Bull. Korean Math. Soc. **43** (2006), No. 3, pp. 611–617

ON THE ALTERNATING SUMS OF POWERS OF CONSECUTIVE q-INTEGERS

Seog-Hoon Rim, Taekyun Kim, and Cheon Seoung Ryoo

Abstract. In this paper we construct q-Genocchi numbers and polynomials. By using these numbers and polynomials, we investigate the q-analogue of alternating sums of powers of consecutive integers due to Euler.

1. Introduction

The Genocchi numbers G_m are defined by the generating function:

$$F(t) = \frac{2t}{e^t+1} = \sum_{m=0}^{\infty} G_m \frac{t^m}{m!}, (|t| < \pi) \text{ (cf. [3])}, \tag{1}$$

where we use the technique method notation by replacing G^m by $G_m(m \geq 0)$ symbolically. It satisfies $G_1 = 1, G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given $G_m = 2(1-2^{2m})B_{2m} = 2mE_{2m-1}$, where B_m are Bernoulli numbers and E_m are Euler numbers which are defined by

$$\frac{2}{e^t+1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \quad \text{(cf. [5, 6, 8])}.$$

For $x \in \mathbb{R}$ (= the field of real numbers), we consider the Genocchi polynomials as follows:

$$F(x,t) = F(t)e^{xt} = \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \tag{2}$$

Received October 11, 2005.

2000 Mathematics Subject Classification: 11S80, 11B68.

Key words and phrases: Genocchi numbers and polynomials, q-Genocchi numbers and polynomials, alternating sums of powers.

Note that $G_m(x) = \sum_{k=0}^{m} \binom{m}{k} G_k x^{m-k}$. Let us also define the Genocchi polynomials of order r as follows:

$$2\left(\frac{1}{1+e^t}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!} \text{ (cf. [3]).}$$

In the special case $x = 0$, we define $G_n^{(r)}(0) = G_n^{(r)}$. What is the value of the following sum for a given positive integer k ?

$$1^k + 2^k + 3^k + \cdots + n^k.$$

Let us denote this sum by $f_k(n)$. Finding formulas for $f_k(n)$ has interested mathematicians for more than 300 years since the time of Jacob Bernoulli (cf. [1, 7, 9, 11]). It was well known that

$$\text{(3)} \qquad f_n(k-1) = \frac{1}{n+1}\sum_{i=0}^{n}\binom{n+1}{i}B_i k^{n+1-i} \text{ (cf. [9, 11]),}$$

where $\binom{n}{k}$ is binomial coefficients.

Let n, k be positive integers ($k > 1$), and let

$$T_n(k) = -1^k + 2^k - 3^k + 4^k - 5^k + \cdots + (-1)^{k-1}(n-1)^k.$$

Following an idea due to Euler, it was known that

$$\text{(4)} \quad T_n(k) = \frac{(-1)^{k+1}}{2}\sum_{l=0}^{k-1}\binom{n}{l}E_l k^{n-l} + \frac{E_n}{2}\left(1 + (-1)^{k+1}\right) \text{ (cf. [5]).}$$

Let q be an indeterminate which can be considered in complex number field, and for any integer k define the q-integer as

$$[k]_q = \frac{q^k - 1}{q - 1} = 1 + q + \cdots + q^{k-1}.$$

Throughout this paper we assume that $q \in \mathbb{C}$ with $0 < q < 1$. Recently many authors studied q-analogue of the sums of powers of consecutive integers. In [2], Garrett and Hummel gave a combinatorial proof of a q-analogue of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$ as follows:

$$\sum_{k=1}^{n} q^{k-1}[k]_q^2\left(\left[\frac{k-1}{2}\right]_{q^2} + \left[\frac{k+1}{2}\right]_{q^2}\right) = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k}\frac{[n+1-j]_q}{[j]_q} \text{ denotes the } q\text{-binomial coefficients.}$$

Garrett and Hummel, in their paper, asked for a simpler q-analogue of the sums of cubes. As a response to Garrett and Hummel 's question, Warnaar gave a simple q-analogue of the sums of cubes as follows:

$$\sum_{k=1}^{n} q^{2n-2k}[k]_q^2[k]_{q^2} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2 \text{ (cf. [12, 13]).}$$

Let

$$f_{m,q}(n) = \sum_{k=1}^{n} [k]_{q^2}[k]_q^{m-1} q^{(n-k)\frac{m+1}{2}}.$$

Then we note that $\lim_{q\to 1} f_{m,q}(n) = f_m(n)$ (cf. [10, 12]).

Warnaar [13] (for $m = 3$) and Schlosser [10] gave formulae for $m = 1, 2, 3, 4, 5$ as the meaning of the q-analogues of the sums of consecutive integers, squares, cubes, quarts and quints. Let $n, k(> 1)$ be positive integers. In the recent paper, it was known that

$$\sum_{j=0}^{k-1} q^j[j]_q^n = \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}\beta_{j,q}q^{kj}[k]_q^{n+1-j} - \frac{1}{n+1}(1-q^{(n+1)k})\beta_{n+1,q} \text{ (see [4, 6, 7])},$$

where $\beta_{j,q}$ are called Carlitz's q-Bernoulli numbers. Originally q-Genocchi numbers and polynomials were introduced by Kim-Jang-Pak in 2001 [3], but they do not seem to be the most natural ones. In this paper we give another construction of q-Genocchi numbers and polynomials which are different than q-Genocchi numbers and polynomials of Kim-Jang-Pak in 2001 [3]. By using these numbers and polynomials, we investigate the q-analogue of alternating sums of powers of consecutive integers.

2. q-Genocchi numbers and polynomials

Let $F_{q,k}(t)$ be the generating functions of the q-Genocchi numbers as follows:

$$F_{q,k}(t) = [2]_q t\sum_{j=0}^{\infty} q^{k-j}[j]_{q^2}(-1)^{j-1}\exp\left(t[j]_{q^2}q^{\frac{k-j}{2}}\right) = \sum_{j=0}^{\infty} G_{n,q}\frac{t^n}{n!}$$

By using Taylor expansion in the above, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,k,q}\frac{t^n}{n!}\\
&= [2]_q t\sum_{j=0}^{\infty} q^{k-j}[j]_{q^2}(-1)^{j-1}\sum_{n=0}^{\infty}\frac{[j]_q^n q^{\frac{n(k-j)}{2}}}{n!}t^n\\
&= [2]_q t\sum_{j=0}^{\infty} q^{k-j}[j]_{q^2}(-1)^{j-1}\\
&\quad\times\sum_{n=0}^{\infty}\left\{\frac{1}{(1-q)^n}q^{\frac{n(k-j)}{2}}\sum_{m=0}^{n}\binom{n}{m}(-1)^m q^{jm}\right\}\frac{t^n}{n!}\\
&= \frac{[2]_q t}{1-q^2}\sum_{n=0}^{\infty}\frac{1}{(1-q)^n}q^{k+\frac{nk}{2}}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\\
&\quad\times\left(\sum_{j=0}^{\infty}(-1)^{j-1}q^{mj-j-\frac{nj}{2}}(1-q^{2j})\right)\frac{t^n}{n!}\\
&= \frac{[2]_q t}{1-q^2}\sum_{n=0}^{\infty}\frac{1}{(1-q)^n}\sum_{m=0}^{n}\binom{n}{m}(-1)^m\frac{q^{m-1-\frac{n}{2}+k+\frac{nk}{2}}(1-q^2)}{(1+q^{m-1-\frac{n}{2}})(1+q^{m+1-\frac{n}{2}})}\frac{t^n}{n!}.
\end{aligned}$$

Note that $G_{0,k,q}=0$. Hence, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} G_{n,k,q}\frac{t^n}{n!}\\
&= t\sum_{n=1}^{\infty}\frac{1}{(1-q)^n}\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{(-1)^{m-1}q^{m+k+\frac{(n-1)k}{2}-\frac{n-1}{2}-2}}{(1+q^{-2+m-\frac{n-1}{2}})(1+q^{m-\frac{n-1}{2}})}\frac{t^{n-1}}{(n-1)!}\\
&= \sum_{n=1}^{\infty}\frac{1}{(1-q)^n}\sum_{m=1}^{n}\binom{n}{m}\frac{(-1)^{m-1}mq^{m+k+\frac{(n-1)(k-1)}{2}-2}}{(1+q^{-2+m-\frac{n-1}{2}})(1+q^{m-\frac{n-1}{2}})}\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the below:

THEOREM 1. *Let $k, n (n \geq 1)$ be positive integers. Then we have*

$$G_{n,k,q}=\left(\frac{1}{1-q}\right)^n\sum_{m=1}^{n}\binom{n}{m}\frac{(-1)^{m-1}mq^{m+k+\frac{(n-1)(k-1)}{2}-2}}{(1+q^{-2+m-\frac{n-1}{2}})(1+q^{m-\frac{n-1}{2}})}.$$

We also define the generating function.

$$F_{q,k}(t,k) = [2]_q t \sum_{j=0}^{\infty} q^{-j}[j+k]_{q^2}(-1)^{j+k-1} \exp\left(t[j+k]_q q^{-\frac{j}{2}}\right)$$
$$= \sum_{n=0}^{\infty} G_{n,k,q}(k)\frac{t^n}{n!}.$$

By using the binomial theorem and some elementary calculations in the above equation, we have

$$\sum_{n=0}^{\infty} G_{n,k,q}(k)\frac{t^n}{n!}$$
$$= \frac{[2]_q t}{1-q^2}\sum_{j=0}^{\infty} q^{-j}(1-q^{2j+2k})(-1)^{j+k-1}\sum_{n=0}^{\infty}[j+k]_q^n q^{-\frac{j}{2}n}\frac{t^n}{n!}$$
$$= \frac{[2]_q t}{1-q^2}\sum_{j=0}^{\infty} q^{-j}(1-q^{2j+2k})(-1)^{j+k}$$
$$\times \sum_{n=0}^{\infty}\left\{\left(\frac{1}{1-q}\right)^n \sum_{m=0}^{n}\binom{n}{m}(-1)^m q^{jm+km} q^{-\frac{j}{2}n}\right\}\frac{t^n}{n!}$$
$$= \frac{[2]_q t}{1-q^2}\sum_{n=0}^{\infty}\left(\frac{1}{1-q}\right)^n$$
$$\times \sum_{m=0}^{n}\binom{n}{m}(-1)^{m+k}q^{mk}\sum_{j=0}^{\infty}(1-q^{2j+2k})(-1)^j q^{-j+jm-\frac{n}{2}j}\frac{t^n}{n!}$$
$$= \frac{[2]_q t}{1-q^2}\sum_{n=0}^{\infty}\left(\frac{1}{1-q}\right)^n \sum_{m=0}^{n}\binom{n}{m}(-1)^{m+k}q^{mk}$$
$$\times\left\{\frac{1}{1+q^{m-\frac{n}{2}-1}} - \frac{q^{2k}}{1+q^{1+m-\frac{n}{2}}}\right\}\frac{t^n}{n!}$$
$$= t\sum_{n=0}^{\infty}\left(\frac{1}{1-q}\right)^{n+1}\sum_{m=0}^{n}\binom{n}{m}(-1)^{m+k}$$
$$\times\left(\frac{q^{mk}}{1+q^{m-\frac{n}{2}-1}} - \frac{q^{(m+2)k}}{1+q^{1+m-\frac{n}{2}}}\right)\frac{t^n}{n!}$$
$$= t\sum_{n=1}^{\infty}\left(\frac{1}{1-q}\right)^n\sum_{m=1}^{n}\binom{n-1}{m-1}(-1)^{m-1+k}$$

$$\times\left(\frac{q^{(m-1)k}}{1+q^{m-2-\frac{n-1}{2}}}-\frac{q^{(m+1)k}}{1+q^{m-\frac{n-1}{2}}}\right)\frac{t^{n-1}}{(n-1)!}$$
$$=\sum_{n=1}^{\infty}\left(\frac{1}{1-q}\right)^n\sum_{m=1}^{n}\binom{n}{m}(-1)^{m-1+k}$$
$$\times\left(\frac{mq^{(m-1)k}}{1+q^{m-2-\frac{n-1}{2}}}-\frac{mq^{(m+1)k}}{1+q^{m-\frac{n-1}{2}}}\right)\frac{t^n}{n!}.$$

Note that $G_{0,k,q}(k)=0$. Therefore we obtain the following theorem.

THEOREM 2. *Let $k, n (n \geq 1)$ be positive integers. Then we have*

$$G_{n,k,q}(k)=\left(\frac{1}{1-q}\right)^n\sum_{m=1}^{n}\binom{n}{m}(-1)^{m-1+k}$$
$$\times\left(\frac{mq^{(m-1)k}}{1+q^{m-2-\frac{n-1}{2}}}-\frac{mq^{(m+1)k}}{1+q^{m-\frac{n-1}{2}}}\right).$$

REMARK 3. Note that

$$(1)\lim_{q\to 1}G_{n,k,q}=G_n^{(2)},\quad (2)\lim_{q\to 1}G_{n,k,q}(k)\neq G_n^{(2)}(k).$$

It is easy to see that

$$[2]_q t\sum_{j=0}^{\infty}q^{k-j}[j]_{q^2}(-1)^{j-1}\exp\left(t[j]_{q^2}q^{\frac{k-j}{2}}\right)$$
$$-[2]_q t\sum_{j=0}^{\infty}q^{-j}[j+k]_{q^2}(-1)^{j-1+k}\exp\left(t[j+k]_q q^{-\frac{j}{2}}\right)$$
$$=[2]_q t\sum_{j=0}^{k-1}(-1)^{j-1}[j]_{q^2}q^{k-j}\exp\left(t[j]_q q^{\frac{k-j}{2}}\right).$$

Thus, we easily see that

$$[2]_q\sum_{j=0}^{k-1}[j]_{q^2}(-1)^{j-1}[j]_q^{n-1}q^{\frac{(k-j)(n+1)}{2}}=\frac{G_{n,k,q}-G_{n,k,q}(k)}{n}.$$

Therefore we obtain the following theorem.

THEOREM 4. *Let $k, n (n \geq 1)$ be positive integers. Then we have*

$$\sum_{j=0}^{k-1}[j]_{q^2}(-1)^{j-1}[j]_q^{n-1}q^{\frac{(k-j)(n+1)}{2}}=\frac{G_{n,k,q}-G_{n,k,q}(k)}{n[2]_q}.$$

References

[1] J. Faulhaber, *Academia Algebrae, Darinnen die miraculosische inventiones zu den höchsten Cossen weiters continuirt und profitiert werden*, Augspurg, bey Johann Ulrich Schönigs, 1631.

[2] K. C. Garrett and K. Hummel, *A combinatorical proof of the sum of q-cubes*, Electro. J. Combin. **11** (2004), no. 1, Research Paper 9, 6pp.

[3] T. Kim, L.-C. Jang, and H. K. Pak, *A note on q-Euler and Genocchi numbers*, Proc. Japan Acad. Ser. A Math. Sci. **77 A** (2001), no. 8, 139–141.

[4] T. Kim, *Sums of powers of consecutive q-integers*, Adv. Stud. Contemp. Math. (Kyungshang) **9** (2004), no. 1, 15–18.

[5] ———, *A note on the alternating sums of powers of consecutive integers*, arXiv.org:math/0508233 **1** (2005), 1–4.

[6] ———, *A note on exploring the sums of powers of consecutive q-integers*, Advan. Stud. Contemp. Math. (Kyungshang) **11** (2005), no. 1, 137–140.

[7] ———, *Analytic continuation of multiple q-Zeta functions and their values at negative integers*, Russ. J. Math. Phys. **11** (2004), no. 1, 71–76.

[8] T. Kim, C. S. Ryoo, L. C. Jang, and S. H. Rim, *Exploring the sums of powers of consecutive q-integers*, Inter. J. Math. Edu. Sci. Tech. **36** (2005), no. 8, 947–956.

[9] D. E. Knuth, *Johann Faulhaber and sums of powers*, Math. Comput. **61** (1993), no. 203, 277–294.

[10] M. Schlosser, *q-analogues of the sums of consecutive integers squares, cubes, quarts, quints*, Electro. J. Comb. **11** (2004), no. 1, Reserch Paper 71, 12pp.

[11] Y.-Y. Shen, *A note on the sums of powers of consecutive integers*, Tunghai Science, **5** (2003), 101–106.

[12] Y. Simsek, D. Kim, T. Kim, and S. H. Rim, *A note on the sums of powers of consecutive q-integers*, J. Appl. Funct. Different. Equat. **1** (2006), no. 1, 10–25.

[13] S. O. Warnaar, *On the q-analogues of the sums of cubes*, Electro. J. Comb. **11** (2004), no. 1, Note 13, 2pp.

Seog-Hoon Rim, Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Korea
E-mail: shrim@knu.ac.kr

Taekyun Kim, Jangjon Research Institute for Mathematical Science & Physics, 252-5 Hapcheon-Dong Hapcheon-Gun, Kyungnam 678-801, Korea
E-mail: tkim@kongju.ac.kr/ tkim64@hanmail.net

Cheon Seoung Ryoo, Department of Mathematics, Hannam University, Daejeon 306-791, Korea
E-mail: ryoocs@hannam.ac.kr