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ON A TOPOLOGICAL DIVISOR OF ZERO IN THE CALKIN ALGEBRA

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ABSTRACT. We give a simple proof of the statement that if T is a bounded linear operator on a complex Hilbert space then T is Fredholm if and only if $\pi(T)$ is not a TDZ, where $\pi(\cdot)$ is the Calkin homomorphism.

An element x of a Banach algebra A is called a *left* [*right*] *divisor of zero* if there exists a nonzero element $y \in A$ such that $xy = 0$ [$yx = 0$] and is called a *left* [*right*] *topological divisor of zero* (briefly, TDZ) if there exists a sequence $y_n \in A$ with $||y_n|| = 1$ for all $n \in \mathbf{N}$ such that $xy_n \to 0$ [$y_n x \to 0$]. Let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. The algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is called the *Calkin algebra* and let π denote the canonical homomorphism of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, which is often called the Calkin homomorphism. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *left Fredholm* if it has closed range with finite dimensional null space and *right Fredholm* if it has closed range with its range of finite co-dimension. If T is both left and right Fredholm we call it *Fredholm*. The Atkinson's theorem says that T is Fredholm if and only if $\pi(T)$ is invertible. In [1, Theorem 8.7.3] it was shown, using the argument of enlargements, that T is Fredholm if and only if $\pi(T)$ is not a divisor of zero. In this paper we give a simple proof of the statement that T is Fredholm if and only if $\pi(T)$ is not a TDZ.

We begin with:

LEMMA 1. *Let $T \in \mathcal{B}(\mathcal{H})$. If $\pi(T)$ is not a left divisor of zero then* $\dim T^{-1}(0) < \infty$.

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Proof. If we write $T = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}$ with respect to the decomposition $T^{-1}(0) \oplus T^{-1}(0)^{\perp}$, put $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$: then $TS = 0$, but $\pi(S) \neq 0$ if $T^{-1}(0)$ is infinite dimensional. □

THEOREM 2. *Let $T \in \mathcal{B}(\mathcal{H})$. If $\pi(T)$ is not a left TDZ, then T has closed range.*

Proof. Suppose that $\operatorname{ran}(T)$ is not closed. Let $T = U|T|$ be the polar decomposition of T and let E be the spectral measure on the Borel subsets of $\sigma(|T|)$ such that $|T| = \int z\,dE(z)$. Since $\operatorname{ran}(T)$ is not closed, we have that $\inf(\sigma(|T|)\setminus\{0\}) = 0$. Thus we can find a strictly decreasing sequence $\{\alpha_n\}$ of elements in $\sigma(|T|)\setminus\{0\}$ such that $\alpha_n \to 0$. Since the α_n are distinct there exists a sequence $\{U_n\}$ of mutually distinct open intervals such that $\alpha_n \in U_n$ for each $n \in \mathbf{N}$. Define

$$F_n := U_n \cap \sigma(|T|).$$

Then the F_n are non-empty relatively open subsets of $\sigma(|T|)$. Thus $E(F_n)\mathcal{H} \neq \{0\}$ for each $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, choose a unit vector x_n in $E(F_n)\mathcal{H}$. Since $\sigma(|T||_{E(F_n)\mathcal{H}}) \subset \mathrm{cl}F_n$ for each $n \in \mathbf{N}$, it follows that $|T||_{E(F_n)\mathcal{H}}$ is invertible for every $n \in \mathbf{N}$. Define S_n on $E(F_n)\mathcal{H}$ by

$$S_n z_n = \alpha_{n-1}\left(|T||_{E(F_n)\mathcal{H}}\right)^{-1} P_n z_n \quad \text{for } z_n \in E(F_n)\mathcal{H},$$

where $\alpha_0 := 1$ and P_n is the orthogonal projection of $E(F_n)\mathcal{H}$ onto $\bigvee\{x_n\}$. Then $||S_n|| \geq 1$ for all $n \in \mathbf{N}$. Define the operator A_n on $\mathcal{H}$ by

$$A_n := \left(\bigoplus_{k=1}^{n} S_k\right) \oplus \left(\bigoplus_{k=n+1}^{\infty} I|_{E(F_k)\mathcal{H}}\right).$$

Since $\bigoplus_{k=n=1}^{\infty} I|_{E(F_k)\mathcal{H}}$ is not compact, A_n is not compact. Moreover $||\pi(A_n)|| \geq 1$ for all $n \in \mathbf{N}$. But

$$|T|A_n = \left(\bigoplus_{k=1}^{n} \alpha_{n-1} I|_{\vee\{x_n\}}\right) \oplus \left(\bigoplus_{k=n+1}^{\infty} |T|_k\right),$$

where $\alpha_0 := 1$ and $|T|_k := |T|_{E(F_k)\mathcal{H}}$. Put $K := \bigoplus_{k=1}^{\infty} \alpha_{n-1} I|_{\vee\{x_n\}}$.

Then K is compact and

$$\begin{aligned} \| \, |T|A_n - K \| &= \| \bigoplus_{k=n+1}^{\infty} |T|_k + \bigoplus_{k=n+1}^{\infty} \alpha_{n-1} I|_{\vee\{x_n\}} \| \\ &\le \| \bigoplus_{k=n+1}^{\infty} |T|_k \| + \| \bigoplus_{k=n+1}^{\infty} \alpha_{n-1} I|_{\vee\{x_n\}} \| \\ &\le 2\alpha_n \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore

$$\|TA_n - UK\| = \|U(|T|A_n - K)\| \le \| \, |T|A_n - K\| \to 0 \quad \text{as } n \to \infty,$$

which implies that $\pi(T)$ is a left TDZ, a contradiction. □

COROLLARY 3. *Let* $T \in \mathcal{B}(\mathcal{H})$. *Then*

(3.1) $\pi(T)$ *is not a left TDZ* $\Longleftrightarrow$ T *is left Fredholm.*

Hence, in particular,

(3.2) $\pi(T)$ *is not a TDZ* $\Longleftrightarrow$ T *is Fredholm.*

Proof. The forward implication of (3.1) follows from Lemma 1 and Theorem 2, and the backward implication comes from the Atkinson's theorem. The implication (3.2) follows at once from (3.1) together with the dual argument. □

References

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