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FUZZY 2-(0- OR 1-)PRIME IDEALS IN SEMIRINGS

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ABSTRACT. In this paper three different types of fuzzy prime ideals are introduced. Condition is obtained for a fuzzy 2-prime ideal will have two elements in its range. It has been shown that A is fuzzy 2-prime ideal of the semiring R if and only if $1 - A$ is a fuzzy m_2-system in R.

1. Introduction

In 1965, Zadeh [19] introduced the concept of fuzzy subsets and studied their properties on the parallel lines to set theory. In 1967, Rosenfeld [14] defined the fuzzy subgroup and gave some of its properties. Rosenfeld's definition of a fuzzy group is a turning point for pure mathematicians. Since then, the study of fuzzy algebraic structure has been pursued in many directions such as groups, rings, modules, vector spaces and so on. In 1981, Das [3] explained the inter-relationship between the fuzzy subgroups and its t-level subsets. Fuzzy subrings and ideals are of comparatively recent origin and were first introduced by Wang-jin Liu [10] in the year 1982. Subsequently, Mukherjee and Sen [12], Swamy and Swamy [16], Yue [18], Dixit et al [4] and Rajesh Kumar [8] applied some basic concepts pertaining to ideals from classical ring theory and developed a theory of fuzzy. T. K. Dutta and B. K. Biswas [5] studied fuzzy ideals, fuzzy prime ideals of semirings, and they defined fuzzy k-ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers.

This paper contains four sections, the first section is merely introduction. In section 2, we have initiate a notion of fuzzy 2-(0- or 1-)prime ideal, fuzzy k-closure and fuzzy m_2 (m_0 or m_1) - system and also give some basic definitions and results which will be used later. In section 3,

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we have shown that if R is a semiring containing a proper k-ideal A_m with $m \neq A(0)$ and if A is a fuzzy 2-prime ideal of R, then $|Im(A)| = 2$. The condition that the semiring R contains a proper k-ideal A_m with $m \neq A(0)$ is necessary. By an example, we have shown that theorem will fail if we drop that condition. In section 4, we have come across that a fuzzy subset μ in semiring R holds a property like subgroup, ideal etc., if and only if its level subset μ_t, for all $t \in [0, 1]$, also satisfies the same property in R. However if μ is a fuzzy subset of R such that level subset μ_t is an m_2 (m_0 or m_1) - system in R, for all $t \in [0, 1]$, then μ is not necessarily a fuzzy m_2 (m_0 or m_1) - system of R. Nevertheless we have shown that if μ is a fuzzy subset in R with $x_1 \in \langle x \rangle_k$ ($x_1 \in \langle x \rangle$) implies $\mu(x_1) \geq \mu(x)$, then μ is a fuzzy m_2 (m_0 or m_1) in R if and only if $\mu^r = \{x \in R/\mu(x) > r\}$ is an m_2 (m_0 or m_1) system in R.

2. Preliminary notes

We would like to reproduce some definitions and results proposed by the pioneers in this field earlier.

An algebra $(R, +, \cdot)$ is said to be a semiring [15] if $(R, +)$ and $(R, \cdot)$ are semigroups satisfying $a.(b + c) = a.b + a.c$ and $(b + c).a = b.a + c.a$ for all $a, b, c \in R$. A semiring R may have an identity 1, defined by $1.a = a = a.1$, and a zero 0, defined by $0 + a = a = a + 0$ and $a.0 = 0 = 0.a$ for all $a \in R$.

From now on we write R for semirings. A nonempty subset I of R is said to be a left (resp., right) ideal if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $rx \in I$ (resp., $xr \in I$). If I is both left and right ideal of R, we say I is a two-sided ideal, or simply ideal of R. A left ideal I of a semiring R is said to be a left k-ideal if $a \in I$ and $x \in R$ and if $a + x \in I$ or $x + a \in I$ then $x \in I$. Right k-ideal is defined dually, and two sided k-ideal or simply a k-ideal is both a left and a right k-ideal. The ideal generated by a, $a \in R$, is defined as the smallest ideal of R, which contains a and is denoted by $\langle a \rangle$. The k-ideal generated by a, $a \in R$, is defined as the smallest k-ideal of R, which contains a and is denoted by $\langle a \rangle_k$.

DEFINITION 2.1. Let S be any set. A mapping $\mu : S \to [0, 1]$ is called a *fuzzy subset* of S.

A fuzzy subset $\mu : S \to [0, 1]$ is nonempty if μ is not the constant map which assumes the value 0. For any two fuzzy subsets λ and μ of S, $\lambda \subseteq \mu$ means that $\lambda(a) \leq \mu(a)$ for all $a \in S$. $1 - f$ is a fuzzy subset of

S defined by $(1-f)(x) = 1 - f(x)$ for all $x \in S$. If μ is a fuzzy subset of R, then the image of μ denoted by $Im(\mu) = \{\mu(r) \mid r \in R\}$ and $|Im\,\mu|$ denotes the cardinality of $Im\,\mu$. For any fuzzy subset μ of R, R_μ denotes the subset $\{x \in R \mid \mu(x) = \mu(0)\}$ of R. The characteristic function of a subset I of R is denoted by χ_I.

DEFINITION 2.2. Let μ be any fuzzy subset of R. For $t \in [0, 1]$, the set $\mu_t = \{x \in R/\mu(x) \geq t\}$ is called a *level subset* of μ.

DEFINITION 2.3. Let f and g be any two fuzzy subsets of R. Then $f \cap g$, $f \cup g$, $f + g$ and $f \cdot g$ are fuzzy subsets of R defined by

$(f \cap g)(x) = \min\{f(x), g(x)\}$

$(f \cup g)(x) = \max\{f(x), g(x)\}$. The *sum* $f + g$ is defined by

$$(f+g)(x) = \begin{cases} \sup\limits_{x=y+z} \{\min\{f(y), g(z)\}\} & \text{if } x \text{ is expressed as } x = y + z \\ 0 & \text{otherwise.} \end{cases}$$

The *product* $f \circ g$ is defined by

$$(f \cdot g)(x) = \begin{cases} \sup\limits_{x=yz} \{\min\{f(y), g(z)\}\} & \text{if } x \text{ is expressed as } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION 2.4. For any $x \in R$ and $t \in [0, 1]$, we define the *fuzzy point* x_t as $x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$

If x_t is a fuzzy point and μ is any fuzzy subset of R and $x_t \subseteq \mu$, then we write $x_t \in \mu$. Note that $x_t \in \mu$, if and only if $x \in \mu_t$, where μ_t is a level subset of μ. For any fuzzy subset f of R, it is obvious that $f = \bigcup\limits_{a_t \in f} a_t$.

DEFINITION 2.5. A fuzzy subset μ of R is said to be a *fuzzy left* (resp., *right*) *ideal* of R if

$\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, and

$\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) \geq \mu(x)$) for all x, y in R.

μ is a *fuzzy ideal* of R if it is both a fuzzy left and a fuzzy right ideal of R.

LEMMA 2.6. *Let I be an ideal of R and $\alpha < \beta \neq 0 \in [0, 1]$. Then the fuzzy subset defined by*

$$A(x) = \begin{cases} \beta & \text{if } x \in I \\ \alpha & \text{if otherwise,} \end{cases}$$ *is a fuzzy ideal of R.*

Proof of the Lemma is a routine matter of checking, so we omit it.

DEFINITION 2.7. A fuzzy ideal μ of R is said to be a *fuzzy k-ideal* of R if $\mu(x) \geq \min\{\max\{\mu(x+y), \mu(y+x)\}, \mu(y)\}$ for all $x, y \in R$.

If R is additively commutative, then the condition reduces to $\mu(x) \geq \min\{\mu(x+y), \mu(y)\}$ for all $x, y \in R$.

Note that every fuzzy ideal of a ring is a fuzzy k-ideal. Here after we consider only additively commutative semiring R and nonempty fuzzy subsets of R.

EXAMPLE 2.8. [5] Let μ be a fuzzy subset of N of natural numbers defined by

$$\mu(x) = \begin{cases} 0.3, & \text{if } x \text{ is odd}, \\ 0.5, & \text{if } x \text{ is nonzero even}, \\ 1, & \text{if } x = 0. \end{cases}$$

Then μ is a fuzzy k-ideal of N.

EXAMPLE 2.9. [5] Let μ be a fuzzy subset of the semiring N of natural numbers defined by

$$\mu(x) = \begin{cases} 1, & \text{if } 7 \leq x, \\ 0.5, & \text{if } 5 \leq x < 7, \\ 0, & \text{if } 0 \leq x < 5. \end{cases}$$

Then it is easy to show that μ is a fuzzy ideal of N, but not a fuzzy k-ideal of N.

THEOREM 2.10. [2] *A fuzzy subset μ of R is a fuzzy left (resp., right) k-ideal of R if and only if, for any $t \in [0,1]$ such that $\mu_t \neq \emptyset$, μ_t is a left (resp., right) k-ideal of R.*

THEOREM 2.11. [6] *Let R be a semiring and $I \subseteq R$. Then I is left (resp., right) k-ideal of R if and only if χ_I is a fuzzy left (resp., right) k-ideal of R.*

THEOREM 2.12. *Let μ and λ be fuzzy k-ideals of R. Then $\mu \cap \lambda$ is also fuzzy k-ideal of R.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}(\mu \cap \lambda)(x+y) &= \min\{\mu(x+y), \lambda(x+y)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\ &\geq \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\ &= \min\{(\mu \cap \lambda)(x), (\mu \cap \lambda)(y)\},\end{aligned}$$

$$\begin{aligned}(\mu \cap \lambda)(xr) &= \min\{\mu(xr), \lambda(xr)\} \\ &\geq \min\{\mu(x), \lambda(x)\} \text{ [As } \mu \text{ and } \lambda \text{ are fuzzy right ideals of } R] \\ &= (\mu \cap \lambda)(x).\end{aligned}$$

Similarly, we can prove that $(\mu \cap \lambda)\,(r\,x) \geq (\mu \cap \lambda)(x)$

$$\begin{aligned}(\mu \cap \lambda)\,(a+x) \wedge (\mu \cap \lambda)\,(x) &= \mu(a+x) \wedge \lambda(a+x) \wedge \mu(x) \wedge \lambda(x) \\ &\geq \mu(a) \wedge \lambda(a) \\ &\geq (\mu \cap \lambda)(a).\end{aligned}$$

Thus $\mu \cap \lambda$ is a fuzzy k-ideal of R. □

THEOREM 2.13. [5] *Let μ be a fuzzy k-ideal of semiring with zero. Then R_μ is a k-ideal of R.*

DEFINITION 2.14. If A is an ideal of R, then $\overline{A} = \{a \in R |\; a+x \in A$ for some $x \in A\}$ is called *k-closure* of A.

LEMMA 2.15. *If A is an ideal of R, then $\overline{A}$ is a k-ideal of R.*

Proof. Let $a_1,\; a_2 \in \overline{A}$. Now $a_1+x_1,\; a_2+x_2 \in A$ for some $x_1,\; x_2 \in A$. Since A is an ideal, we have $(a_1+a_2)+(x_1+x_2) \in A$, where $x_1+x_2 \in A$ and so $a_1+a_2 \in \overline{A}$ so that $\overline{A}$ is closed under addition. Let $a \in \overline{A}$. Hence $a+x \in A$ for some $x \in A$. For any $r \in R$, we claim that $a\,r,\; r\,a \in \overline{A}$. Since A is an ideal, we have $(a+x)\,r \in A$ and so $a\,r + x\,r \in A$. Clearly $x\,r \in A$. Hence $a\,r \in \overline{A}$. Similarly we can prove that $r\,a \in \overline{A}$. Let $a,\; a+b \in \overline{A}$. Now $a+x,\; (a+b)+y \in A$ for some $x,\; y \in A$. Since A is an ideal, we have $[(a+b)+y]+x \in A$ implies $b+[(a+x)+y] \in A$. Since $(a+x)+y \in A$ we have $b \in \overline{A}$. Thus $\overline{A}$ is a k-ideal. □

LEMMA 2.16. *Let A be an ideal of a semiring R. Then A is a k-ideal if and only if $A = \overline{A}$.*

Proof. Assume that A is a k-ideal. Clearly $A \subseteq \overline{A}$. Let $a \in \overline{A}$. Then $a+x \in A$ for some $x \in A$. Since $x,\; a+x \in A$ and since A is a k-ideal we have $a \in A$. Therefore $\overline{A} \subseteq A$ and hence $A = \overline{A}$.

Conversely, let us assume that $A = \overline{A}$. By Lemma 2.15, $\overline{A}$ is a k-ideal and hence A is a k-ideal. □

DEFINITION 2.17. If f is any fuzzy subset of R, then $\overline{f}$ is defined as, for any $a \in R$, $\overline{f}(a) = \sup\limits_{x \in R}\{\min\{f(a+x),\; f(x)\}\}$. $\overline{f}$ is called *fuzzy k-closure* of f.

Clearly, $f \leq \overline{f}$. When f is fuzzy ideal of ring R, then $f = \overline{f}$.

LEMMA 2.18. *If f is any fuzzy ideal of R. Then $\overline{f}$ is a fuzzy k-ideal of R.*

Proof. Consider

$$\begin{aligned}
&\overline{f}(x) \cap \overline{f}(y) \\
&= \{\bigcup_{z_1 \in R}\{f(x+z_1) \cap f(z_1)\} \cap \{\bigcup_{z_2 \in R}\{f(y+z_2) \cap f(z_2)\}\} \\
&= \bigcup_{z_1,\, z_2 \in R}\{f(x+z_1) \cap f(z_1)\} \cap \{f(y+z_2) \cap f(z_2)\} \\
&= \bigcup_{z_1,\, z_2 \in R}\{f(x+z_1) \cap f(y+z_2) \cap f(z_1) \cap f(z_2)\} \\
&\leq \bigcup_{z_1, z_2 \in R}\{f(x+y+\{z_1+z_2\}) \cap f(z_1+z_2)\} \\
&\quad [\text{Since } f \text{ is a fuzzy ideal of } R] \\
&= \overline{f}(x+y).
\end{aligned}$$

$$\begin{aligned}
\overline{f}(ar) &= \sup_{x \in R}\{\min\{f(ar+x),\ f(x)\}\} \\
&\geq \sup_{x \in R}\{\min\{f(ar+xr),\ f(xr)\}\} \\
&\geq \sup_{x \in R}\{\min\{f(a+x),\ f(x)\}.
\end{aligned}$$

Similarly, we can prove that $\overline{f}(ra) \geq \overline{f}(a)$. Hence $\overline{f}$ is a fuzzy ideal of R. □

LEMMA 2.19. *Suppose f is a fuzzy k-ideal. Then $f = \overline{f}$.*

Proof. Since f is a fuzzy k-ideal of R, $f(a) \geq \min\{f(a+x),\ f(x)\}$ for all $x,\ a \in R$. So

$$\begin{aligned}
f(a) &\geq \sup_{x \in R}\{\min\{f(a+x),\ f(x)\}\} \\
&= \overline{f}(a) \text{ for all } a \in R.
\end{aligned}$$

Thus $\overline{f} \leq f$. But $f \leq \overline{f}$. Hence $f = \overline{f}$. □

LEMMA 2.20. *Let B and C be any fuzzy ideals of R and A be a fuzzy k-ideal of R. If $C\,B \subseteq A$ implies $\overline{C}\,B \subseteq A$, $C\,\overline{B} \subseteq A$ and $\overline{C}\,\overline{B} \subseteq A$*

Proof. Suppose $C\,B \subseteq A$. Let $a \in R$. Consider

$$\begin{aligned}
&\overline{C}\,B(a) \\
&= \bigcup_{a=x_1y_1}\{\overline{C}(x_1) \cap B(y_1)\} \\
&= \bigcup_{a=x_1y_1}\{\bigcup_{z_1 \in R}\{C(x_1+z_1) \cap C(z_1)\} \cap B(y_1)\} \\
&= \bigcup_{a=x_1y_1}\{\bigcup_{z_1 \in R}\{C(x_1+z_1) \cap B(y_1) \cap C(z_1) \cap B(y_1)\}\} \\
&= \bigcup_{a=x_1y_1}\{\{\bigcup_{z_1 \in R}\{C(x_1+z_1) \cap B(y_1)\}\} \cap \{\bigcup_{z_1 \in R}\{C(z_1) \cap B(y_1)\}\}\}
\end{aligned}$$

$$\leq \underset{a=x_1y_1}{\cup} \{\{ \underset{z_1\in R}{\cup} A((x_1+z_1)y_1)\} \cap \{ \underset{z_1\in R}{\cup} A(z_1y_1)\}\}$$
$$\leq \underset{a=x_1y_1}{\cup} \{ \underset{z_1\in R}{\cup} A(x_1\, y_1 + z_1\, y_1)\} \cap \{ \underset{z_1\in R}{\cup} A(z_1y_1)\}\}$$
$$\leq \underset{a=x_1y_1}{\cup} \{A(x_1\, y_1)\}$$
[Since A is fuzzy k-ideal of R]
$$\leq A(a).$$

Thus $\overline{C}\, B \subseteq A$. Similarly, we can prove that $C\, \overline{B} \subseteq A$ and $\overline{C}\, \overline{B} \subseteq A$. □

DEFINITION 2.21. An ideal P of R is called a 0-(2-)*prime ideal* if for any ideals (k-ideals) A, B of R, $A\, B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

An ideal P of R is called a 1-*prime ideal* if for any k-ideal A of R and for any ideal B of R, $A\, B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

DEFINITION 2.22. A subset M of R is called an m_0-*system* if for every a, $b \in M$, there exists $x \in R$ such that $a\, x\, b \in M$.

A subset M of R is called an m_1-*system* if for every a, $b \in M$, there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle$ such that $a_1\, b_1 \in M$.

A subset M of R is called an m_2-*system* if for every a, $b \in M$ there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle_k$ such that $a_1\, b_1 \in M$.

Now we introduce the different types of fuzzy prime ideals in semiring. These fuzzy prime ideals coincide in rings.

DEFINITION 2.23. A fuzzy ideal P of R is called a *fuzzy* 0-(2-)*prime ideal* if for any fuzzy ideals (k-ideals) A, B of R, $A\, B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

A fuzzy ideal P of R is called a 1-*prime ideal* if for any fuzzy k-ideal A, and for any ideal B of R, $A\, B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

LEMMA 2.24. *If P is a fuzzy 0-prime ideal of R, then P is a fuzzy 2-prime ideal (fuzzy 1-prime ideal) of R.*

Proof is obvious.

Now we give an example of a fuzzy 2-prime ideal which is not a fuzzy 0-prime ideal.

EXAMPLE 2.25. Consider the semiring $R = \{0,\ 1,\ 2,\ 3\}$, where "+" and "•" are defined as follows:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	3
2	2	3	3	3
3	3	3	3	3

•	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

Define $P : R \to [0,\ 1]$ by

$$P(x) = \begin{cases} 1 & \text{if } x = 0,\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

Define $f : R \to [0,\ 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 0,\ 2,\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly P and f are fuzzy ideals and $f\,f \subseteq P$. But $f \nsubseteq P$. Hence P is not fuzzy 0-prime. However, P is a fuzzy 2-prime ideal of R.

THEOREM 2.26. *If P is a fuzzy k-ideal of R, then P is a fuzzy 0-prime ideal if and only if P is a fuzzy 2-prime ideal of R.*

Proof. Assume that P is a fuzzy 2-prime ideal and P is a fuzzy k-ideal of R. Let us assume that A and B are fuzzy ideals of R such that $A\,B \subseteq P$. By Lemma 2.20, $\overline{A}\,\overline{B} \subseteq P$. As P is a fuzzy k-ideal of R, $\overline{A} \subseteq P$ or $\overline{B} \subseteq P$. But $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ Thus $A \subseteq P$ or $B \subseteq P$. Hence P is a fuzzy 0-prime ideal of R. □

DEFINITION 2.27. A fuzzy subset f of R is said to be a *fuzzy m_0-system* if for any $t,\ s \in [0,\ 1)$ and $a,\ b \in R$, $f(a) > t$, $f(b) > s$ implies that there exists $x \in R$ such that $f(a\,x\,b) > \max\{t,\ s\}$.

A fuzzy subset f of R is said to be a *fuzzy m_1-system* if for any $t,\ s \in [0,\ 1)$ and $a,\ b \in R$, $f(a) > t$, $f(b) > s$ implies that there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle$ such that $f(a_1\,b_1) > \max\{t,\ s\}$.

A fuzzy subset f of R is said to be a *fuzzy m_2-system* if for any $t,\ s \in [0,\ 1)$ and $a,\ b \in R$, $f(a) > t$, $f(b) > s$ implies that there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle_k$ such that $f(a_1\,b_1) > \max\{t,\ s\}$.

LEMMA 2.28. *Every fuzzy m_0- system of R is an m_1- system and m_2-system of R.*

Proof. Let f be a fuzzy m_0- system of R. Let $a,\ b \in R$ such that $f(a) > t$ and $f(b) > s$, with $t,\ s \in [0,\ 1)$. As f is a fuzzy m_0- system there exists $x \in R$ such that $f(a\,x\,b) > \max\{t,\ s\}$. Now $a\,x = a_1 \in \langle a \rangle_k$, $b_1 = b \in \langle b \rangle_k$. Thus $f(a_1\,b_1) > \max\{t,\ s\}$. Hence f is a m_2- system. Similarly, we can prove if f is a m_0- system, then f is a m_1- system. □

The following two Lemmas are easily seen.

LEMMA 2.29. *Let f_1 and f_2 be any two fuzzy subsets of R. If $f_1 \leq f$ and $f_2 \leq g$, then $f_1\,f_2 \leq f\,g$ for any fuzzy subsets f and g.*

LEMMA 2.30. *Let a_r and b_s be any two fuzzy points of R such that $a_r \in f$ and $b_s \in g$, where f and g are any fuzzy subset of R. Then $a_r\, b_s \in f\, g$.*

3. Fuzzy 2-(0- or 1-)prime ideal of R

LEMMA 3.1. [*Lemma 1.1,* [16]] *If μ is a fuzzy ideal of R and $a \in R$ then $\mu(x) \geq \mu(a)$ for all $x \in \langle a \rangle$.*

LEMMA 3.2. *If μ is a fuzzy k-ideal of R and $a \in R$, then $\mu(x) \geq \mu(a)$ for all $x \in \langle a \rangle_k$.*

Proof. Suppose $x \in \langle a \rangle_k$. Then by Lemma 2.16, $x + y \in \langle a \rangle$ for some $y \in \langle a \rangle$. By the above Lemma 3.1, $\mu(x+y) \geq \mu(a)$ and $\mu(y) \geq \mu(a)$. Thus $\min\{\mu(x+y),\ \mu(y)\} \geq \mu(a)$. Since μ is a fuzzy k-ideal, we have $\mu(x) \geq \mu(a)$. □

LEMMA 3.3. *Let I be a 2-(0- or 1-) prime ideal of R and $\alpha \in [0,\ 1)$. If μ is a fuzzy subset of R defined by*

$$\mu(x) = \begin{cases} 1 & if\, x \in I, \\ \alpha & otherwise, \end{cases}$$

then μ is a fuzzy 2-(0- or 1-) prime ideal of R.

Proof. Let I be a 2- prime ideal of R. By Lemma 2.6, μ is a nonconstant fuzzy ideal of R. Suppose λ and θ are two fuzzy k-ideals of R such that $\lambda\, \theta \subseteq \mu$, and $\lambda \not\subseteq \mu$ and $\theta \not\subseteq \mu$. Then there exist $x,\ y \in R$ such that $\lambda(x) > \mu(x)$ and $\theta(y) > \mu(y)$. These imply that $\mu(x) = \mu(y) = \alpha$. Therefore, $x,\ y \notin I$. Since I is a 2-prime ideal of R, $\langle x \rangle_k \langle y \rangle_k \not\subseteq I$. Hence there exist $c \in \langle x \rangle_k$ and $d \in \langle y \rangle_k$ such that $c \cdot d \notin I$. Let $a = c\,d$. So $\mu(a) = \alpha$. Hence $(\lambda\, \theta)(a) \leq \mu(a) = \alpha$. Now

$$\begin{aligned}(\lambda\, \theta)(a) &= \sup_{a=pq}\{\min\{\lambda(p),\ \theta(q)\}\} \\ &\geq \min\{\lambda(c),\ \theta(d)\} \\ &\geq \min\{\lambda(x),\ \theta(y)\} \text{ [by Lemma 3.2]} \\ &> \min\{\mu(x),\ \mu(y)\} \\ &= \alpha,\end{aligned}$$

which contradicts the fact that $\lambda\, \theta \subseteq \mu$. Hence μ is a fuzzy 2-prime ideal of R. □

THEOREM 3.4. *Let A be any fuzzy subset of R and let R contain a proper k-ideal A_m with $m \neq A(0)$. If A is a fuzzy 2-prime ideal of R, then $|Im(A)| = 2$.*

Proof. Since A is not constant, $|Im(A)| \geq 2$. Suppose that $|Im(A)| \geq 3$. Let $A(0) = s$, $k = \text{g.l.b}\ \{A(x)|x \in R\}$. Then there exists $t \in [0,\ 1]$ such that $t < m < s$ and $t \geq k$. Let B and C be fuzzy subsets of R such that $B(x) = \frac{1}{2}(t+m)$ for all $x \in R$ and $C(x) = k$ if $x \notin A_m = \{x \in R|A(x) \geq m\}$; $C(x) = s$ if $x \in A_m$. Clearly B is a fuzzy k-ideal of R. Now we show that C is a fuzzy k-ideal of R. Clearly C is a fuzzy ideal of R. Let $x,\ y \in R$. Let us show that $C(x) \geq \min\{C(x+y),\ C(y)\}$. If $C(x) = s$, there is nothing to prove. If $C(x) = k$, let us show that $\min\{C(x+y),\ C(y)\} = k$. If not, $C(x+y) = s$, $C(y) = s$, then $y, x+y \in A_m$. As A_m is a k-ideal of R, $x \in A_m$, which is a contradiction. Thus, $C(x+y) = C(y) = k$. Consequently C is a fuzzy k-ideal of R.

We now claim that $BC \subseteq A$. Let $x \in R$. Consider the following cases

(i) $x = 0$. Then $BC(x) = \sup\limits_{x=uv}\{\min\{B(u),\ C(v)\}\} \leq \frac{1}{2}(t+m) < s = A(0)$.

(ii) $x \neq 0$, $x \in A_m$. Then $A(x) \geq m$, and $BC(x) = \sup\limits_{x=uv}\{\min\{B(u),\ C(v)\}\} \leq \frac{1}{2}(t+m) < m = A(x)$.

(iii) $x \neq 0$, $x \notin A_m$. Then for any $u,\ v \in R$ such that $x = uv$, $u \notin A_m$ and $v \notin A_m$. Then $C(v) = k$. Hence $BC(x) = \sup\limits_{x=uv}\{\min\{B(u),\ C(v)\}\} = k \leq A(x)$.

Thus in any case, $BC(x) \leq A(x)$. Hence $BC \subseteq A$. Now there exists $u \in R$ such that $A(u) = t$. Then $B(u) = \frac{1}{2}(t+m) > A(u)$. Hence $B \nsubseteq A$. Also there exists $x \in R$ such that $A(x) = m$. Then $x \in A_m$ and thus $C(x) = s > m = A(x)$. Hence $C \nsubseteq A$. Thus neither $B \subseteq A$ nor $C \subseteq A$. This shows that A is not a fuzzy 2-prime ideal of R, which is a contradiction of the hypothesis. Hence $|Im(A)| = 2$. □

In Theorem 3.4, the condition that the semiring R contains a proper k-ideal A_m with $m \neq A(0)$, is necessary. The following example shows that the theorem will fail if we drop that condition.

EXAMPLE 3.5. Consider the semiring $R = \{0,\ 1,\ 2,\ 3\}$, where $+$ and $\bullet$ are defined as with Example 2.25. Define $\mu\ R \to [0,\ 1]$ by

$$\mu(x) := \begin{cases} 1 & \text{if } x = 0, \\ 1/3 & \text{if } x = 3, \\ 1/4 & \text{if } x = 2, \\ 0 & \text{if } x = 1. \end{cases}$$

Since

$\mu_0 = \{0,\ 1,\ 2,\ 3\}$

$\mu_{1/4} = \{0,\ 2,\ 3\}$

$\mu_{1/3} = \{0,\ 3\}$
$\mu_1 = \{0\},$

as μ_0, μ_1, $\mu_{1/3}$, $\mu_{1/4}$ are ideals and k-ideals are μ_1 and μ_0. There is no proper k-ideal $\mu_m \neq \mu_0$. But μ is a fuzzy 2-prime ideal of R such that $|Im(\mu)| \geq 2$.

THEOREM 3.6. *Let A be any fuzzy subset of R and let R contain a proper k-ideal A_m with $m \neq A(0)$. If A is a fuzzy 2-prime ideal of R, then $A(0) = 1$.*

Proof. Since A is a fuzzy 2-prime ideal, by Theorem 3.4, $|Im(A)| = 2$. Let $Im(A) = \{t,\ s\}$ and $t < s$. Then $A(0) = s$. Suppose that $s \neq 1$. Let $s < n \leq 1$. Let B and C be fuzzy subsets of R such that $B(x) = \frac{1}{2}(t+n)$ for all $x \in R$ and $C(x) = t$ if $x \notin A_m$ $C(x) = n$ if $x \in A_m$, where $A_m = \{x \in R | A(x) \geq m\}$. Clearly B is a fuzzy k-ideal of R. Since A_m is a k-ideal in R, C is a fuzzy k-ideal of R. It can be easily checked that $BC \subseteq A$. As $A(0) = s < n = C(0)$. This implies that $C \nsubseteq A$. Also there exists $x \in R$ such that $A(x) = t < \frac{1}{2}(t+n) = B(x)$. Hence $B \nsubseteq A$. Thus neither $B \subseteq A$ nor $C \subseteq A$. This is a contradiction to the hypothesis that A is a fuzzy 2-prime ideal of R. Hence $A(0) = 1$. □

THEOREM 3.7. *Let A be any fuzzy subset of R. If $|Im(A)| = 2$, $A(0) = 1$ and the set $R_A = \{x \in R | A(x) = A(0)\}$ is a 2-prime ideal of R, then A is a fuzzy 2-prime ideal of R.*

Proof. Let $Im(A) = \{t,\ 1\}$, $t < 1$. Then $A(0) = 1$. Let $x, y \in R$. If $x, y \in R_A$. Then $x + y \in R_A$ and $A(x + y) = 1 = \min\{A(x),\ A(y)\}$. If $x \in R_A$ and $y \notin R_A$, then we have two cases,viz, $x + y \in R_A$ or $x + y \notin R_A$. In both cases, $A(x + y) \geq \min\{A(x),\ A(y)\}$. If $x \notin R_A$ and $y \notin R_A$, then $A(x) = A(y) = t$ and thus $A(x + y) \geq \min\{A(x),\ A(y)\}$. Hence $A(x + y) \geq \min\{A(x),\ A(y)\}$ for all $x, y \in R$. Now if $x \in R_A$, then $xy, yx \in R_A$ and $A(xy) = A(yx) = 1$. If $x \notin R_A$, then $A(xy) \geq A(x) = t$ and $A(yx) \geq A(x) = t$. Hence A is a fuzzy ideal of R. Let B and C be fuzzy k-ideals of R such that $BC \subseteq A$. Suppose that $B \nsubseteq A$ and $C \nsubseteq A$, then there exist $x, y \in R$ such that $B(x) > A(x)$ and $C(y) > A(y)$. Clearly $A(x) = A(y) = t$ implies $x \notin R_A$ and $y \notin R_A$. Now, since R_A is a 2-prime ideal of R, there exist $x_1 \in \langle x \rangle_k$ and $y_1 \in \langle y \rangle_k$ such that $x_1 y_1 \notin R_A$. Thus $A(x) = A(y) = A(x_1 y_1) = t$. Now

$$BC(x_1 y_1) = \sup_{x_1 y_1 = ab} \{\min\{B(a),\ C(b)\}\}$$

$$\geq \min\{B(x_1),\, C(y_1)\}$$
$$> \min\{B(x),\, C(y)\}[\text{ Lemma 3.2 }]$$
$$> t = A(x_1 y_1).$$

Hence $BC \not\subseteq A$, which is a contradiction of the fact that $BC \subseteq A$. Thus either $B \subseteq A$ or $C \subseteq A$. This implies that A is a fuzzy 2-prime ideal of R. □

Theorem 3.8 is an immediate consequence of Lemma 3.3, Theorem 3.6 and Theorem 3.7.

THEOREM 3.8. *Let A be any fuzzy subset of R and R contain a proper k-ideal A_m with $m \neq A(0)$. A is a fuzzy 2-prime ideal of R if and only if $Im(A) = \{1,\, \alpha\}$, where $\alpha \in [0,\, 1)$ and the ideal R_A is a 2-prime ideal of R.*

4. Fuzzy $m_2(m_0$ or $m_1)$-systems

EXAMPLE 4.1. A constant fuzzy subset is a fuzzy $m_2(m_0$ or $m_1)$-system.

THEOREM 4.2. *Let M be a subset of R. M is an m_2 (m_0 or m_1)-system in R if and only if the characteristic function of M, f_M is a fuzzy m_2 (m_0 or m_1)-system in R.*

Proof. Let M be an m_2-system in R. For any $t,\, s \in [0,\, 1)$, suppose there exist $a,\, b \in R$ such that $f_M(a) > t$, $f_M(b) > s$. Hence $a,\, b \in M$. As M is an m_2-system in R, there exist $a_1 \in \langle a \rangle_k$, $b_1 \in \langle b \rangle_k$ such that $a_1 b_1 \in M$, and hence $f_M(a_1 b_1) = 1$. Thus $f_M(a_1 b_1) > \max\{t,\, s\}$.

Conversely, let us assume that f_M is a fuzzy m_2-system in R. Let $a,\, b \in M$. Then $f_M(a) = 1 = f_M(b)$. Thus for any $t,\, s \in [0,\, 1)$ $f_M(a) > t$, $f_M(b) > s$. Hence there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle_k$ such that $f_M(a_1 b_1) > \max\{t,\, s\}$. Therefore $f_M(a_1 b_1) = 1$ and hence $a_1 b_1 \in M$. □

REMARK 4.3. Let μ be a fuzzy subset in R. μ holds a property like subgroup, ideal etc., if and only if its level subset μ_t in R also satisfies the same property in R. However, μ is a fuzzy subset in R such that the level subset μ_t in R is an m_2 (m_0 or m_1)-system in R, for all $t \in [0,\, 1]$, does not imply μ is a fuzzy m_2 (m_0 or m_1)-system of R as the following example shows.

EXAMPLE 4.4. Consider the semiring $R = (Z_6, \oplus_6, \otimes_6)$. Define $\mu : R \to [0,\ 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = 1 \\ .5 & \text{if} \quad x = 3 \\ 0 & \text{if} \quad x = 0,\ 2,\ 4,\ 5. \end{cases}$$

For any $t \in [0,\ 1]$, $\mu_t = \{1\}$ or $\{1,\ 3\}$ or $\{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$. Hence μ_t is an $m_2(m_0$ or $m_1)$-system in R for all t. But μ is not a fuzzy $m_2(m_0$ or $m_1)$-system in R, since $\mu(1) > .9$, $\mu(3) > .4$ but there is no $a_1 \in \langle 1 \rangle_k$ and $b_1 \in \langle 3 \rangle_k$ such that $\mu(a_1 b_1) > \max\{.9,\ .4\}$.

However we have the following Theorem.

THEOREM 4.5. *Let μ be a fuzzy subset in R with $x_1 \in \langle x \rangle_k$ ($x_1 \in \langle x \rangle$) implies $\mu(x_1) \geq \mu(x)$. μ is a fuzzy $m_2(m_0$ or $m_1)$-system in R if and only if $\mu^r = \{x \in R/\mu(x) > r\}$ is an m_2 (m_0 or m_1)-system in R for all $r \in [0,\ 1)$*

Proof. Let μ be a fuzzy m_2-system in R. Let $x, y \in \mu^r$, for some $r \in [0, 1)$. This implies that $\mu(x) > r$ and $\mu(y) > r$. As μ is a fuzzy m_2-system in R, there exist $x_1 \in \langle x \rangle_k$ and $y_1 \in \langle y \rangle_k$ such that $\mu(x_1 y_1) > r$ implies $x_1 y_1 \in \mu^r$. Thus μ^r is an m_2-system in R.

Conversely, let us assume that μ^r is an m_2-system in R for all $r \in [0,\ 1)$. Suppose that $\mu(x) > r$ and $\mu(y) > s$, for some $r,\ s \in [0,\ 1)$ and $x,\ y \in R$. If $r = s$, the result is immediate. Without loss of generality let us assume that $s > r$. Now $\mu(x) > r$ and $\mu(y) > s > r$. Since μ^r is an m_2-system in R, then there exist $x_1 \in \langle x \rangle_k$ and $y_1 \in \langle y \rangle_k$ such that $\mu(x_1 y_1) > r$. Now $x_1 y_1 \in \langle y \rangle_k$ and $\mu(x_1 y_1) \geq \mu(y) > s$. Thus μ is a fuzzy m_2-system of R. □

THEOREM 4.6. *Let A be a fuzzy ideal of R and R contain a proper k-ideal A_m with $m \neq A(0)$. A is a fuzzy 2-(0- or 1-) prime ideal of R if and only if $1 - A$ is a fuzzy m_2 (m_0 or m_1)-system of R.*

Proof. Let us assume that A is a fuzzy 2-prime ideal of R. For any $t,\ s \in [0,\ 1)$, suppose there exist $a,\ b \in R$ such that $(1 - A)(a) > t$ and $(1 - A)(b) > s$. Hence $A(a) < 1 - t$ and $A(b) < 1 - s$. As A is a fuzzy 2-prime ideal of R, $Im(A) = \{1,\ \alpha\}$, $\alpha \in [0,\ 1)$. Thus $\alpha < 1 - t$, $\alpha < 1 - s$ and $A(a) = A(b) = \alpha$. Let $P = \{x \in R/A(x) = 1\}$. Then by Theorem 3.8, P is a 2-prime ideal in R and $a,\ b \notin P$. This implies $a,\ b \in R|P$ which is an m_2-system in R. Thus there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle_k$ such that $A(a_1 b_1) = \alpha$. Now $A(a_1 b_1) = \alpha < \min\{(1-t),\ (1-s)\} = 1 - \max\{t,\ s\}$. Now $\max\{t,\ s\} < 1 - A(a_1 b_1) = (1 - A)(a_1 b_1)$.

Conversely, let us assume that $1 - A$ is a fuzzy m_2-system of R. Let A_1, A_2 be two fuzzy k-ideals such that $A_1\ A_2 \subseteq A$. Suppose that $A_1 \nsubseteq A$ and $A_2 \nsubseteq A$. Now $A_1 = \bigcup_{a_p \in A_1} a_p$ and $A_2 = \bigcup_{b_q \in A_2} b_q$. Then there exist $a_s \in A_1$ and $b_t \in A_2$, $s, t \in [0, 1)$, such that $A(a) < s$ and $A(b) < t$. This implies that $1 - A(a) > 1 - s$, $1 - A(b) > 1 - t$. As $1 - A$ is a fuzzy m_2-system of R, there exist $a_1 \in \langle a \rangle_k$ and $b_1 \in \langle b \rangle_k$ such that $(1 - A)(a_1 b_1) > \max\{(1 - s), (1 - t)\} = 1 - \min\{s, t\}$. Thus $A(a_1 b_1) < \min\{s, t\}$ and $(a_1 b_1)_{\min\{s,t\}} \notin A$. Now by Lemma 2.29 and Lemma 2.30, $(a_1 b_1)_{\min\{s,t\}} = (a_1)_s\ (b_1)_t \in A_1\ A_2 \subseteq A$, a contradiction. Therefore A is a fuzzy 2-prime ideal of R. □

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