Bull. Korean Math. Soc. **43** (2006), No. 3, pp. 635–644

DERIVATIONS ON SUBRINGS OF MATRIX RINGS

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ABSTRACT. For a lower niltriangular matrix ring $A = NT_n(K)$ $(n \geq 3)$, we show that every derivation of A is a sum of certain diagonal, trivial extension and strongly nilpotent derivation. Moreover, a strongly nilpotent derivation is a sum of an inner derivation and an uaz-derivation.

1. Introduction

Let $NT_n(K)(n \geq 3)$ be the ring of all (lower niltriangular) $n \times n$ matrices over an associative ring with identity K which are all zeros on and above the main diagonal.

It is well-known(see [4], p.100) that if F is a field, then any F-derivation of $M_n(F)$ is inner. Moreover, Amitsur [1] showed that any derivation of $M_n(K)$ is a sum of an inner derivation and a trivial extension and Nowicki [8] characterized derivations of special subrings of $M_n(K)$.

Dubish and Perlis [3] classified automorphisms on $NT_n(F)$ over a field F. Every automorphism on $NT_n(F)$ is equal to a product of certain diagonal automorphism, inner automorphism and nil automorphism. Moreover, Levchuk ([6], [7]) characterized automorphisms of $NT_n(K)$ and Kuzucuoglu and Levchuk [5] characterized automorphisms on $R_n(K, J) = NT_n(K) + M_n(J)$.

In this paper, we will characterize derivations of $NT_n(K)$. In section 2, we characterize ideals and characteristic ideals of $NT_n(K)$. In section 3, we show that for a derivation δ on $NT_n(K)$, $\delta = i_d + \bar{\sigma} + s_t$ where i_d is a diagonal inner, $\bar{\sigma}$ is a trivial extension of K and s_t is a strongly nilpotent derivation. In section 4, we have that for a strongly

Received November 21, 2005.
2000 Mathematics Subject Classification: 16S32, 16D25.
Key words and phrases: derivations, diagonal derivations, strongly nilpotent derivations, inner derivations.

nilpotent derivation s_t, $s_t = s_i + s_{uaz}$ where s_i is an inner derivation and s_{uaz} is an uaz-derivation. Moreover, we characterize the difference between uaz-derivations and az-derivations.

For a ring R, not necessarily contains 1, a derivation δ is an additive map on R which satisfies

$$\delta(ab) = \delta(a)b + a\delta(b) \qquad (a, b \in R).$$

We say that δ is an inner derivation if there exist $r \in R$ such that $\delta(x) = rx - xr$ for all $x \in R$.

For the convenience we have the followings :

(1) $NT_n(K) \equiv A_n \equiv A$.

(2) e_{ij} : matrix units of $M_n(K)$.

(3) A^k : k-th product of A.

(4) Any derivation σ of K can be extended to A by putting

$$\bar{\sigma}(\sum_{i>j} r_{ij}e_{ij}) = \sum_{i>j} \sigma(r_{ij})e_{ij} \qquad (r_{ij} \in K).$$

It is easy to show that $\bar{\sigma}$ is also a derivation of A. We call $\bar{\sigma}$ a trivial extension of σ.

(5) Let B_n be the set of all matrices in $M_n(K)$ with zeros above the diagonal. Then each diagonal matrix $d = \sum d_i e_{ii} (d_i \in K)$ determines a derivation $i_d(x) = [d, x]$ of B_n and the derivation i_d induces on A. We call i_d a diagonal derivation.

(6) Since we can regard A as a K-module, we define a K-derivation on A by $\delta(rx) = r\delta(x) (r \in K, x \in A)$.

(7) For all $x \in A$, we denote $\{x\}_{ij} = \pi_{ij}(x)$.

2. Ideals of A

The ideals of $NT_n(F)$ are characterized in Dubisch and Perlis [3], which are referred to "staircase open polygon". Also, the ideals of A can be regarded similarly. But we characterize ideals of A another way. For any subset H of A, trivially $\sum \pi_{ij}(H)e_{ij} \supseteq H$. If $H = \sum \pi_{ij}(H)e_{ij}$, we call H a direct subset of A.

PROPOSITION 2.1. *Let H be a subset of A. If H is an ideal of A, then the followings hold;*

(1) *$\pi_{ij}(H)$ is a subgroup of K.*

(2) *For all* $s > i$, $\pi_{sj}(H) \supseteq K\pi_{ij}(H)$.

(3) *For all* $t < j$, $\pi_{it}(H) \supseteq \pi_{ij}(H)K$.

Conversely, if H *is a direct subset of* A *and satisfies above* (1), (2) *and* (3), *then* H *is an ideal of* A.

Proof. The proof of the first statement is obvious.

Conversely, by (2)

$$\begin{aligned}\pi_{ij}(NT_n(K)H) &= \sum_{\lambda=1}^{n} \pi_{i\lambda}(NT_n(K))\pi_{\lambda j}(H) \\ &= \sum_{\lambda=j+1}^{i-1} \pi_{i\lambda}(NT_n(K))\pi_{\lambda j}(H) \\ &= \sum_{\lambda=j+1}^{i-1} K\pi_{\lambda j}(H) = K\pi_{i-1,j}(H) \subseteq \pi_{ij}(H).\end{aligned}$$

Thus, by $\sum \pi_{ij}(H)e_{ij} = H$, $NT_n(K)H \subseteq H$, that is, $AH \subseteq H$.

Similarly, by (3) and $\sum \pi_{ij}(H)e_{ij} = H$, $HA \subseteq H$. Therefore, H is an ideal of A. □

Next example shows that an ideal of A is not necessarily direct and for noetherian ring K, A is not noetherian in general.

EXAMPLE 2.2. For the rational number field $\mathbf{Q}$ and the ring of integers $\mathbf{Z}$, let $K = M_2(\mathbf{Q})$ and $A = NT_3(K)$. Denote $f_{ij} (i, j = 1, 2)$ by matrix units of $M_2(\mathbf{Q})$. Set

$$H_k = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \frac{n}{2^k} f_{21} & 0 & 0 \\ T & \frac{n}{2^k} f_{21} & 0 \end{pmatrix} \middle| n \in \mathbf{Z} \right\}, \qquad k = 1, 2, \ldots$$

where $T = \mathbf{Q}f_{11} + \mathbf{Q}f_{21} + \mathbf{Q}f_{22}$. Then we have the following properties;

(1) H_k are ideals but not direct.

(2) T is not an ideal of K.

(3) For a trivial extension δ of an inner derivation of K, H_k is not invariant in general.

(4) K is noetherian. But since $H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq \cdots$, A is not noetherian.

DEFINITION 2.3. Let C be a subring of a ring R. C is called characteristic if every derivation δ on R induces a derivation on C.

Obviously for $k(1 < k < n)$, the k-th powers A^k are characteristic ideals of A.

For $x \in A$, it is important to find characteristic ideals of A which contain $\delta(x)$. We introduce certain characteristic ideals of A which contains matrix unit $e_{ij}(i > j)$.

Let C_l be the totality of matrices in A whose columns beyond the l-th are zero. Then C_l is an ideal of A. Likewise, an ideal is given by the set R_k of all matrices in A whose first $k-1$ rows are zero.

PROPOSITION 2.4 [3]. *C_l is the left annihilator of A^l and R_k is the right annihilator of A^k.*

THEOREM 2.5. *C_l and R_k are characteristic ideals. Moreover, for each derivation δ on A and each matrix unit $e_{kl} \in A$, $\delta(e_{kl}) \in C_l \cap R_k$.*

Proof. For arbitrary derivation δ of A, let $c \in C_l$ and $x \in A^l$. Then by Proposition 2.4

$$0 = \delta(cx) = \delta(c)x + c\delta(x).$$

Since A^l is a characteristic ideal $\delta(x) \in A^l$ and $c\delta(x) = 0$. So, $\delta(c)x = 0$. This means $\delta(c)A^l = 0$. Thus $\delta(c) \in C_l$. Therefore C_l is a characteristic ideal.

Similarly, R_k is a characteristic ideal.

Moreover, $e_{kl} \in C_l \cap R_k$. So $\delta(e_{kl}) \in C_l \cap R_k$. □

From the Theorem 2.5, $\delta(e_{kl}) \in C_l \cap R_k$. So we have the following;

$$(*) \qquad \delta(e_{k,k-1}) = \sum_{i \geq k} \sum_{j \leq k-1} \beta_{ij}^{(k)} e_{ij}, \qquad \beta_{ij}^{(k)} \in K.$$

Now we characterize the characteristic ideals of A

THEOREM 2.6. *Let H be a characteristic ideal of A. Then the followings hold;*

(1) $\pi_{ij}(H)$ is an ideal of K.

(2) H is direct.

(3) $\pi_{ij}(H)$ is a characteristic ideal of K.

Proof. (1) For $r \in K$, let i_d be a diagonal derivation induced by $d = re_{ii}$, that is, $i_d(x) = dx - xd$ for all $x \in A$. Then

$$r\pi_{ij}(H) = \pi_{ij}(rH) = \pi_{ij}([d,H]) = \pi_{ij}(i_d(H)) \subseteq \pi_{ij}(H).$$

So, $\pi_{ij}(H)$ is a left ideal of K.

To show that $\pi_{ij}(H)$ is a right ideal, take $d = -re_{jj}$. Then

$$\pi_{ij}(H)r = \pi_{ij}(Hr) = \pi_{ij}([d,H]) = \pi_{ij}(i_d(H)) \subseteq \pi_{ij}(H).$$

So, $\pi_{ij}(H)$ is a right ideal of K. Therefore, $\pi_{ij}(H)$ is an ideal of K.

(2) Since H is a characteristic ideal,

$$[-e_{jj}, [e_{ii}, H]] = \pi_{ij}(H)e_{ij} \subseteq H.$$

So, $\sum \pi_{ij}(H)e_{ij} \subseteq H$, that is, $\sum \pi_{ij}(H)e_{ij} = H$.

(3) Let σ be a derivation of K. Then trivial extension $\bar{\sigma}$ of σ is a derivation of A. So, $\bar{\sigma}(H) \subseteq H$. Thus, $\sigma(\pi_{ij}(H)) \subseteq \pi_{ij}(H)$. Therefore, $\pi_{ij}(H)$ is a characteristic ideal. □

COROLLARY 2.7. *Let H be a characteristic ideal of A. If $\pi_{ij}(H) = K$ or 0, then H is generated by $C_{l_1} \cap R_{k_1}, C_{l_2} \cap R_{k_2}, \dots, C_{l_t} \cap R_{k_t}$.*

3. Characterizations of derivations

Since A is a free K-module with basis $\{e_{ij}\}(i > j)$, derivations of A highly depends on the image of e_{ij}. Every K-module derivation of A is determined by the image of e_{ij}, but in general every derivation of A is not determined by the image of e_{ij}. However, we get a useful lemma which says that for any derivation δ of $\{\delta(e_{ij})\}_{ij} = 0$, the coordinate function of δ is also a derivation of K.

LEMMA 3.1. *Suppose δ is a derivation of A and $\{\delta(e_{ij})\}_{ij} = 0$ for all $i > j$. Define the coordinate function $\delta_{ij} : K \longrightarrow K$ such that $\delta_{ij}(r) = \{\delta(re_{ij})\}_{ij} (r \in K)$. Then $\delta_{ij} = \delta_{21}$ and δ_{ij} is a derivation of K.*

Proof. For $r \in K$, we get

$$\begin{aligned}\delta_{31}(r) &= \{\delta(re_{31})\}_{31} = \{\delta(re_{32}e_{21})\}_{31} \\ &= \{\delta(re_{32})e_{21} + re_{32}\delta(e_{21})\}_{31} \\ &= \{\delta(re_{32})e_{21}\}_{31} \\ &= \{\delta_{32}(r)e_{31}\}_{31} = \delta_{32}(r).\end{aligned}$$

On the other hand,

$$\begin{aligned}\delta_{31}(r) &= \{\delta(re_{31})\}_{31} = \{\delta(re_{32}e_{21})\}_{31} = \{\delta(e_{32}re_{21})\}_{31}\\ &= \{\delta(e_{32})re_{21} + e_{32}\delta(re_{21})\}_{31} = \{\delta_{21}(r)e_{31}\}_{31}\\ &= \delta_{21}(r).\end{aligned}$$

Hence, $\delta_{32} = \delta_{21}$. Similarly, we can show that for $4 \leq k \leq n$, $\delta_{k,k-1} = \delta_{21}$.

If $i - j \geq 2$,

$$\begin{aligned}\delta_{ij}(r) &= \{\delta(re_{ij})\}_{ij} = \{\delta(re_{i,i-1}\cdots e_{j+1,j})\}_{ij}\\ &= \{\delta_{i,i-1}(r)e_{ij}\}_{ij} = \delta_{i,i-1}(r).\end{aligned}$$

Therefore, for all $i > j, \delta_{ij} = \delta_{21}(i > j)$.

Now we will show that δ_{31} is a derivation of K. For arbitrary $r, r' \in K$,

$$\begin{aligned}\delta_{31}(rr') &= \{\delta(rr'e_{31})\}_{31} = \{\delta(re_{32}r'e_{21})\}_{31}\\ &= \{\delta(re_{32})r'e_{21} + re_{32}\delta(r'e_{21})\}_{31}\\ &= \{\delta(re_{32})\}_{32}r' + r\{\delta(r'e_{21})\}_{21}\\ &= \delta_{32}(r)r' + r\delta_{21}(r') = \delta_{31}(r)r' + r\delta_{31}(r).\end{aligned}$$

So, δ_{31} is a derivation of K. This means δ_{ij} is a derivation of K. □

COROLLARY 3.2. *Suppose that δ, δ' are derivations of A satisfying $\{\delta(e_{ij})\}_{ij} = \{\delta'(e_{ij})\}_{ij}$ for all $i > j$. Then $\delta_{ij} - \delta'_{ij} = \delta_{21} - \delta'_{21}$ and $\delta_{ij} - \delta'_{ij}$ is a derivation of K.*

COROLLARY 3.3. *Let $\delta_{ij} : K \longrightarrow K(i > j)$ be derivations. Define $\delta : A \longrightarrow A$ by $\delta(\sum_{i>j} r_{ij}e_{ij}) = \sum_{i>j}\delta_{ij}(r_{ij})e_{ij}$. If δ is a derivation of A, then for all $i > j$, $\delta_{ij} = \delta_{21}$.*

Proof. Since $\delta(e_{ij}) = 0$, $\{\delta(e_{ij})\}_{ij} = 0$. And for all $r \in K$, $\delta_{ij}(r) = \{\delta(re_{ij})\}_{ij}$. So, by Lemma 3.1, $\delta_{ij} = \delta_{21}$. □

LEMMA 3.4. *If δ is a diagonal derivation of A. Then*

(1) $\delta(e_{k,k-1}) = \alpha_k e_{k,k-1}$, where $\alpha_k \in K$ and $2 \leq k \leq n$.

(2) $\delta(e_{kl}) = \alpha_{kl}e_{kl}$, where $\alpha_{kl} = \alpha_k + \alpha_{k-1} + \cdots + \alpha_{l+1}$.

Conversely, if δ is a derivation of A satisfying (1) and (2), then $(\delta - i_d)(e_{kl}) = 0$, where i_d is a diagonal derivation induced by $d = \alpha_2 e_{22} + \cdots + (\alpha_2 + \cdots + \alpha_n)e_{nn}$.

Proof. The proof of the first statement is obvious.

Conversely, for all $k > l$,

$$\begin{aligned} i_d(e_{kl}) &= de_{kl} - e_{kl}d \\ &= (\alpha_2 + \cdots + \alpha_k)e_{kl} - (\alpha_2 + \cdots + \alpha_l)e_{kl} \\ &= (\alpha_k + \cdots + \alpha_{l+1})e_{kl}. \end{aligned}$$

So, $(\delta - i_d)(e_{kl}) = 0$. □

LEMMA 3.5. *Let δ be a derivation on A. Then there exists a diagonal derivation i_d such that $\{(\delta - i_d)(e_{ij})\}_{ij} = 0$.*

Proof. The quantities $\delta(e_{k,k-1}), \ldots, \delta(e_{l+1,l})$ can be denoted as equations (*) in section 2 with corner coefficients $\beta^{(k)}_{k,k-1} \equiv \alpha_k, \ldots, \beta^{(l+1)}_{l+1,l} \equiv \alpha_{l+1}$. By multiplying the equations, we find that

$$\delta(e_{kl}) = \sum_{i \geq k} \sum_{j \leq l} \beta^{(kl)}_{ij} e_{ij} \qquad (\beta^{(kl)}_{ij} \in K)$$

with corner coefficients $\beta^{(kl)}_{kl} \equiv \alpha_{kl} = \alpha_k + \alpha_{k-1} + \cdots + \alpha_{l+1}$.

The quantities α_l, α_{kl} thus fulfill the conditions (1) and (2) of Lemma 3.4. So, the correspondence $e_{kl} \mapsto \alpha_{kl}e_{kl}$ generates a diagonal derivation i_d such that $\{\delta(e_{ij})\}_{ij} = \{i_d(e_{ij})\}_{ij}$, that is, $\{(\delta - i_d)(e_{ij})\}_{ij} = 0$.□

DEFINITION 3.6. Let s_t be a derivation on A. s_t is called a strongly nilpotent derivation if for all $x \in A^k, s_t(x) \in A^{k+1}$.

Obviously every strongly nilpotent derivation of A is nilpotent and every inner derivation of A is strongly nilpotent.

PROPOSITION 3.7. *Suppose that δ is a derivation on A such that $\{\delta(e_{ij})\}_{ij} = 0$ for all $i > j$. Then there exists a trivial extension $\bar{\sigma}$ of A such that $\delta - \bar{\sigma}$ is strongly nilpotent.*

Proof. By Lemma 3.1, it is obvious. □

THEOREM 3.8. *Let δ be a derivation of A. Then $\delta = i_d + \bar{\sigma} + s_t$ where i_d is a diagonal inner, $\bar{\sigma}$ is a trivial extension of K and s_t is a strongly nilpotent derivation.*

Proof. By Lemma 3.5, there exists a diagonal derivation i_d such that $\{(\delta - i_d)(e_{ij})\}_{ij} = 0$. And by Proposition 3.7, there exists a trivial extension $\bar{\sigma}$ of A such that $(\delta - i_d) - \bar{\sigma}$ is strongly nilpotent. □

4. *uaz*- derivations of A

Matrix units $e_{21}, e_{31}, \ldots, e_{n1}$ are left annihilators of A and matrix units $e_{n1}, e_{n2}, \ldots, e_{n,n-1}$ are right annihilators of A. There exist derivations that the images of these matrix units are zero. These derivations are important role to characterize strongly nilpotent derivations.

DEFINITION 4.1. A strongly nilpotent derivation δ of A is called a *uaz*-derivation if $\delta(u) = 0$ for every matrix unit u which is an absolute left or right divisor of zero.

THEOREM 4.2. *Let δ be a strongly nilpotent derivation of A. Then δ is a uaz-derivation of A if and only if $\delta(e_{k,k-1}) = \gamma_k e_{n1} (k = 2, \cdots, n)$ where $\gamma_2 = \gamma_n = 0$ and the remaining γ_k are arbitrary scalars.*

Proof. ($\Longleftarrow$) By the hypothesis, $\delta(e_{21}) = \delta(e_{n,n-1}) = 0$. Since δ is a derivation of A, we can get

$$\begin{aligned}\delta(e_{k,k-2}) &= \delta(e_{k,k-1}e_{k-1,k-2})\\ &= \delta(e_{k,k-1})e_{k-1,k-2} + e_{k,k-1}\delta(e_{k-1,k-2})\\ &= \gamma_k e_{n1} e_{k-1,k-2} + e_{k,k-1}\gamma_{k-1}e_{n1} = 0.\end{aligned}$$

So, $\delta(e_{kj}) = 0$ for $j < k-1$. Thus, δ is a *uaz*-derivation.

($\Longrightarrow$) i) If $k = 2$ or n, then $\gamma_2 = \gamma_n = 0$ by hypothesis.

ii) Assume $2 < k < n$.

Since δ is a strongly nilpotent derivation, let $\delta(e_{k,k-1}) = t_k$ with $t_k \in A^2$ and $t_k \in C_{k-1} \cap R_k$ by the Theorem 2.5.

Now $0 = \delta(e_{k1}) = \delta(e_{k,k-1}e_{k-1,1}) = \delta(e_{k,k-1})e_{k-1,1} = t_k e_{k-1,1}$. So, $(k-1)$-th column of the matrix $t_k = 0$.

For $1 < j < k-1$, $0 = \delta(e_{k,k-1}e_{j1}) = \delta(e_{k,k-1})e_{j1} = t_k e_{j1}$. So, the j-th column of $t_k = 0$ for all $1 < j < k-1$.

Thus, the j-th column of $t_k = 0$ for all $1 < j \leq k-1$.

On the other hand, $0 = \delta(e_{nk}e_{k,k-1}) = e_{nk}\delta(e_{k,k-1}) = e_{nk}t_k$. So, the k-th row of $t_k = 0$. Also, for $n > j > k$, $0 = \delta(e_{nj}e_{k,k-1}) = e_{nj}\delta(e_{k,k-1}) = e_{nj}t_k$. So, j-th row of $t_k = 0$, for all $k < j < n$.

Thus, j-th row of $t_k = 0$, for all $k \geq j < n$.

Therefore, $t_k = \gamma_k e_{n1}$. □

THEOREM 4.3. *Let s_t be a strongly nilpotent derivation of A. Then $s_t = s_i + s_{uaz}$ where s_i is an inner derivation and s_{uaz} is a uaz-derivation.*

Proof. It is enough to show that for a strongly nilpotent derivation s_t there exist an inner derivation s_i such that $s_t - s_i$ is a *uaz*-derivation.

By (*) and the hypothesis, we can set

$$s_t(e_{k1}) = \sum_{p>k} \alpha_{pk} e_{p1} \qquad (k = 2, \ldots, n-1).$$

The scalars α_{pk} are thus defined for $n \geq p > k > 1$ and $s_t(e_{k1}) = (\sum_{p>q>1} \alpha_{pq} e_{pq}) e_{k1} = [a, e_{k1}]$, where $a = \sum_{p>q>1} \alpha_{pq} e_{pq}$.

So, the inner derivation $s_a(x) = [a, x]$ has the property $s_1 \equiv s_t - s_a$ maps e_{k1} to zero.

Since s_1 is strongly nilpotent, $s_1(e_{nk}) = \sum_{q<k} \beta_{kq} e_{nq}$. And since for all $p(1 < p < k), e_{nk} e_{p1} = 0$. So, we have $0 = s_1(e_{nk} e_{p1}) = s_1(e_{nk}) e_{p1} + e_{nk} s_1(e_{p1}) = s_1(e_{nk}) e_{p1} = (\sum_{q<k} \beta_{kq} e_{nq}) e_{p1} = \beta_{kp} e_{n1}$.

It follows that the coefficients β_{kp} are zero except possibly for $\beta_{k1}(k = 2, \ldots, n-1)$. So, $s_1(e_{nk}) = \beta_{k1} e_{n1} = e_{nk}(\sum_{j=2}^{n-1} \beta_{j1} e_{j1})$.

Let $-b = \sum_{j=2}^{n-1} \beta_{j1} e_{j1}$. Then the inner derivation $s_b(x) = [b, x]$ has the property $(s_1 - s_b) e_{nl} = 0 (l = 1, \ldots, n-1)$ and $(s_1 - s_b) e_{k1} = -s_b(e_{k1}) = 0 (k = 2, \ldots, n)$.

Therefore, $s_1 - s_b = s_t - (s_a + s_b)$ is a *uaz*-derivation and $s_a + s_b$ is an inner derivation. □

COROLLARY 4.4. *Let δ be a derivation of A. Then $\delta = i_d + \bar{\sigma} + s_i + s_{uaz}$ where i_d is a diagonal inner, $\bar{\sigma}$ is a trivial extension of K, s_i is an inner derivation and s_{uaz} is a uaz-derivation.*

The left(right) annihilators of A are the quantities $e_{21}, e_{31}, \ldots, e_{n1}$ $(e_{n1}, e_{n2}, \ldots, e_{n,n-1})$ and their linear combinations.

DEFINITION 4.5. A strongly nilpotent derivation δ of A is called an *az*-derivation (annihilator zero derivation) if $\delta(a) = 0$ for every absolute left or right divisor of zero a.

It is obvious that an *az*-derivation is a *uaz*-derivation. Moreover, for a K-derivation, an *az*-derivation is equal to a *uaz*-derivation.

In general, every derivation cannot be expressed as a sum of diagonal, trivial extension, inner and *az*-derivations. The derivation given in the next example is a *uaz*-derivation, but not an *az*-derivation.

EXAMPLE 4.6. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{Z}[X] & 0 & 0 \\ \mathbf{Z}[X] & \mathbf{Z}[X] & 0 \end{pmatrix}$$

where $\mathbf{Z}[X]$ is a polynomial ring over an integer $\mathbf{Z}$.

Define $\delta : A \longrightarrow A$ by

$$\delta(\sum f_{ij}e_{ij})(i > j) = \frac{d}{dx} f_{21}e_{31}.$$

Then δ is strongly nilpotent and inner part of δ is 0, that is, δ is a uaz-derivation. But $\delta \neq az$-derivation.

References

[1] S. A. Amitsur, *Extension of derivations to central simple algebras*, Comm. Algebra **10** (1982), no. 8, 797–803.

[2] J. H. Chun and J. W. Park, *Prime ideals of subrings of matrix rings*, Commun. Korean Math. Soc. **19** (2004), no. 2, 211–217.

[3] R. Dubisch and S. Perlis, *On total nilpotent algebra*, Amer. J. Math. **73** (1951), 439–452.

[4] I. N. Herstein, *Noncommutative rings*, The Mathematical Association of America, 1968.

[5] F. Kuzucuoglu and V. M. Levchuk, *The automorphism group of certain radical matrix rings*, J. Algebra **243** (2001), no. 2, 473–485.

[6] V. M. Levchuk, *Automorphisms of certain nilpotent matrix groups and rings*, Dokl. Akad. Nauk SSSR **222** (1975), no. 6, 1279–1282.

[7] ———, *Connections between the unitriangular group and certain rings.II.Groups of automorphisms*, Sibirsk. Mat. Zh. **24** (1983), no. 4, 64–80.

[8] A. Nowicki, *Derivations of special subrings of matrix rings and regular graph*, Tsukuba J. Math. **7** (1983), no. 2, 281–297.

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