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A STRONG SOLUTION FOR THE WEAK TYPE II GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS

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ABSTRACT. The aim of this paper is to give an existence theorem for a strong solution of generalized vector quasi-equilibrium problems of the weak type II due to Hou et al. using the equilibrium existence theorem for 1-person game, and as an application, we shall give a generalized quasivariational inequality.

1. Introduction

Equilibrium problems include various problems related to optimization theory. In particular, equilibrium problems contain a number of important problems as fixed point problems, coincidence point problems, Nash equilibria problems, variational inequalities, complementarity problems, maximization problems and so on. Till now equilibrium problems have been generalized in diverse directions, and generalized vector quasi-equilibrium problems (GVQEP) have been extensively studied by many authors recently (e.g., see [1, 3-5, 7-10]). In most papers, main interests have been paid to get such sufficient conditions for more general problem settings and under weaker assumptions about continuity, convexity, compactness and monotonicity.

Until now, only a few papers deal with these problems in the strong sense, and most of results need the various conditions of pseudomonotonicity and convexity to obtain the solutions of weak sense. Also in most proofs of previous existence theorems on equilibrium problems, the proving tools are either the Fan-Glicksberg fixed point theorem or the (generalized) KKM theorem, and using those theorems, we shall

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need some additional assumptions to find solutions for (GVQEP) as in [1, 3-5, 7-10].

In a recent paper [9], Hou et al. investigated the four types of generalized vector quasi-equilibrium problems, and using weak type of C-diagonal quasiconvexity, they obtained an existence theorem of solution for the weak type II generalized vector quasi-equilibrium problem.

In this paper, we shall use the equilibrium existence theorem for 1-person game due to Yuan [18] as a basic tool for finding a strong solution of the weak type II (GVQEP). And as an application, we shall give a generalized quasivariational inequality using the monotone assumption.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by int A the interior of A in X. If A is a subset of a vector space, we shall denote by $co\,A$ the convex hull of A. If A is a nonempty subset of a topological vector space X and $S, T : A \to 2^X$ are multifunctions (or correspondences), then $co\,T$, $T \cap S : A \to 2^X$ are multifunctions defined by $(co\,T)(x) = co\,T(x)$, $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

Let X, Y and Z be real Hausdorff topological vector spaces, and $K \subseteq X$, $D \subseteq Y$. Let $C : K \to 2^Z$ be a multifunction such that for each $x \in K$, $C(x)$ is a closed convex solid cone in Z with $C(x) \neq Z$ (i.e., int $C(x) \neq \emptyset$ for each $x \in K$). Let $S : K \to 2^K, A : K \to 2^D$ and $F : K \times D \times K \to 2^Z$ be given multifunctions.

In a recent paper [9], Hou et al. introduced the following generalized vector quasi-equilibrium problems (GVQEP) with multifunctions *(S, A, F)* as follow:

Weak Type I (GVQEP): Find a pair of points $\bar{x} \in K$ and $\bar{y} \in A(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, u) \cap -\text{int } C(\bar{x}) = \emptyset \quad \text{for all} \quad u \in S(\bar{x});$$

Weak Type II (GVQEP): Find a pair of points $\bar{x} \in K$ and $\bar{y} \in A(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, u) \not\subseteq -\text{int } C(\bar{x}) \quad \text{for all} \quad u \in S(\bar{x}).$$

In the above two types of (GVQEP), we may call an $\bar{x} \in K$ *a strong solution* of the problem of the weak type I (resp., weak type II) (GVQEP) if $\bar{y}$ does not depend on $\bar{x} \in K$ and $u \in S(\bar{x})$, i.e., there exists $\bar{x} \in K$ such that $F(\bar{x}, \bar{y}, u) \cap -\text{int } C(\bar{x}) = \emptyset$ (resp., $F(\bar{x}, \bar{y}, u) \not\subseteq -\text{int } C(\bar{x})$) for all $u \in S(\bar{x})$ and $\bar{y} \in A(\bar{x})$. On the other hand, an $\bar{x} \in K$ is said to be *a weak solution* of the problem of the weak type I (resp., weak type II) (GVQEP) if there exists $\bar{y} \in A(\bar{x})$ depending on $\bar{x} \in K$ such that $F(\bar{x}, \bar{y}, u) \cap -\text{int } C(\bar{x}) = \emptyset$ (resp., $F(\bar{x}, \bar{y}, u) \not\subseteq -\text{int } C(\bar{x})$) for all $u \in S(\bar{x})$.

Until now, most of papers deal with these problems in the weak sense, and existence results need the various conditions of monotonicity and general convexity assumptions to obtain the solutions of weak sense. In a recent paper [9], Hou et al. investigated the four types of generalized vector quasi-equilibrium problems, and they obtained an existence theorem of solution in the weak sense for the weak type II generalized vector quasi-equilibrium problem by using weak type II C-diagonal quasiconvexity; on the other hand, we are now interested in the existence theorem of solutions in the strong sense of the weak type II (GVQEP) by using $C(x)$-quasiconvex-like condition.

For more special forms of vector equilibrium problem and vector quasi-equilibrium problems about its weak solutions, readers can refer to [1, 3-5,7-11].

In this paper, we shall prove the existence theorem of strong solutions of the problem (GVQEP) of the weak type II, which includes many kinds of vector variational inequalities as special cases. In particular, when the multifunction F is a single-valued function and the moving cone $C(x)$ is a constant cone, the problem (GVQEP) of the weak type II reduces to the generalized vector equilibrium problem studied in [1].

Let K be a nonempty convex subset of a real vector space X. Then a multifunction $F : K \times K \to 2^Z$ is said to be *$C(x)$-quasiconvex-like* [11] if for any $x \in K$, $y_1, y_2 \in K$ and $t \in [0, 1]$, we have either $F(x, ty_1 + (1 - t)y_2) \subseteq F(x, y_1) - C(x)$ or $F(x, ty_1 + (1 - t)y_2) \subseteq F(x, y_2) - C(x)$.

When F is single-valued and the moving cone $C(x)$ is a constant cone C, the $C(x)$-quasiconvex-like multifunction reduces to a properly quasi-convex function defined in [1].

Now we shall need the $C(x)$-quasiconvex-like condition for two variables as follows:

DEFINITION 1. Let K be a nonempty convex subset of a real vector space X, D be a nonempty convex subset of a real vector space Y, and C a multifunction of a set $K \subseteq X$ into Z such that for any $x \in K$, $C(x)$ is a closed convex solid cone in Z with $C(x) \neq Z$. Let $F : K \times D \times K \to 2^Z$ be a multifunction. Then F is called *$C(x)$-quasiconvex-like* on $K \times D$ if for each $x \in K$, $(x_1, y_1), (x_2, y_2) \in K \times D$, and for each $t \in [0,1]$, we have either

$$F\big(t(x_1, y_1) + (1-t)(x_2, y_2), x\big) \subseteq F(x_1, y_1, x) - C(x)$$

or

$$F\big(t(x_1, y_1) + (1-t)(x_2, y_2), x\big) \subseteq F(x_2, y_2, x) - C(x).$$

It is well-known that the quasiconvexity for each variable can not imply the quasiconvexity for two variables simultaneously. In fact, when F is single-valued and the moving cone $C(x) \equiv [0, \infty)$ for each $x \in K$, the following simple example confirms this fact:

EXAMPLE. If $f : K \times D \to \mathbb{R}$ is a quasiconvex function on $K \times D$, then it is easy to see that f is a quasiconvex function on K by letting $y_1 = y_2$ in the definition, and also f is quasiconvex on D. However, the converse can not be true. In fact, in the case $K = D = [0,1], Z = \mathbb{R}$, and $C(x) \equiv [0, \infty)$, we consider a real-valued function $f : K \times D \times K \to \mathbb{R}$ defined by

$$f(x, y, z) := xy + z \quad \text{for each} \quad (x, y, z) \in K \times D \times K.$$

Then it is easy to see that f is a quasiconvex function on K and D, respectively. However, f is not a quasiconvex function on $K \times D$. In fact, when $(x_1, y_1) = (0,1), (x_2, y_2) = (1,0)$, we can see that $f(\frac{1}{2}, \frac{1}{2}, z) > f(0,1,z)$ and $f(\frac{1}{2}, \frac{1}{2}, z) > f(1,0,z)$ for all $z \in [0,1]$.

Let X, Y be nonempty topological spaces and $T : X \to 2^Y$ be a multifunction. A multifunction $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subseteq V$, there exists an open neighborhood U of x in X such that $T(y) \subseteq V$ for each $y \in U$; and a multifunction $T : X \to 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. And f is said to be *continuous* if f is both lower semicontinuous and upper semicontinuous. It is also known that $T : X \to 2^Y$ is lower semicontinuous if and only if for each closed set V in Y, the set $\{x \in X \mid T(x) \subseteq V\}$ is closed in X.

A topological space is *perfectly normal* if it is normal, and if every open subset is an F_σ. Then, by Proposition 3 due to Michael in [12], we know that every F_σ subset of a paracompact space is also paracompact.

The following equilibrium existence theorem for 1-person game is essential in proving our main results, and is the particular case of Theorem 2.4 due to Yuan [18]:

LEMMA 1. *Let* $\Gamma = (X, A, P)$ *be an 1-person game such that*

(1) X *is a nonempty compact convex subset of a locally convex Hausdorff topological vector space* E;

(2) *the correspondence* $A : X \to 2^X$ *is continuous such that for each* $x \in X, A(x)$ *is nonempty closed convex*;

(3) *the correspondence* $P : X \to 2^X$ *is lower semicontinuous such that* $x \notin coP(x)$ *for each* $x \in X$;

(4) *the set* $\{x \in X \mid A(x) \cap coP(x) \neq \emptyset\}$ *is open and paracompact.*

Then Γ *has an equilibrium* $\bar{x} \in X$, *i.e.,*

$$\bar{x} \in A(\bar{x}) \quad \text{and} \quad A(\bar{x}) \cap P(\bar{x}) = \emptyset.$$

3. A strong solution of (GVQEP) and its applications

We now prove a new existence theorem for a strong solution for (GVQEP) of the weak type II as follows:

THEOREM 1. *Let* X, Y *and* Z *be locally convex real Hausdorff topological vector spaces, and* K *be a nonempty compact convex perfectly normal subset of* X, D *a nonempty closed convex subset of* Y. *Let* $C : K \to 2^Z$ *be a multifunction such that for any* $x \in K$, $C(x)$ *is a closed convex solid cone in* Z *with* $C(x) \neq Z$, *and let* $F : K \times D \times K \to 2^Z$ *be a multifunction.*

Assume that

(1) *the multifunction* $S : K \to 2^K$ *be continuous such that each* $S(x)$ *is a nonempty closed convex subset of* K;

(2) *the multifunction* $A : K \to 2^D$ *be lower semicontinuous such that each* $A(x)$ *is a nonempty convex subset of* D;

(3) *for each* $x \in K$, $F(x, y, x) \not\subseteq -\text{int } C(x)$ *for all* $y \in A(x)$;

(4) *for each* $u \in K$, $(x, y) \mapsto F(x, y, u)$ *is a* $C(u)$*-quasiconvex-like function of* (x, y) *on* $K \times D$;

(5) *for each net* $(x_\alpha, y_\alpha, z_\alpha) \in K \times D \times K$ *converging to* $(\bar{x}, \bar{y}, \bar{z})$, *if* $F(x_\alpha, y_\alpha, z_\alpha) \not\subseteq -\text{int } C(z_\alpha)$ *for each* α, *then* $F(\bar{x}, \bar{y}, \bar{z}) \not\subseteq -\text{int } C(\bar{z})$.

Then there exists a strong solution $\hat{x} \in K$ *of the weak type II (GVQEP) such that* $\hat{x} \in S(\hat{x})$ *and*

$$F(x, y, \hat{x}) \not\subseteq -\text{int } C(\hat{x}) \quad \text{for all } x \in S(\hat{x}) \text{ and } y \in A(\hat{x}).$$

Proof. We first define a multifunction $T : K \to 2^K$ by, for each $x \in K$,

$$T(x) := \{x' \in K \mid F(x', y, x) \subseteq -\text{int } C(x) \text{ for some } y \in A(x)\}.$$

Note that $T(x)$ might be an empty-set. By the assumption (3), it is easy to see that $x \notin T(x)$ for each $x \in K$. We now show that each $T(x)$ is convex. Suppose the contrary, i.e., let $x \in K$ be fixed such that there exist $x_1, x_2 \in T(x)$ and $\bar{x} = tx_1 + (1-t)x_2 \notin T(x)$ for some $t \in (0,1)$. Since $x_1, x_2 \in T(x)$, there exist $y_1, y_2 \in A(x)$, respectively, such that $F(x_1, y_1, x) \subseteq -\text{int } C(x)$ and $F(x_2, y_2, x) \subseteq -\text{int } C(x)$. Since $A(x)$ is convex, $\bar{y} := ty_1 + (1-t)y_2 \in A(x)$. By the assumption (4), F is a $C(x)$-quasiconvex-like function on $K \times D$ so that for each $t \in [0,1]$, we have either

$$F\big(t(x_1, y_1) + (1-t)(x_2, y_2), x\big) \subseteq F(x_1, y_1, x) - C(x)$$

or

$$F\big(t(x_1, y_1) + (1-t)(x_2, y_2), x\big) \subseteq F(x_2, y_2, x) - C(x).$$

Since $C(x)$ is a closed convex solid cone, we have $C(x) + \text{int } C(x) \subseteq \text{int } C(x)$. Hence, for $\bar{y} \in A(x)$, we have either

$$F(\bar{x}, \bar{y}, x) \subseteq F(x_1, y_1, x) - C(x) \subseteq -\text{int } C(x) - C(x) \subseteq -\text{int } C(x)$$

or

$$F(\bar{x}, \bar{y}, x) \subseteq F(x_2, y_2, x) - C(x) \subseteq -\text{int } C(x) + C(x) \subseteq -\text{int } C(x);$$

so that in either cases, we have $F(\bar{x}, \bar{y}, x) \subseteq -\text{int } C(x)$. Hence we have $\bar{x} = tx_1 + (1-t)x_2 \in T(x)$, which is a contradiction.

Next we shall show that the multifunction $T : K \to 2^K$ has open lower sections, i.e., $T^{-1}(x')$ is open in K for each $x' \in K$. If so, T is automatically lower semicontinuous by Lemma 5.1 in [17]. First, note that

$$T^{-1}(x') \equiv \{x \in K \mid F(x', y, x) \subseteq -\text{int } C(x) \text{ for some } y \in A(x)\}$$
$$\text{is open in } K$$
$$\Longleftrightarrow$$
$$K \setminus T^{-1}(x') \equiv \{x \in K \mid F(x', y, x) \not\subseteq -\text{int } C(x) \text{ for all } y \in A(x)\}$$
$$\text{is closed in } K.$$

Let $W := K \setminus T^{-1}(x')$. In order to show that W is closed in K, it suffices to show that if (x_α) is a net in W converging to $\bar{x} \in K$, then $\bar{x} \in W$, i.e., $F(x', y, \bar{x}) \not\subseteq -\text{int } C(\bar{x})$ for all $y \in A(\bar{x})$. Suppose the contrary, i.e., $F(x', y', \bar{x}) \subseteq -\text{int } C(\bar{x})$ for some $y' \in A(\bar{x})$. Since A is lower semicontinuous, and $(x_\alpha) \to \bar{x}$, $y' \in A(\bar{x})$, there exists a net

$y_\alpha \in A(x_\alpha)$ such that $(y_\alpha) \to y'$. Since $F(x', y_\alpha, x_\alpha) \not\subseteq -\text{int } C(x_\alpha)$ for each α, by the assumption (5), we have $F(x', y', \bar{x}) \not\subseteq -\text{int } C(\bar{x})$; which is a contradiction. Hence, $\bar{x} \in W$ so that $T^{-1}(x')$ is open in K for each $x' \in K$. Therefore, by Lemma 5.1 in [17], T is lower semicontinuous.

Next we shall show that the set $\{x \in K | S(x) \cap T(x) \neq \emptyset\}$ is open in K. As in the previous argument, we shall show that the set $W_1 := \{x \in K | S(x) \cap T(x) = \emptyset\}$ is closed in K. Let (x_α) be a net in W_1 converging to $\hat{x} \in K$. Then, for all $x' \in S(x_\alpha)$, $F(x', y, x_\alpha) \not\subseteq -\text{int } C(x_\alpha)$ for all $y \in A(x_\alpha)$. We must show that for all $x \in S(\hat{x}), F(x, y, \hat{x}) \not\subseteq -\text{int}\, C(\hat{x})$ for all $y \in A(\hat{x})$. Suppose the contrary, i.e., there exist $x'' \in S(\hat{x}), y' \in A(\hat{x})$ such that $F(x'', y', \hat{x}) \subseteq -\text{int}\, C(\hat{x})$. Since S and A are lower semicontinuous multifunctions, and $(x_\alpha) \to \hat{x}$, there exist two convergent nets

$$x'_\alpha \in S(x_\alpha) \text{ such that } (x'_\alpha) \to x'', \text{ and } y'_\alpha \in A(x_\alpha) \text{ such that } (y'_\alpha) \to y'.$$

Since $F(x'_\alpha, y'_\alpha, x_\alpha) \not\subseteq -\text{int } C(x_\alpha)$, by the assumption (5) again, we have $F(x'', y', \hat{x}) \not\subseteq -\text{int } C(\hat{x})$; which is a contradiction.

Applying Lemma 1, it remains to show the assumption (4) of Lemma 1. Since K is compact and perfectly normal, $\{x \in K | S(x) \cap T(x) \neq \emptyset\}$ is an open subset of K so that it is an F_σ subset of a compact set K; and hence it is a paracompact set. Therefore, the whole assumptions of Lemma 1 are satisfied so that there exists a solution $\hat{x} \in K$ such that $\hat{x} \in S(\hat{x})$ and $S(\hat{x}) \cap T(\hat{x}) = \emptyset$, i.e., for all $x \in S(\hat{x})$, $x \notin T(\hat{x})$. This implies that $F(x, y, \hat{x}) \not\subseteq -\text{int } C(\hat{x})$ for all $x \in S(\hat{x})$ and $y \in A(\hat{x})$. This completes the proof. □

REMARKS. (i) If K is not assumed to be perfectly normal, we shall need the following additional assumption to assure the same conclusion:

(5′) the set $\{x' \in S(x') \,|\, F(x', y, x) \subseteq -\text{int } C(x) \text{ for some } y \in A(x)\}$ is (possibly empty) paracompact.

(ii) Theorem 1 is comparable to Theorem 1 in [7] and Theorem 3.2 in [1], and the assumptions are different from each other. In fact, we do not need the quasi-completeness of D and upper semicontinuity of A. However, we do need the lower semicontinuity of A and the perfectly normality of K, and the quasiconvexity for the counterpart variables of f. Hence, in some sense, Theorem 1 is a dual vector form of Theorem 1 in [7].

(iii) If F is a single-valued and real-valued upper semicontinous function and the moving cone $C(x)$ is a constant cone $C(x) \equiv [0, \infty)$ for each $x \in K$, then the assumption (5) is automatically satisfied.

The following result due to Yang-Liu [16] is a useful characterization of quasiconvexity for real-valued functions:

LEMMA 2. *Let X be a nonempty convex set in a locally convex Hausdorff topological vector space and $f : X \to \mathbb{R}$ be a lower semicontinuous function. If for every $x, y \in X$, there exists $t \in (0,1)$ such that*

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\},$$

then f is a quasiconvex function on X.

Here we note that in Theorem 2.1 in [16], X is assumed to be a nonempty convex set in R^n; however X might be a nonempty convex set in a locally convex Hausdorff topological vector space without affecting the conclusion.

As we mentioned before, in the scalar case ($Z \equiv \mathbb{R}$ and $C(x) \equiv [0, \infty)$ for each $x \in K$), the $C(x)$-quasiconvex-like condition and the quasiconvexity are equivalent to each other. Hence the following is a direct consequence of Lemma 2 :

LEMMA 3. *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space E and D be a nonempty convex set of a locally convex Hausdorff topological vector space F. Let $f : K \times D \to \mathbb{R}$ be a lower semicontinuous function on $K \times D$ such that each $(x_1, y_1), (x_2, y_2) \in K \times D$, there exists a real number $t \in (0,1)$ satisfying $f(t(x_1, y_1) + (1-t)(x_2, y_2)) \leq \max\{f(x_1, y_1), f(x_2, y_2)\}$. Then f is a quasiconvex function on $K \times D$.*

For some applications of (GVQEP) on generalized quasi-variational inequalities, we shall need a scalarization procedure as in Oettli [13]. Let E be a real Hausdorff topological vector space. We shall denote by E' the dual space of E (i.e., the vector space of all continuous linear functionals on E). We denote the pairing between E' and E by $\langle w, x \rangle$ for $w \in E'$ and $x \in E$. Let K be a nonempty compact convex set of E and D be a nonempty convex set of a dual space E'.

Let us introduce some notions. A nonempty subset $K \times D$ in $E \times E'$ is called *monotone* if for each $(x_1, y_1), (x_2, y_2) \in K \times D$, $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$. And we say that $T : K \to 2^D$ is a *monotone* mapping on K if the graph of T is a monotone subset of $E \times E'$, i.e., for each $y_1 \in T(x_1), y_2 \in T(x_2)$, we have $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$. When $T : K \to \mathbb{R}$ is a single-valued and real-valued function, it is easy to see that T is a monotone mapping if and only if T is nondecreasing. And it is also known that the subdifferential ∂f of any proper convex function f is a monotone mapping.

Let $f : K \times D \times K \to \mathbb{R}$ be a real-valued function defined by

$$f(x,y,u) := \langle y, x-u\rangle \quad \text{for each} \quad (x,y,u) \in K \times D \times K.$$

Then, we may call that f is *monotone* on $K \times D$ if for each $u \in K$, $f(x_1,y_1,u)+f(x_2,y_2,u) \geq f(x_1,y_2,u)+f(x_2,y_1,u)$ for all (x_1,y_1), $(x_2,y_2) \in K \times D$.

Let K be a nonempty compact convex subset of a real Hausdorff topological vector space E and D be a nonempty convex subset of a dual space E'.

Then we can see the following

LEMMA 4. *Let $u \in K$ be fixed, and $f : K \times D \to \mathbb{R}$ be a real-valued function defined by $f(x,y,u) := \langle y, x-u\rangle$ for each $(x,y) \in K \times D$. Then*

$K \times D \subseteq E \times E^*$ is monotone $\iff$ f is monotone on $K \times D$.

Proof. For each $(x_1,y_1),(x_2,y_2) \in K \times D$, we have the following

$$\begin{aligned} &f(x_1,y_1,u)+f(x_2,y_2,u)-f(x_1,y_2,u)-f(x_2,y_1,u) \\ &= \langle y_1, x_1-u\rangle + \langle y_2, x_2-u\rangle - \langle y_1, x_2-u\rangle - \langle y_2, x_1-u\rangle \\ &= \langle y_1-y_2, x_1-x_2\rangle. \end{aligned}$$

Therefore, we obtain the conclusion. □

Now, using a scalarization procedure, we can interpret the strong solution $\hat{x} \in K$ of the problem (GVQEP) in Theorem 1 into the following:

$$\begin{aligned} &\hat{x} \in S(\hat{x}) \text{ and } f(\hat{x},y,x) \geq 0 \quad \text{for all } x \in S(\hat{x}) \text{ and } y \in A(\hat{x}) \\ \Leftrightarrow\ &\hat{x} \in S(\hat{x}) \text{ and } \langle y, \hat{x}-x\rangle \geq 0 \quad \text{for all } x \in S(\hat{x}) \text{ and } y \in A(\hat{x}) \\ \Leftrightarrow\ &\hat{x} \in S(\hat{x}) \text{ and } \inf_{x\in S(\hat{x})} \inf_{y\in A(\hat{x})} \langle y, \hat{x}-x\rangle \geq 0. \end{aligned}$$

In order to apply Theorem 1 to f, we shall need the following

THEOREM 2. *Let K be a nonempty compact convex subset of a real Hausdorff topological vector space E and D be a nonempty convex subset of a dual space E'. Let $f : K \times D \times K \to \mathbb{R}$ be a real-valued function defined by*

$$f(x,y,u) := \langle y, x-u\rangle \quad \textit{for each} \quad (x,y,u) \in K \times D \times K.$$

If $K \times D$ is a monotone subset of $E \times E'$, then f is quasiconvex on $K \times D$.

Proof. By Lemma 4, f is monotone on $K \times D$, for each $u \in K$, and hence we have

$$\langle y_1, x_1 - u\rangle + \langle y_2, x_2 - u\rangle \ \geq \ \langle y_1, x_2 - u\rangle + \langle y_2, x_1 - u\rangle$$

for all $(x_1, y_1), (x_2, y_2) \in K \times D$. Here, if we let $a := \langle y_1, x_1 - u\rangle, b := \langle y_2, x_2 - u\rangle, c := \langle y_1, x_2 - u\rangle, d := \langle y_2, x_1 - u\rangle$, then we have $a+b \geq c+d$. By Lemma 3, in order to prove the quasiconvexity of f on $K \times D$, we must find a real number $t \in (0,1)$ such that $f\big(t(x_1, y_1)+(1-t)(x_2, y_2)\big) \leq \max\{f(x_1, y_1), f(x_2, y_2)\}$. That is, we must find $t \in (0,1)$ satisfying the quadratic inequality

$$(*) \qquad t^2(a+b-c-d)+t(c+d-2b)+b \leq \max\{a, b\}.$$

By simple calculations, we can show that there always exists a real number $t \in (0,1)$ satisfying the quadratic inequality $(*)$ in either case of (i) $a+b > c+d$ or (ii) $a+b = c+d$. Therefore, f is quasiconvex on $K \times D$. □

Finally, when F is a single-valued and real-valued continuous function, as an application of Theorem 1, we can obtain a generalized quasivariational inequality as follows :

THEOREM 3. *Let K be a nonempty compact convex subset of a locally convex real Hausdorff topological vector space E and D be a nonempty convex subset of a dual space E'. Let $S : K \to 2^K$ be continuous such that each $S(x)$ is a nonempty closed convex subset of K, and $A : K \to 2^D$ be lower semicontinuous such that each $A(x)$ is a nonempty convex subset of D. Assume that $K \times D$ is a monotone subset of $E \times E'$.*

Then there exists a solution $\hat{x} \in K$ such that $\hat{x} \in S(\hat{x})$ and

$$\inf_{y \in A(\hat{x})} \langle y, \hat{x} - x\rangle \ \geq \ 0 \quad \text{for all } \ x \in S(\hat{x}).$$

Proof. We first define a real-valued function $f : K \times D \times K \to \mathbb{R}$ by

$$f(x, y, u) := \langle y, x - u\rangle \quad \text{for each } \ (x, y, u) \in K \times D \times K.$$

Then, f is clearly continuous on $K \times D \times K$, and by Lemma 4, $f(\cdot, \cdot, u)$ is monotone on $K \times D$ for each $u \in K$. Hence, by Theorem 2, f is quasiconvex on $K \times D$. Also, for each $x \in K$, we have $f(x, y, x) = \langle y, x - x\rangle = 0$ for all $y \in A(x)$. Therefore, the whole assumptions of Theorem 1 are satisfied so that there exists a point $\hat{x} \in K$ such that

$$\hat{x} \in S(\hat{x}) \ \text{ and } \ f(x, y, \hat{x}) \geq 0 \quad \text{for all } \ x \in S(\hat{x}) \text{ and } \ y \in A(\hat{x}).$$

Hence we have $\inf_{y \in A(\hat{x})} \langle y, \hat{x} - x\rangle \ \geq \ 0$ for all $x \in S(\hat{x})$, which completes the proof. □

REMARK. Theorem 3 can be comparable to Theorem 2 of Shih-Tan [15]. In fact, A need not be monotone but we shall need a monotone condition for the set $K \times D$.

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