ON THE ALTERNATING SUMS OF POWERS OF CONSECUTIVE $q$-INTEGERS

SEOG-HOON RIM, TAEKYUN KIM, AND CHEON SEOUNG RYOO

ABSTRACT. In this paper we construct $q$-Genocchi numbers and polynomials. By using these numbers and polynomials, we investigate the $q$-analogue of alternating sums of powers of consecutive integers due to Euler.

1. Introduction

The Genocchi numbers $G_m$ are defined by the generating function:

$$F(t) = \frac{2t}{e^t + 1} = \sum_{m=0}^{\infty} G_m \frac{t^m}{m!}, (|t| < \pi) \ (\text{cf. [3]}),$$

where we use the technique method notation by replacing $G^m$ by $G_m(m \geq 0)$ symbolically. It satisfies $G_1 = 1, G_3 = G_5 = G_7 = \cdots = 0,$ and even coefficients are given $G_m = 2(1 - 2^m)B_{2m} = 2mE_{2m-1},$ where $B_m$ are Bernoulli numbers and $E_m$ are Euler numbers which are defined by

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \ (\text{cf. [5, 6, 8]}).$$

For $x \in \mathbb{R}$ (= the field of real numbers), we consider the Genocchi polynomials as follows:

$$F(x, t) = F(t)e^{xt} = \frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.$$
Note that \( G_m(x) = \sum_{k=0}^{m} \binom{m}{k} G_k x^{m-k} \). Let us also define the Genocchi polynomials of order \( r \) as follows:

\[
2 \left( \frac{1}{1 + e^t} \right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{cf. [3]}).
\]

In the special case \( x = 0 \), we define \( G_n^{(r)}(0) = G_n^{(r)} \). What is the value of the following sum for a given positive integer \( k \)?

\[
1^k + 2^k + 3^k + \cdots + n^k.
\]

Let us denote this sum by \( f_k(n) \). Finding formulas for \( f_k(n) \) has interested mathematicians for more than 300 years since the time of Jacob Bernoulli (cf. [1, 7, 9, 11]). It was well known that

\[
f_n(k-1) = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i k^{n+1-i} \quad (\text{cf. [9, 11]}),
\]

where \( \binom{n}{k} \) is binomial coefficients.

Let \( n, k \) be positive integers \( (k > 1) \), and let

\[
T_n(k) = -1^k + 2^k - 3^k + 4^k - 5^k + \cdots + (-1)^{k-1}(n-1)^k.
\]

Following an idea due to Euler, it was known that

\[
T_n(k) = \frac{(-1)^{k+1}}{2} \sum_{l=0}^{k-1} \binom{n}{l} E_l k^{n-l} + \frac{E_n}{2} \left( 1 + (-1)^{k+1} \right) \quad (\text{cf. [5]}).
\]

Let \( q \) be an indeterminate which can be considered in complex number field, and for any integer \( k \) define the \( q \)-integer as

\[
[k]_q = \frac{q^k - 1}{q - 1} = 1 + q + \cdots + q^{k-1}.
\]

Throughout this paper we assume that \( q \in \mathbb{C} \) with \( 0 < q < 1 \). Recently many authors studied \( q \)-analogue of the sums of powers of consecutive integers. In [2], Garrett and Hummel gave a combinatorial proof of a \( q \)-analogue of \( \sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2 \) as follows:

\[
\sum_{k=1}^{n} q^{k-1} [k]_q^2 \left( \left[ \frac{k-1}{2} \right]_{q^2} + \left[ \frac{k+1}{2} \right]_{q^2} \right) = \left[ \frac{n+1}{2} \right]_q^2,
\]

where

\[
\left[ \frac{n}{k} \right]_q = \prod_{j=1}^{k} \frac{[n+1-j]_q}{[j]_q} \quad \text{denotes the } q \text{-binomial coefficients}.
\]
Garrett and Hummel, in their paper, asked for a simpler $q$-analogue of the sums of cubes. As a response to Garrett and Hummel's question, Warnaar gave a simple $q$-analogue of the sums of cubes as follows:

$$\sum_{k=1}^{n} q^{2n-2k}[k]_q^2 [k]_q = \left[ \frac{n+1}{2} \right]_q^2 \text{(cf. [12, 13]).}$$

Let

$$f_{m,q}(n) = \sum_{k=1}^{n} [k]_q^2 [k]_q^{m-1} q^{(n-k)\frac{m+1}{2}}.$$

Then we note that \( \lim_{q \to 1} f_{m,q}(n) = f_m(n) \) (cf. [10, 12]).

Warunaar [13] (for \( m = 3 \)) and Schlosser [10] gave formulae for \( m = 1, 2, 3, 4, 5 \) as the meaning of the $q$-analogues of the sums of consecutive integers, squares, cubes, quarts and quints. Let \( n, k(>1) \) be positive integers. In the recent paper, it was known that

$$\sum_{j=0}^{k-1} q^j[j]_q^n = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \beta_j q^j [k]_q^{n+1-j} - \frac{1}{n+1} (1 - q^{(n+1)k}) \beta_{n+1,q} \text{ (see [4, 6, 7])},$$

where \( \beta_j \) are called Carlitz's $q$-Bernoulli numbers. Originally $q$-Genocchi numbers and polynomials were introduced by Kim-Jang-Pak in 2001 [3], but they do not seem to be the most natural ones. In this paper we give another construction of $q$-Genocchi numbers and polynomials which are different than $q$-Genocchi numbers and polynomials of Kim-Jang-Pak in 2001 [3]. By using these numbers and polynomials, we investigate the $q$-analogue of alternating sums of powers of consecutive integers.

2. $q$-Genocchi numbers and polynomials

Let \( F_{q,k}(t) \) be the generating functions of the $q$-Genocchi numbers as follows:

$$F_{q,k}(t) = [2]_q t \sum_{j=0}^{\infty} q^{k-j}[j]_q [q^{-2}]_q (-1)^{j-1} \exp \left( t[j]_q q^{k-j} \right) = \sum_{j=0}^{\infty} G_{n,q} \frac{t^n}{n!}.$$
By using Taylor expansion in the above, we see that

\[
\sum_{n=0}^{\infty} G_{n,k,q} \frac{t^n}{n!} = [2]_q t \sum_{j=0}^{\infty} q^{k-j} [j] q^j (-1)^{j-1} \sum_{n=0}^{\infty} \frac{[j]^n q^{n(k-j)}}{n!} t^n
\]

\[
= [2]_q t \sum_{j=0}^{\infty} q^{k-j} [j] q^j (-1)^{j-1}
\times \sum_{n=0}^{\infty} \left\{ \frac{1}{(1-q)^n} q^{\frac{n(k-j)}{2}} \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{jm} \right\} \frac{t^n}{n!}
\]

\[
= \frac{[2]_q t}{1-q^2} \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} q^{k+\frac{n^2}{2}} \sum_{m=0}^{n} \binom{n}{m} (-1)^m
\times \left( \sum_{j=0}^{\infty} (-1)^{j-1} q^{mj-j} \frac{n^j}{2} \right) \frac{t^n}{n!}
\]

\[
= \frac{[2]_q t}{1-q^2} \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{m-1-k+\frac{n^2}{2}} (1-q^2) t^n
\]

\[
= \frac{[2]_q t}{1-q^2} \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{q^{m-1-k+\frac{n^2}{2}} (1-q^2)}{(1+q^{m-1-k+\frac{n^2}{2}})(1+q^{m+1-k+\frac{n^2}{2}})} \frac{t^n}{n!}.
\]

Note that \( G_{0,k,q} = 0 \). Hence, we have

\[
\sum_{n=1}^{\infty} G_{n,k,q} \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \frac{1}{(1-q)^n} \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{(-1)^{m-1} q^{m+k+\frac{(n-1)k}{2}} - 2}{(1+q^{2+m-\frac{n-1}{2}})(1+q^{m-\frac{n-1}{2}})} \frac{t^{n-1}}{(n-1)!}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(1-q)^n} \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{m-1} mq^{m+k+\frac{(n-1)(k-1)}{2}} - 2}{(1+q^{2+m-\frac{n-1}{2}})(1+q^{m-\frac{n-1}{2}})} \frac{t^n}{n!}.
\]

By comparing the coefficients of \( \frac{t^n}{n!} \) on both sides of the above equation, we obtain the below:

**Theorem 1.** Let \( k, n (n \geq 1) \) be positive integers. Then we have

\[
G_{n,k,q} = \left( \frac{1}{1-q} \right)^n \sum_{m=1}^{n} \binom{n}{m} \frac{(-1)^{m-1} mq^{m+k+\frac{(n-1)(k-1)}{2}} - 2}{(1+q^{2+m-\frac{n-1}{2}})(1+q^{m-\frac{n-1}{2}})}. \]
We also define the generating function.

\[ F_{q,k}(t, k) = [2]_q t \sum_{j=0}^{\infty} q^{-j}[j + k]q^2(-1)^{j+k-1} \exp \left( t[j + k]q^{-\frac{j}{2}} \right) \]

\[ = \sum_{n=0}^{\infty} G_{n,k,q}(k) \frac{t^n}{n!}. \]

By using the binomial theorem and some elementary calculations in the above equation, we have

\[ \sum_{n=0}^{\infty} G_{n,k,q}(k) \frac{t^n}{n!} = \frac{[2]_q t}{1 - q^2} \sum_{j=0}^{\infty} q^{-j}(1 - q^{2j+2k})(-1)^{j+k-1} \sum_{n=0}^{\infty} [j + k]_q q^{-\frac{j}{2}n} \frac{t^n}{n!} \]

\[ = \frac{[2]_q t}{1 - q^2} \sum_{j=0}^{\infty} q^{-j}(1 - q^{2j+2k})(-1)^{j+k} \]

\[ \times \sum_{n=0}^{\infty} \left\{ \left( \frac{1}{1 - q} \right)^n \sum_{m=0}^{n} \binom{n}{m} (-1)^m q^{jm+km} q^{-\frac{j}{2}n} \right\} \frac{t^n}{n!} \]

\[ = \frac{[2]_q t}{1 - q^2} \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \]

\[ \times \sum_{m=0}^{n} \binom{n}{m} (-1)^{m+k} q^{km} \sum_{j=0}^{\infty} (1 - q^{2j+2k})(-1)^j q^{-j+km - \frac{j}{2}j} \frac{t^n}{n!} \]

\[ = \frac{[2]_q t}{1 - q^2} \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^n \sum_{m=0}^{n} \binom{n}{m} (-1)^{m+k} q^{mk} \]

\[ \times \left\{ \frac{1}{1 + q^m - \frac{q^2}{2} - 1} - \frac{q^{2k}}{1 + q^{1+m - \frac{n}{2}}} \right\} \frac{t^n}{n!} \]

\[ = t \sum_{n=0}^{\infty} \left( \frac{1}{1 - q} \right)^{n+1} \sum_{m=0}^{n} \binom{n}{m} (-1)^{m+k} \]

\[ \times \left( \frac{q^{mk}}{1 + q^{m - \frac{n}{2} - 1}} - \frac{q^{(m+2)k}}{1 + q^{1+m - \frac{n}{2}}} \right) \frac{t^n}{n!} \]

\[ = t \sum_{n=1}^{\infty} \left( \frac{1}{1 - q} \right)^n \sum_{m=1}^{n} \binom{n}{m-1} (-1)^{m-1+k} \]
\begin{align*}
&\times \left(\frac{q^{(m-1)k}}{1 + q^{m-2} - \frac{n-1}{2}} - \frac{q^{(m+1)k}}{1 + q^{m+2} - \frac{n-1}{2}}\right) (n-1)! \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{1 - q}\right) \sum_{m=1}^{n} \binom{n}{m} (-1)^{m-1+k} \\
&\times \left(\frac{mq^{(m-1)k}}{1 + q^{m-2} - \frac{n-1}{2}} - \frac{mq^{(m+1)k}}{1 + q^{m+2} - \frac{n-1}{2}}\right) \frac{t^n}{n!}.
\end{align*}

Note that \( G_{0,k,q}(k) = 0 \). Therefore we obtain the following theorem.

**Theorem 2.** Let \( k, n(n \geq 1) \) be positive integers. Then we have

\[
G_{n,k,q}(k) = \left(\frac{1}{1 - q}\right) \sum_{m=1}^{n} \binom{n}{m} (-1)^{m-1+k} \\
\times \left(\frac{mq^{(m-1)k}}{1 + q^{m-2} - \frac{n-1}{2}} - \frac{mq^{(m+1)k}}{1 + q^{m+2} - \frac{n-1}{2}}\right).
\]

**Remark 3.** Note that

1. \( \lim_{q \to 1} G_{n,k,q} = G_n^{(2)} \),
2. \( \lim_{q \to 1} G_{n,k,q}(k) \neq G_n^{(2)}(k) \).

It is easy to see that

\[
[2]_q \sum_{j=0}^{\infty} q^{k-j}[j]_q^2 (-1)^{j-1} \exp \left(t[j]_q q^{\frac{k-j}{2}}\right)
\]

\[
- [2]_q \sum_{j=0}^{\infty} q^{-j}[j+k]_q^2 (-1)^{j-1+k} \exp \left(t[j+k]_q q^{-\frac{k}{2}}\right)
\]

\[
= [2]_q \sum_{j=0}^{k-1} (-1)^{j-1}[j]_q^2 q^{k-j} \exp \left(t[j]_q q^{\frac{k-j}{2}}\right).
\]

Thus, we easily see that

\[
[2]_q \sum_{j=0}^{k-1} [j]_q^2 (-1)^{j-1}[j]_q^{n-1} q^{(k-j)(n+1)} = \frac{G_{n,k,q} - G_{n,k,q}(k)}{n}.
\]

Therefore we obtain the following theorem.

**Theorem 4.** Let \( k, n(n \geq 1) \) be positive integers. Then we have

\[
\sum_{j=0}^{k-1} [j]_q^2 (-1)^{j-1}[j]_q^{n-1} q^{\frac{(k-j)(n+1)}{2}} = \frac{G_{n,k,q} - G_{n,k,q}(k)}{n[2]_q}.
\]

References


Seog-Hoon Rim, Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Korea
E-mail: shrim@knu.ac.kr

TaeKyun Kim, Jangjon Research Institute for Mathematical Science & Physics, 252-5 Hapcheon-Dong Hapcheon-Gun, Kyungnam 678-801, Korea
E-mail: tkim@kongju.ac.kr/ tkim64@hanmail.net

Cheon Seoung Ryoo, Department of Mathematics, Hannam University, Daejeon 306-791, Korea
E-mail: ryoo0c@hannam.ac.kr