ON A TOPOLOGICAL DIVISOR OF ZERO IN THE CALKIN ALGEBRA

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ABSTRACT. We give a simple proof of the statement that if $T$ is a bounded linear operator on a complex Hilbert space then $T$ is Fredholm if and only if $\pi(T)$ is not a TDZ, where $\pi(\cdot)$ is the Calkin homomorphism.

An element $x$ of a Banach algebra $A$ is called a left [right] divisor of zero if there exists a nonzero element $y \in A$ such that $xy = 0$ [$yx = 0$] and is called a left [right] topological divisor of zero (briefly, TDZ) if there exists a sequence $y_n \in A$ with $||y_n|| = 1$ for all $n \in \mathbb{N}$ such that $xy_n \to 0$ [$y_n x \to 0$]. Let $\mathcal{H}$ be a complex Hilbert space, let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$, and let $K(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. The algebra $B(\mathcal{H})/K(\mathcal{H})$ is called the Calkin algebra and let $\pi$ denote the canonical homomorphism of $B(\mathcal{H})$ onto $B(\mathcal{H})/K(\mathcal{H})$, which is often called the Calkin homomorphism. An operator $T \in B(\mathcal{H})$ is called left Fredholm if it has closed range with finite dimensional null space and right Fredholm if it has closed range with its range of finite co-dimension. If $T$ is both left and right Fredholm we call it Fredholm. The Atkinson’s theorem says that $T$ is Fredholm if and only if $\pi(T)$ is invertible. In [1, Theorem 8.7.3] it was shown, using the argument of enlargements, that $T$ is Fredholm if and only if $\pi(T)$ is not a divisor of zero. In this paper we give a simple proof of the statement that $T$ is Fredholm if and only if $\pi(T)$ is not a TDZ.

We begin with:

**Lemma 1.** Let $T \in B(\mathcal{H})$. If $\pi(T)$ is not a left divisor of zero then $\dim T^{-1}(0) < \infty$.

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Proof. If we write $T = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}$ with respect to the decomposition $T^{-1}(0) \oplus T^{-1}(0)^{\perp}$, put $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$: then $TS = 0$, but $\pi(S) \neq 0$ if $T^{-1}(0)$ is infinite dimensional. \hfill \Box

Theorem 2. Let $T \in \mathcal{B}(\mathcal{H})$. If $\pi(T)$ is not a left TDZ, then $T$ has closed range.

Proof. Suppose that $\text{ran}(T)$ is not closed. Let $T = U |T|$ be the polar decomposition of $T$ and let $E$ be the spectral measure on the Borel subsets of $\sigma(|T|)$ such that $|T| = \int z \ dE(z)$. Since $\text{ran}(T)$ is not closed, we have that $\inf(\sigma(|T|) \setminus \{0\}) = 0$. Thus we can find a strictly decreasing sequence $\{\alpha_n\}$ of elements in $\sigma(|T|) \setminus \{0\}$ such that $\alpha_n \to 0$. Since the $\alpha_n$ are distinct there exists a sequence $\{U_n\}$ of mutually distinct open intervals such that $\alpha_n \in U_n$ for each $n \in \mathbb{N}$. Define

$$F_n := U_n \cap \sigma(|T|).$$

Then the $F_n$ are non-empty relatively open subsets of $\sigma(|T|)$. Thus $E(F_n)\mathcal{H} \neq \{0\}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, choose a unit vector $x_n$ in $E(F_n)\mathcal{H}$. Since $\sigma(|T|)_{E(F_n)\mathcal{H}} \subset \text{cl}F_n$ for each $n \in \mathbb{N}$, it follows that $|T|_{E(F_n)\mathcal{H}}$ is invertible for every $n \in \mathbb{N}$. Define $S_n$ on $E(F_n)\mathcal{H}$ by

$$S_n z_n = \alpha_{n-1} (|T|_{E(F_n)\mathcal{H}})^{-1} P_n z_n \quad \text{for } z_n \in E(F_n)\mathcal{H},$$

where $\alpha_0 := 1$ and $P_n$ is the orthogonal projection of $E(F_n)\mathcal{H}$ onto $\bigvee\{x_n\}$. Then $\|S_n\| \geq 1$ for all $n \in \mathbb{N}$. Define the operator $A_n$ on $\mathcal{H}$ by

$$A_n := \left( \bigoplus_{k=1}^{n} S_k \right) \oplus \left( \bigoplus_{k=n+1}^{\infty} I_{E(F_k)\mathcal{H}} \right).$$

Since $\bigoplus_{k=1}^{\infty} I_{E(F_k)\mathcal{H}}$ is not compact, $A_n$ is not compact. Moreover $\|\pi(A_n)\| \geq 1$ for all $n \in \mathbb{N}$. But

$$|T| A_n = \left( \bigoplus_{k=1}^{n} \alpha_{n-1} |T|_{\bigvee\{x_n\}} \right) \oplus \left( \bigoplus_{k=n+1}^{\infty} |T|_k \right),$$

where $\alpha_0 := 1$ and $|T|_k := |T|_{E(F_k)\mathcal{H}}$. Put $K := \bigoplus_{k=1}^{\infty} \alpha_{n-1} |T|_{\bigvee\{x_n\}}$. 


Then $K$ is compact and
\[
\|\|TA_n - K\|\| = \|\bigoplus_{k=n+1}^{\infty} T|_k + \bigoplus_{k=n+1}^{\infty} \alpha_{n-1}I|_{\mathcal{V}\{x_n\}}\| \\
\leq \|\bigoplus_{k=n+1}^{\infty} T|_k\| + \|\bigoplus_{k=n+1}^{\infty} \alpha_{n-1}I|_{\mathcal{V}\{x_n\}}\| \\
\leq 2\alpha_n \to 0 \quad \text{as } n \to \infty.
\]
Therefore
\[
\|\|TA_n - UK\|\| = \|\|U(T|A_n - K)\|\| \leq \|\|T|A_n - K\|\| \to 0 \quad \text{as } n \to \infty,
\]
which implies that $\pi(T)$ is a left TDZ, a contradiction. \qed

**Corollary 3.** Let $T \in \mathcal{B}(\mathcal{H})$. Then

\[(3.1) \quad \pi(T) \text{ is not a left TDZ } \iff T \text{ is left Fredholm.}\]

Hence, in particular,

\[(3.2) \quad \pi(T) \text{ is not a TDZ } \iff T \text{ is Fredholm.}\]

**Proof.** The forward implication of (3.1) follows from Lemma 1 and Theorem 2, and the backward implication comes from the Atkinson’s theorem. The implication (3.2) follows at once from (3.1) together with the dual argument. \qed

**References**