_1$ and $(\lambda - T_2)f_2(\lambda) \equiv 0$. Since T_2 has the single valued extension property, $f_2(\lambda) \equiv 0$. Thus $(\lambda - T_1)f_1(\lambda) \equiv x_1$. Hence $\lambda_0 \in \rho_T(x_1)$.

Conversely, let $\lambda_0 \in \rho_{T_1}(x_1)$. Then there exist a neighborhood D of λ_0 and an analytic function $f_1 : D \to H$ such that $(\lambda - T_1)f_1(\lambda) \equiv x_1$. Then

$$\begin{aligned} (\lambda - T) \begin{pmatrix} f_1(\lambda) \\ 0 \end{pmatrix} &= \begin{pmatrix} \lambda - T_1 & -S \\ 0 & \lambda - T_2 \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - T_1)f_1(\lambda) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence $\lambda_0 \in \rho_T(x_1 \oplus 0)$. $\qquad\square$

COROLLARY 14. *If $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, then $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix} \in \mathcal{L}(H \oplus H)$ has the single valued extension property if and only if T_1 and T_2 have the single valued extension property.*

Proof. ($\Leftarrow$) It is clear from the proof of Lemma 9. ($\Rightarrow$) Let $f, g :$
$D \to H(D \subset \mathbf{C}$ open) be analytic functions such that $(\lambda - T_1)f(\lambda) \equiv 0$
and $(\lambda - T_2)g(\lambda) \equiv 0$. Since

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 & -S \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

if $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, we get

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda - T_1 & -S \\ 0 & \lambda - T_2 \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f(\lambda) \\ g(\lambda) \end{pmatrix} \equiv 0.$$

Since T has the single valued extension property,

$$\begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f(\lambda) \\ g(\lambda) \end{pmatrix} \equiv 0.$$

Thus $f(\lambda) \equiv 0$ and $g(\lambda) \equiv 0$. Hence T_1 and T_2 have the single valued
extension property. $\square$

COROLLARY 15. *If T_1 and T_2 in $\mathcal{L}(H)$ have the single valued exten-
sion property, then*

$$H_{T_1}(F) \oplus 0 \subset H_T(F),$$

*where $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ and $H_T(F) = \{x \oplus y \in H : \sigma_T(x \oplus y) \subset F\}$ for
$F \subset \mathbf{C}$.*

Proof. If $x \in H_{T_1}(F)$, then $\sigma_{T_1}(x) \subset F$. Since $\sigma_{T_1}(x) = \sigma_T(x \oplus 0)$ by
Corollary 13, $\sigma_T(x \oplus 0) \subset F$. Thus $x \oplus 0 \in H_T(F)$. Hence $H_{T_1}(F) \oplus 0 \subset$
$H_T(F)$. $\square$

PROPOSITION 16. *For S and T in $\mathcal{L}(H)$, if ST has the single valued
extension property if and only if TS has.*

Proof. ($\Rightarrow$) If $(TS - \lambda)f(\lambda) = 0$, then

$$(ST - \lambda)Sf(\lambda) = 0.$$

Since ST has the single valued extension property, $Sf(\lambda) = 0$ and
$TSf(\lambda) = 0$. Since $(TS - \lambda)f(\lambda) = 0$, $\lambda f(\lambda) = 0$. Since $f(\lambda) = 0$
if $\lambda \neq 0$ and $f(\lambda)$ is analytic, $f \equiv 0$ from the behavior of an analytic
function. Hence TS has the single valued extension property.

($\Leftarrow$) The converse implication is similar. $\square$

COROLLARY 17. *Let $T = U|T|$ be the polar decomposition of T in
$\mathcal{L}(H)$. Then $T = U|T|$ has the single valued extension property if and
only if $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ has.*

Proof. Since $T = (U|T|^{\frac{1}{2}})|T|^{\frac{1}{2}}$ and $\widetilde{T} = |T|^{\frac{1}{2}}(U|T|^{\frac{1}{2}})$, the proof follows from Proposition 16. $\qquad\square$

COROLLARY 18. *If ST has the single valued extension property for S and T in $\mathcal{L}(H)$, then $\sigma_{TS}(Tx) \subset \sigma_{ST}(x)$ and $\sigma_{ST}(Sx) \subset \sigma_{TS}(x)$.*

Proof. Let $\lambda_0 \in \rho_{ST}(x)$. Then there exist a neighborhood D of λ_0 and an analytic function $f : D \to H$ such that $(\lambda - ST)f(\lambda) \equiv x$. Then $(\lambda - TS)Tf(\lambda) \equiv Tx$. Thus $\lambda_0 \in \rho_{TS}(Tx)$. Hence $\sigma_{TS}(Tx) \subset \sigma_{ST}(x)$. Similarly we obtain $\sigma_{ST}(Sx) \subset \sigma_{TS}(x)$. $\qquad\square$

COROLLARY 19. *For S and T in $\mathcal{L}(H)$, if ST has the single valued extension property, then $SH_{TS}(F) \subset H_{ST}(F)$ and $TH_{ST}(F) \subset H_{TS}(F)$, where $F \subset \mathbf{C}$.*

PROPOSITION 20. *Let $T \in \mathcal{L}(H)$ have the single valued extension property. If $S = VTV^*$, where V is an isometry, then S has the single valued extension property.*

Proof. Assume that f is an analytic H-valued function defined on an open set D such that $(\lambda - S)f(\lambda) \equiv 0$. Then $(\lambda - T)V^*f(\lambda) = 0$. Since T has the single valued extension property, $V^*f(\lambda) = 0$. Since $S = VTV^*$, $Sf(\lambda) = 0$. Thus $\lambda f(\lambda) = 0$. Since $f(\lambda) = 0$ if $\lambda \neq 0$ and $f(\lambda)$ is analytic, $f \equiv 0$ from the behavior of an analytic function. Hence we complete the proof. $\qquad\square$

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