

A FIXED POINT APPROACH TO THE STABILITY OF QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. Cădariu and Radu applied the fixed point method to the investigation of Cauchy and Jensen functional equations. In this paper, we adopt the idea of Cădariu and Radu to prove the Hyers-Ulam-Rassias stability of the quadratic functional equation for a large class of functions from a vector space into a complete β -normed space.

1. Introduction

In 1940, S. M. Ulam [17] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by D. H. Hyers [7] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, for all x and y , can be approximated by an exact solution, say an additive function. Th. M. Rassias [14] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

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and generalized the result of Hyers. Since then, the stability of several functional equations has been extensively investigated.

The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [6, 8, 9, 10, 12, 15].

Let E_1 and E_2 be real vector spaces. A function $f : E_1 \rightarrow E_2$ is called a quadratic function if and only if f is a solution function of the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y).$$

It is well known that for each quadratic function $f : E_1 \rightarrow E_2$, there exists a unique symmetric biadditive function $B : E_1 \times E_1 \rightarrow E_2$ satisfying $f(x) = B(x, x)$ for all $x \in E_1$.

The Hyers-Ulam stability of the quadratic functional equation was first proved by F. Skof [16] for functions $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. P. W. Cholewa [3] demonstrated that Skof's theorem is also valid if E_1 is replaced by an abelian group G (cf. [11].)

THEOREM 1. *Let G be an abelian group and let E be a real Banach space. If a function $f : G \rightarrow E$ satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in G$, then there exists a unique quadratic function $q : G \rightarrow E$ such that

$$\|f(x) - q(x)\| \leq \frac{1}{2}\delta$$

for any $x \in G$.

Thereafter, S. Czerwik [4] proved the Hyers-Ulam-Rassias stability of quadratic functional equation.

THEOREM 2. *Let E_1 and E_2 be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a function $f : E_1 \rightarrow E_2$ satisfies the inequality*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that

$$\|f(x) - q(x)\| \leq \frac{2\varepsilon}{|4 - 2^p|} \|x\|^p$$

for any $x \in E_1$.

Recently, L. Cădariu and V. Radu [2] applied the fixed point method to the investigation of the Cauchy additive functional equation ([1, 13]). Using such a clever idea, they could present a short, simple proof and extend the range of relevant functions to the complete β -normed space.

In this paper, we will adopt the idea of Cădariu and Radu to prove the Hyers-Ulam-Rassias stability of the quadratic functional equation for a large class of functions between a vector space and a complete β -normed space.

An advantage of our result is that the range of relevant functions is extended to any complete (real or complex) β -normed space, while the existing results concern only the real Banach space as we see in the preceding theorems.

2. Preliminaries

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if and only if d satisfies

- (M₁) $d(x, y) = 0$ if and only if $x = y$;
- (M₂) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (M₃) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [5].

THEOREM 3. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \rightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^kx) < \infty$ for some $x \in X$, then the followings are true:*

- (a) *The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ ;*
- (b) *x^* is the unique fixed point of Λ in*

$$X^* = \{y \in X \mid d(\Lambda^k x, y) < \infty\};$$

- (c) *If $y \in X^*$, then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y).$$

Throughout this paper, we fix a real number β with $0 < \beta \leq 1$ and let \mathbb{K} denote either \mathbb{R} or \mathbb{C} . Suppose E is a vector space over \mathbb{K} . A function $\|\cdot\|_\beta : E \rightarrow [0, \infty)$ is called a β -norm if and only if it satisfies

- (N₁) $\|x\|_\beta = 0$ if and only if $x = 0$;
- (N₂) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$;
- (N₃) $\|x + y\|_\beta \leq \|x\|_\beta + \|y\|_\beta$ for all $x, y \in E$.

3. Main results

In the following theorems, by using the idea of Cădariu and Radu (see [1, 2]), we will prove the Hyers-Ulam-Rassias stability of the quadratic functional equation in a more general setting.

THEOREM 4. *Let E_1 and E_2 be vector spaces over \mathbb{K} . In particular, let E_2 be a complete β -normed space, where $0 < \beta \leq 1$. Suppose $\varphi : E_1 \times E_1 \rightarrow [0, \infty)$ is a given function and there exists a constant L , $0 < L < 1$, such that*

$$(1) \quad \varphi(2x, 2x) \leq 4^\beta L \varphi(x, x)$$

for all $x \in E_1$. Furthermore, let $f : E_1 \rightarrow E_2$ be a function with $f(0) = 0$ which satisfies

$$(2) \quad \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|_\beta \leq \varphi(x, y)$$

for all $x, y \in E_1$. If φ satisfies

$$(3) \quad \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{4^{n\beta}} = 0$$

for any $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that

$$(4) \quad \|f(x) - q(x)\|_\beta \leq \frac{1}{4^\beta} \frac{1}{1-L} \varphi(x, x)$$

for all $x \in E_1$.

Proof. If we define

$$X = \{h : E_1 \rightarrow E_2 \mid h(0) = 0\}$$

and introduce a generalized metric on X as follows

$$d(g, h) = \inf \{C \in [0, \infty) \mid \|g(x) - h(x)\|_\beta \leq C\varphi(x, x) \text{ for all } x \in E_1\},$$

then (X, d) is complete. (See the proof of [2, Theorem 2.5].)

We define an operator $\Lambda : X \rightarrow X$ by

$$(\Lambda h)(x) = \frac{1}{4}h(2x)$$

for all $x \in E_1$.

First, we assert that Λ is strictly contractive on X . Given $g, h \in X$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C$, i.e.,

$$\|g(x) - h(x)\|_\beta \leq C\varphi(x, x)$$

for all $x \in E_1$. If we replace x in the last inequality by $2x$ and make use of (1), then we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\|_\beta \leq LC\varphi(x, x)$$

for every $x \in E_1$, i.e., $d(\Lambda g, \Lambda h) \leq LC$. Hence, we conclude that $d(\Lambda g, \Lambda h) \leq Ld(g, h)$ for any $g, h \in X$.

Next, we assert that $d(\Lambda f, f) < \infty$. If we substitute x for y in (2) and we divide both sides by 4^β , then (1) establishes

$$\|(\Lambda f)(x) - f(x)\|_\beta \leq \frac{1}{4^\beta}\varphi(x, x)$$

for any $x \in E_1$, i.e.,

$$(5) \quad d(\Lambda f, f) \leq \frac{1}{4^\beta} < \infty.$$

Then, it follows from Theorem 3 (a) that there exists a function $q : E_1 \rightarrow E_2$ with $q(0) = 0$, which is a fixed point of Λ , such that $\Lambda^n f \rightarrow q$, i.e.,

$$(6) \quad \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x) = q(x)$$

for all $x \in E_1$.

Since the integer k of Theorem 3 is 0 and $f \in X^*$ (see Theorem 3 for the definition of X^*), by Theorem 3 (c) and (5), we obtain

$$(7) \quad d(f, q) \leq \frac{1}{1-L}d(\Lambda f, f) \leq \frac{1}{4^\beta} \frac{1}{1-L},$$

i.e., the inequality (4) is true for all $x \in E_1$.

Now, substitute $2^n x$ and $2^n y$ for x and y in (2), respectively. If we divide both sides of the resulting inequality by $4^{n\beta}$, and letting n go to infinity, it follows from (3) and (6) that q is a quadratic function.

Assume that inequality (4) is also satisfied with another quadratic function $q_1 : E_1 \rightarrow E_2$ besides q . (As q_1 is a quadratic function, q_1

satisfies that $q_1(x) = \frac{1}{4}q_1(2x) = (\Lambda q_1)(x)$ for all $x \in E_1$. That is, q_1 is a fixed point of Λ .) In view of (4) and the definition of d , we know that

$$d(f, q_1) \leq \frac{1}{4^\beta} \frac{1}{1-L} < \infty,$$

i.e., $q_1 \in X^* = \{y \in X \mid d(\Lambda f, y) < \infty\}$. (In view of (5), the integer k of Theorem 3 is 0.) Thus, Theorem 3 (b) implies that $q = q_1$. This proves the uniqueness of q . \square

We will now generalize the above theorem by removing the hypothesis $f(0) = 0$ and we get the following theorem.

THEOREM 5. *Let E_1 be a vector space over \mathbb{K} and let E_2 be a complete β -normed space over \mathbb{K} , where $0 < \beta \leq 1$. Suppose a function $\varphi : E_1 \times E_1 \rightarrow [0, \infty)$ satisfies the condition (3) for all $x, y \in E_1$ and there exists a constant $L, \frac{1}{4^\beta} \leq L < 1$, for which the inequality (1) holds for any $x \in E_1$. If a function $f : E_1 \rightarrow E_2$ satisfies the inequality (2) for all $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that*

$$\|f(x) - f(0) - q(x)\|_\beta \leq \frac{1}{4^\beta} \frac{1}{1-L} [\inf\{\varphi(z, 0) \mid z \in E_1\} + \varphi(x, x)]$$

for all $x \in E_1$.

Proof. Putting $y = 0$ in (2) yields

$$\|2f(0)\|_\beta \leq \varphi(x, 0)$$

for any $x \in E_1$. We define a function $g : E_1 \rightarrow E_2$ by $g(x) = f(x) - f(0)$. If we set

$$\psi(x, y) = \varphi_0 + \varphi(x, y)$$

for each $x, y \in E_1$, where $\varphi_0 = \inf\{\varphi(x, 0) \mid x \in E_1\}$, it then follows from (2) that

$$\|g(x+y) + g(x-y) - 2g(x) - 2g(y)\|_\beta \leq \psi(x, y)$$

for all $x, y \in E_1$.

Considering (1) and $L \geq \frac{1}{4^\beta}$, we see that

$$\psi(2x, 2x) = \varphi_0 + \varphi(2x, 2x) \leq \varphi_0 + 4^\beta L \varphi(x, x) \leq 4^\beta L \psi(x, x)$$

for any $x \in E_1$.

Moreover, we make use of (3) to verify that

$$\lim_{n \rightarrow \infty} \frac{\psi(2^n x, 2^n y)}{4^{n\beta}} = \lim_{n \rightarrow \infty} \frac{\varphi_0 + \varphi(2^n x, 2^n y)}{4^{n\beta}} = 0$$

for every $x, y \in E_1$.

According to Theorem 4, there exists a unique quadratic function $q : E_1 \rightarrow E_2$ satisfying inequality (4) with g instead of f . This completes our proof. \square

By a similar way as in the proof of Theorem 4, we also apply Theorem 3 and prove the following theorem.

THEOREM 6. *Let E_1 and E_2 be a vector space over \mathbb{K} and a complete β -normed space over \mathbb{K} , respectively. Assume that $\varphi : E_1 \times E_1 \rightarrow [0, \infty)$ is a given function and there exists a constant L , $0 < L < 1$, such that*

$$(8) \quad \varphi(x, x) \leq \frac{1}{4^\beta} L \varphi(2x, 2x)$$

for all $x \in E_1$. Furthermore, assume that $f : E_1 \rightarrow E_2$ is a given function with $f(0) = 0$ and satisfies the inequality (2) for all $x, y \in E_1$. If φ satisfies

$$\lim_{n \rightarrow \infty} 4^{n\beta} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for every $x, y \in E_1$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that

$$(9) \quad \|f(x) - q(x)\|_\beta \leq \frac{1}{4^\beta} \frac{L}{1-L} \varphi(x, x)$$

for any $x \in E_1$.

Proof. We use the definitions for X and d , the generalized metric on X , as in the proof of Theorem 4. Then, (X, d) is complete.

We define an operator $\Lambda : X \rightarrow X$ by

$$(\Lambda h)(x) = 4h\left(\frac{x}{2}\right)$$

for all $x \in E_1$. We apply the same argument as in the proof of Theorem 4 and prove that Λ is a strictly contractive operator. Moreover, we prove

$$(10) \quad d(\Lambda f, f) \leq \frac{1}{4^\beta} L$$

instead of (5).

According to (a) of Theorem 3, there exists a function $q : E_1 \rightarrow E_2$ with $q(0) = 0$, which is a fixed point of Λ , such that

$$\lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right) = q(x)$$

for each $x \in E_1$.

Since the integer k of Theorem 3 is 0 and $f \in X^*$ (see Theorem 3 for the definition of X^*), using Theorem 3 (c) and (10), we get

$$d(f, q) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{4^\beta} \frac{L}{1-L},$$

which implies the validity of inequality (9).

In the last part of proof of Theorem 4, if we replace $2^n x$, $2^n y$ and $4^{n\beta}$ by $\frac{x}{2^n}$, $\frac{y}{2^n}$ and $\frac{1}{4^{n\beta}}$, respectively, then we can prove that q is a unique quadratic function satisfying inequality (9) for all $x \in E_1$. \square

Theorem 6 cannot be generalized to the case without the condition $f(0) = 0$. For example, if φ is continuous at $(0, 0)$, then the condition (8) implies

$$\varphi(x, x) \geq \left(\frac{4^\beta}{L}\right)^n \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}\right)$$

for any $n \in \mathbb{N}$. By letting $n \rightarrow \infty$, we conclude that $\varphi(0, 0) = 0$. And if we put $x = y = 0$ in (2), then we get $f(0) = 0$.

4. Applications

In the following corollaries, using Theorems 4, 5 and 6, we will extend Theorems 1 and 2 for a nonnegative real number $p \neq 2$ and for a complete β -normed space as the range space.

COROLLARY 7. *Fix a positive number p less than 2 and choose a constant β with $\frac{p}{2} < \beta \leq 1$. Let E_1 and E_2 be a normed space over \mathbb{K} and a complete β -normed space over \mathbb{K} , respectively. If a function $f : E_1 \rightarrow E_2$ satisfies*

$$(11) \quad \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_\beta \leq \varepsilon (\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$ and for some $\varepsilon > 0$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that

$$(12) \quad \|f(x) - q(x)\|_\beta \leq \frac{2\varepsilon}{|4^\beta - 2^p|} \|x\|^p$$

for any $x \in E_1$.

Proof. Putting $x = y = 0$ in (11), we get $f(0) = 0$. If we set $\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$ and if we set $L = \frac{2^p}{4^\beta}$, then we have $0 < L < 1$ and

$$\varphi(2x, 2x) = 2^{1+p}\varepsilon\|x\|^p = 2^p\varphi(x, x) = 4^\beta L\varphi(x, x)$$

for all $x \in E_1$.

Furthermore, we get

$$\frac{\varphi(2^n x, 2^n y)}{4^{n\beta}} = L^n \varepsilon (\|x\|^p + \|y\|^p) \rightarrow 0, \text{ as } n \rightarrow \infty$$

for any $x, y \in E_1$.

According to Theorem 4, there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that the inequality (12) holds for every $x \in E_1$. \square

In the following corollary, we deal with the inequality (11) for the case $p = 0$. We need only to set $L = \frac{1}{4^\beta}$ and apply Theorem 5 for its proof.

COROLLARY 8. *Let E_1 and E_2 be a vector space over \mathbb{K} and a complete β -normed space over \mathbb{K} respectively, where $0 < \beta \leq 1$. If a function $f : E_1 \rightarrow E_2$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\|_\beta \leq \delta$$

for all $x, y \in E_1$ and for some $\delta > 0$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that

$$\|f(x) - f(0) - q(x)\|_\beta \leq \frac{2\delta}{4^\beta - 1}$$

for any $x \in E_1$.

The last corollary with $\beta = 1$ yields a larger upper bound, $\frac{2\delta}{3}$, than $\frac{\delta}{2}$ of Theorem 1 for the difference between $f(x) - f(0)$ and $q(x)$.

In the following corollary, we assume that β is a constant with $0 < \beta \leq 1$.

COROLLARY 9. *Assume that p is a real constant larger than 2. Let E_1 and E_2 be a normed space over \mathbb{K} and a complete β -normed space over \mathbb{K} , respectively. If $f : E_1 \rightarrow E_2$ is a function with $f(0) = 0$ and satisfies the inequality (11) for all $x, y \in E_1$ and for some $\varepsilon > 0$, then there exists a unique quadratic function $q : E_1 \rightarrow E_2$ such that the inequality (12) holds for all $x \in E_1$.*

Proof. If we set $\varphi(x, y) = \varepsilon(\|x\|^p + \|y\|^p)$ for any $x, y \in E_1$, then we obtain

$$\varphi(x, x) = 2\varepsilon\|x\|^p = \frac{1}{4^\beta} L\varphi(2x, 2x)$$

for each $x \in E_1$, where $L = \frac{4^\beta}{2^p}$ is less than 1 because $0 < \beta \leq 1$ and $p > 2$.

Moreover, we have

$$4^{n\beta} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = L^n \varepsilon (\|x\|^p + \|y\|^p) \rightarrow 0, \text{ as } n \rightarrow \infty$$

for all $x, y \in E_1$.

In view of Theorem 6, there exists a unique quadratic function $q : E_1 \rightarrow E_2$ for which the inequality (12) is true for any $x \in E_1$. \square

We remark that the conclusions of Corollaries 7 and 9 are consistent with that of Theorem 2.

References

- [1] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 1, Art. 4, 7 pp.
- [2] ———, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
- [3] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), no. 1-2, 76–86.
- [4] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
- [5] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
- [6] G. L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), no. 1-2, 143–190.
- [7] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.
- [8] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel/Boston, 1998.
- [9] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), no. 2-3, 125–153.
- [10] S.-M. Jung, *Hyers-Ulam-Rassias stability of functional equations*, Dynam. Systems Appl. **6** (1997), no. 4, 541–566.
- [11] ———, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), no. 1, 126–137.
- [12] ———, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.
- [13] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), no. 1, 91–96.
- [14] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), no. 2, 297–300.
- [15] ———, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), no. 1, 23–130.
- [16] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

- [17] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publisher, New York, 1960.

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