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SOME EXTENSIONAL PROPERTIES OF MODULES

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ABSTRACT. The purpose of this paper is to describe some characterizations on rational and t-rational extensions of modules, to determine the forms of t-rational extensions of given t-torsion free module for a left exact radical t and investigate the maximal t-rational extensions of modules.

1. Preliminaries

Throughout this paper, R means a ring with identity and all modules are unitary left R-modules. First we introduce the notions of rational extensions and essential extensions of modules. This theory will be need in the construction of the maximal right ring of quotients. Much of these research come from the work of Utumi [10] and Findlay-Lambek [3].

Let N be a submodule of a module M. Then M is called a *rational extension* of N or N is a *rational submodule* (or *dense submodule*) of M, if for every module D with $N \subseteq D \subseteq M$ and every R-homomorphism $f : D \longrightarrow M$, the inclusion $N \subseteq Kerf$ implies $f = 0$. This notion is denoted by $N \leq_r M$. Clearly, $\mathbf{Z}$ is a rational submodule of Q as $\mathbf{Z}$-modules.

We shall say that a module E containing a module M *an essential extension* of M if every nonzero submodule of E intersects M nontrivially. An essential extension E of M is said to be *maximal* if no module properly containing E can be an essential extension of M. If E is an essential extension of M, then we shall also say that M is an *essential submodule* (or *large submodule*) of E, and which is denoted by $M \leq_e E$. In Lam's book [5], we get a useful characterization of essential extension, that is, $M \leq_e E$ if and only if for every $x \neq 0$ in E, there exists r in R such that $rx \neq 0$ in M.

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We denote the category of all modules by *R-Mod* and the injective hull of a module M by $E(M)$. A *preradical* t of an abelian category Θ assigns to each object A a subobject $t(A)$ in such a way that every morphism $A \longrightarrow B$ induces $t(A) \longrightarrow t(B)$ by restriction. In other words, a preradical is a subfunctor of the identity functor on Θ.

The class of all preradicals of Θ is a complete lattice, because there is a partial ordering in which $t_1 \leq t_2$ means that $t_1(C) \subseteq t_2(C)$ for all objects C, and any family (t_i) of preradicals has a least upper bound $\sum t_i$ and a greatest lower bound $\cap t_i$, defined in the obvious ways. If t_1 and t_2 are preradicals, one defines preradical $t_1 t_2$ as

$$t_1 t_2(C) \; = \; t_1(t_2(C))$$

for all objects C. A preradical t is called *idempotent* if $tt = t$, and is called a *radical* if $t(C/t(C)) = 0$ for every object C.

For a preradical t of *R-Mod*, a module A is said to be *t-torsion* (resp. *t-torsion free*) if $t(A) = A$ (resp. $t(A) = 0$). The t-torsion class (resp. t-torsion free class) of modules or the class of t-torsion (resp. t-torsion free) modules will be denoted by $T(t)$ (resp. $F(t)$). Also, for two preradicals t and s of *R-Mod*, we shall say that *s is less than t* or *t is larger than s* if $s(A) \subseteq t(A)$ for every module A.

We denote the left linear topology corresponding to a left exact preradical t by $L(t)$, that is,

$$L(t) \; = \; \{{}_R I \leq R \mid R/I \in T(t)\}.$$

We will refer to Lambek [6] and Stenström [8] for the remainder terminologies and basic properties of preradicals and torsion theories.

2. Rational and t-rational extensions

An exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ of homomorphisms of R-modules is said to be a *rational extension* of A if $\alpha(A) \leq_r B$. In this case we also say that B is a *rational extension* of A or A is a *rational submodule* of B. Our exposition here follows in part Bican-Kepka-Nemec [2] and Lam [5].

An exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ of R-homomorphisms is said to be an *essential extension* of A if $\alpha(A) \leq_e B$. We note that $N \leq_r M$ implies $N \leq_e M$. But the converse is not true.

The next proposition provides characterizations for rational extensions in terms of nonexistence of certain types of R-homomorphisms and additional condition.

PROPOSITION 2.1. *Let $N \leq M$ be left R-modules. Then the following conditions are equivalent*:

(1) $N \leq_r M$.

(2) *For every submodule D of M with the relation $N \subseteq D \subseteq M$, $Hom_R(D/N, M) = 0$.*

(3) *If $x, y \in M$ with $y \neq 0$, then there exists $r \in R$ such that $rx \in N$ and $ry \neq 0$.*

(4) $Hom_R(M/N, E(M)) = 0$.

Proof. (1) $\Longrightarrow$ (2) Assume that $N \leq_r M$, that is, for every module D with $N \subseteq D \subseteq M$ and every R-homomorphism $f : D \longrightarrow M$, the inclusion $f(N) = 0$ implies $f = 0$. Let $g : D/N \longrightarrow M$ be an R-homomorphism, that is, $g \in Hom_R(D/N, M)$. Define $h : D \longrightarrow M$ by $h(d) = g(d + N)$, where $d \in D$. Then clearly, h is well defined, R-homomorphism and $h(N) = 0$. Since $N \leq_r M$, $h = 0$. Hence $f = 0$.

(2) $\Longrightarrow$ (1) This is proved by similar method of (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) Suppose for some $x, y \in M$ with $y \neq 0$, we have for all $r \in R$ such that $rx \in N$, $ry = 0$. We define $f : N + Rx \longrightarrow M$ by $f(n + rx) = ry$, $(n \in N,\ r \in R)$. Obviously, f is an R-homomorphism vanishing on N. So by hypothesis of (2), $0 = f(x) = f(0+1x) = 1x = x$. This is contradiction.

(3) $\Longrightarrow$ (4) Assume that there exists a nonzero R-homomorphism $f : M \longrightarrow E(M)$ with $f(N) = 0$. Then since $f(M) \neq 0$, $M \cap f(M) = 0$. Then There exist $x,\ y \in M - \{0\}$ such that $y = f(x)$. By (3), there exists $r \in R$ with $rx \in N$ and $ry \neq 0$. But then $0 = f(rx) = rf(x) = ry$. This fact is contradiction.

(4) $\Longrightarrow$ (2) Suppose that for some D as in (2), there exists a nonzero R-homomorphism $g : D/N \longrightarrow M$. By the injectivity of $E(M)$, we can extend g to a nonzero homomorphism $M/N \longrightarrow E(M)$. □

Let Q be a module. We define a preradical k_Q by

$$k_Q(M) = \cap\{Kerf \mid f \in Hom_R(M, Q)\}$$

for all modules M. As is well known, k_Q is the largest radical among all preradicals t such that $t(Q) = 0$.

Hereafter, we will consider that every preradical means a preradical of R-Mod. Let t be a preradical. We call that an exact sequence $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is a *t-rational extension* of A if B is in $F(t)$ and any submodule of $Coker(\alpha)$ is in $T(t)$. If α is an inclusion map, then we say that A is a *t-rational submodule* of B. Immediately, an exact sequence $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is a rational extension of A if and only if it is a k_B-rational extension of

A. Also, if an exact sequence $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is a t-rational extension of A, then it is a rational extension of A.

The converse of this above remark is not true in general as following:

EXAMPLE 2.2. Let R be the ring $\mathbf{Z}$ of rational integers. We put $A = 12\mathbf{Z}$, $\mathbf{B} = \mathbf{Z}$ and $t = Soc$. Since $Z(B) = 0$ and A is an essential submodule of B, $0 \longrightarrow A \overset{i}{\longrightarrow} B$ is a rational extension of A, where i is an inclusion map. However, $Soc(B/A) = \mathbf{2Z/12Z} \neq \mathbf{B/A}$. Thus $0 \longrightarrow A \overset{i}{\longrightarrow} B$ is not a t-rational extension of A.

On the other hand, for any preradical t, an exact sequence $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is called a *t-essential extension* of A if it is an essential extension of A and any submodule of $Coker(\alpha)$ is in $T(t)$. If α is an inclusion map, we say that A is a *t-essential submodule* of B.

PROPOSITION 2.3. *If $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is a t-rational extension of A, then it is a t-essential extension of A.*

Proof. It is sufficient to show that $\alpha(A)$ is an essential submodule of B. Suppose $\alpha(A) \cap D = 0$ and $D \leq B$. Then

$$D \cong D/(\alpha(A) \cap D) \cong (\alpha(A) + D)/\alpha(A).$$

Since $\alpha(A) + D)/\alpha(A)$ is t-torsion and D is t-torsion free, $\alpha(A) + D) = \alpha(A)$. With the fact $\alpha(A) \cap D = 0$, we get $D = 0$. □

Also the converse of this proposition is not true in general as following:

EXAMPLE 2.4. Let R be the ring $\mathbf{Z}$ of rational integers. We put $A = \mathbf{4Z/8Z}$, $\mathbf{B} = \mathbf{Z/8Z}$ and $t = Z$, where Z is the singular torsion functor. Then $Z(B) = B$, and so $Z(B/A) = B/A$. Also since A is an essential submodule of B, $0 \longrightarrow A \overset{i}{\longrightarrow} B$ is a t-essential extension of A, where i is an inclusion map. However, $0 \longrightarrow A \overset{i}{\longrightarrow} B$ is not a t-rational extension of A.

We note that for any t a left exact preradical, an exact sequence $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is a t-rational extension of A if and only if B is in $F(t)$ and $Coker(\alpha)$ is in $T(t)$.

PROPOSITION 2.5. *Let t be a left exact radical, and $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ and $0 \longrightarrow B \overset{\beta}{\longrightarrow} C$ two exact sequences. Then $0 \longrightarrow A \overset{\alpha}{\longrightarrow} B$ is a t-rational extension of A and $0 \longrightarrow B \overset{\beta}{\longrightarrow} C$ is a t-rational extension of B if and only if $0 \longrightarrow A \overset{\beta\alpha}{\longrightarrow} C$ is a t-rational extension of A.*

Proof. Assume that both $0 \longrightarrow A \xrightarrow{\alpha} B$ and $0 \longrightarrow B \xrightarrow{\beta} C$ are t-rational extensions. Then since

$$0 \longrightarrow \beta(B)/\beta\alpha(A) \longrightarrow C/\beta\alpha(A) \longrightarrow C/\beta(B) \longrightarrow 0$$

is exact,

$$(C/\beta\alpha(A))/(\beta(B)/\beta\alpha(A)) \cong C/\beta(B).$$

On the other hand, since $\beta(B)/\beta\alpha(A) \cong B/\alpha(A)$, $\beta(B)/\beta\alpha(A)$ and $C/\beta(B)$ are in $T(t)$, we see that $C/\beta\alpha(A)$ is in $T(t)$. Thus $0 \longrightarrow A \xrightarrow{\beta\alpha} C$ is a t-rational extension of A.

Conversely, suppose that $0 \longrightarrow A \xrightarrow{\beta\alpha} C$ is a t-rational extension of A. We must show that $B/\alpha(A)$ and $c/\beta(B)$ are in $T(t)$. Since

$$B/\alpha(A) \cong \beta(B)/\beta\alpha(A) \subseteq C/\beta\alpha(A),$$

$B/\alpha(A)$ is in $T(t)$. While since $C/\beta(B) \cong (C/\beta\alpha(A))/(\beta(B)/\beta\alpha(A))$ which is contained in $T(t)$, $C/\beta(B)$ is in $T(t)$. This completes the proof. □

Let t be a left exact preradical and $0 \longrightarrow A \xrightarrow{\alpha} B'$ an essential extension of A. We put that

$$t(B'/\alpha(A)) \;=\; B/\alpha(A),$$

where $\alpha(A) \subseteq B \subseteq B'$. Then $0 \longrightarrow A \xrightarrow{\alpha} B$ is an essential extension of A and $B/\alpha(A)$ is in $T(t)$. Hence we have

THEOREM 2.6. *Let t be a left exact preradical and A a t-torsion free module. Then an essential extension of A $0 \longrightarrow A \xrightarrow{\alpha} B'$ induces a t-rational extension of A.*

PROPOSITION 2.7. *Let t be a preradical and $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A and $0 \longrightarrow A \xrightarrow{\alpha'} B'$ an exact sequence. If there exists a homomorphism $f: B \longrightarrow B'$ such that $f\alpha = \alpha'$, then f is a unique monomorphism.*

Proof. By the hypothesis, the following diagram

$$\begin{array}{ccc} 0 \longrightarrow & A \xrightarrow{\alpha} & B \\ & \| & \downarrow \\ 0 \longrightarrow & A \xrightarrow{\alpha'} & B' \end{array}$$

is commutative.

First, we show that f is monomorphism. If $x \in Ker f \cap \alpha(A)$, then $f(x) = 0$ and $x = \alpha(a)$ for some $a \in A$. Thus $\alpha'(a) = f\alpha(a) = f(x) = 0$

and so $a = 0$ and $x = 0$. Since $\alpha(A)$ is an essential submodule of B, $Ker f = 0$, that is, f is a monomorphism.

Next, if there exists a homomorphism $f' : B \longrightarrow B'$ such that $f'\alpha = \alpha'$, then $f'\alpha = f\alpha$ and so $(f' - f)\alpha(a) = 0$ for all $a \in A$. Since $(f' - f)$ is monic, $\alpha(a) = 0$, that is, $(f' - f)(A) = 0$. Hence $f' = f$. □

THEOREM 2.8. *Let t be a preradical and $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A and $0 \longrightarrow A \xrightarrow{\alpha'} B'$ an exact sequence. If there exist homomorphisms $f : B \longrightarrow B'$ and $g : B' \longrightarrow B$ such that $f\alpha = \alpha'$ and $g\alpha' = \alpha$, then $gf = 1_B$.*

Proof. By the assumption, $gf\alpha = \alpha$ and so $(gf - 1_B)\alpha = 0$. Since

$$B/Ker(gf - 1_B) \cong Im(gf - 1_B) \subseteq B,$$

and $B/Ker(gf - 1_B)$ is t-torsion free. Also since $\alpha(A) \subseteq Ker(gf - 1_B)$ and $B/\alpha(A)$ is in $T(t)$, $B/Ker(gf - 1_B) = 0$, that is, $Ker(gf - 1_B) = B$. Thus $gf - 1_B = 0$, that is, $gf = 1_B$. □

COROLLARY 2.9. *Let t be a preradical, and both $0 \longrightarrow A \xrightarrow{\alpha} B$ and $0 \longrightarrow A \xrightarrow{\alpha'} B'$ t-rational extensions of A. If there exist homomorphisms $f : B \longrightarrow B'$ and $g : B' \longrightarrow B$ such that $f\alpha = \alpha'$ and $g\alpha' = \alpha$, then $B \cong B'$.*

Let t be a preradical. We call an exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ a *maximal t-rational extension* of A if

(i) $0 \longrightarrow A \xrightarrow{\alpha} B$ is a t-rational extension of A,

(ii) For any t-rational extension of A $0 \longrightarrow A \xrightarrow{\alpha'} B'$, there exists a homomorphism $f : B' \longrightarrow B$ such that $f\alpha' = \alpha$.

By the Corollary 2.9, we obtain the following important statement.

THEOREM 2.10. *Let t be a left exact radical and A a t-torsion free module. Then there exists a maximal t-rational extension of A, uniquely up to isomorphism.*

Proof. We put that $t(E(A)/A) = B/A$, where $A \subseteq B \subseteq E(A)$. Then as is easily seen, $0 \longrightarrow A \xrightarrow{i} B$ is a t-rational extension of A, where i is an inclusion map. Let $0 \longrightarrow A \xrightarrow{\alpha'} B'$ be any t-rational extension of A. Then we have the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha'} & B' \\ & & {\scriptstyle i'}\downarrow & \swarrow & \\ & & E(A), & & \end{array}$$

where i' is an inclusion map. Thus $f\alpha' = i'$ and f is monomorphism, we show that $f(B') \subseteq B$. Since $f\alpha'(A) = A$ and $\alpha(A) \subseteq B'$, $A \subseteq Imf$. Also, since

$$(Imf + B)/B \cong Imf/(Imf \cap B) \cong (Imf/A)/((Imf \cap B)/A)$$

and

$$Imf/A \cong B'/\alpha'(A),$$

$(Imf + B)/B$ is in $T(t)$. Since

$$0 \longrightarrow B/A \longrightarrow (Imf + B)/A \longrightarrow (Imf + B)/B \longrightarrow 0$$

is exact, B/A and $(Imf + B)/B$ are in $T(t)$. Thus $(Imf + B)/A$ is in $T(t)$.

Moreover,

$$B/A = t(E(A)/A) \supseteq t((Imf + B)/A) = (Imf + B)/A$$

and

$$t(B/A) = B/A.$$

Thus $B = Imf + B$, that is, $Imf \subseteq B$. Hence $0 \longrightarrow A \xrightarrow{i} B$ is a maximal t-rational extension of A.

Finally, let two exact sequences $0 \longrightarrow A \xrightarrow{\alpha_1} B_1$ and $0 \longrightarrow A \xrightarrow{\alpha_2} B_2$ be maximal t-rational extensions of A. Then by Corollary 2.9, we have $B_1 \cong B_2$. □

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