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A BMO TYPE CHARACTERIZATION OF WEIGHTED LIPSCHITZ FUNCTIONS IN TERMS OF THE BEREZIN TRANSFORM

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ABSTRACT. The Berezin transform is the analogue of the Poisson transform in the Bergman spaces. Dyakonov characterize the holomorphic weighted Lipschitz function in the unit disk in terms of the Poisson integral. In this paper, we characterize the harmonic weighted Lipschitz function in terms of the Berezin transform instead of the Poisson integral.

1. Introduction

A continuous function $\omega : (0,\infty) \to \mathbb{R}$ with $\limsup_{t\to 0^+} \omega(t) = 0$ will be called a modulus of continuity, if $\omega(t)$ is non-negative and non-decreasing. If, in addition, there is a constant $C(\omega) > 0$ such that

$$\int_0^t \frac{\omega(s)}{s} ds + t \int_t^\infty \frac{\omega(s)}{s^2} ds \le C(\omega)\omega(t),$$

where $0 < t < 1$, then we say that it is regular. Given a modulus of continuity ω and $E \subset \mathbb{R}^N$, we define the weighted Lipschitz space by

$$\Lambda_\omega(E) = \{f : |f(x)| \le M,\ |f(x) - f(y)| \le M\omega(|x-y|),\ x,y \in E\}$$

with norm the smallest M. Weighted Lipschitz spaces have been studied by many authors (see [5, 6, 8, 9, 10, 12, 13], and references in their papers).

Let $\mathbf{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ its boundary in [8], Dyakonov proved the following.

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THEOREM 1. *Let f be a holomorphic function on $\mathbf{D}$ that is continuous up to $\mathbf{T}$. If both ω and ω^2 are regular, then*

$$\|f\|_{\Lambda_\omega(\mathbf{T})} \sim \sup_{\mathbf{D}} \frac{1}{\omega(1-|z|)}[\mathcal{P}(|f|^2)(z) - |f(z)|^2]^{1/2}.$$

Here $\mathcal{P}$ is the Poisson integral operator defined by

$$\mathcal{P}g(z) = \int_{\mathbf{T}} g(\zeta)\frac{1-|z|^2}{|\zeta - z|^2}d\sigma(\zeta), \quad z \in \mathbf{D},$$

and by

$$\mathcal{P}g(\zeta) = g(\zeta), \quad \zeta \in \mathbf{T}.$$

For every function $h \in L^1(\mathbf{D}, dA)$, we define

$$\mathbb{B}h(z) = \int_{\mathbf{D}} \frac{(1-|z|^2)^2}{|1-z\bar{w}|^4} h(w)dA(w), \quad z \in \mathbf{D},$$

where dA is the normalized area measure on $\mathbf{D}$. The operator $\mathbb{B}$ will be called the Berezin transform. It is remarkable that $h \in L^1(\mathbf{D}, \mathbf{dA})$ is harmonic if and only if $\mathbb{B}h = h$ (see [1]).

In this paper, we prove the Berezin transform version of Theorem 1 as following.

THEOREM 2. *Let ω be a regular modulus of continuity. Let h be harmonic in $\mathbf{D}$. Consider the equalities*

$$M_0(h) = \sup_{z\in\mathbf{D}} \left[\frac{(1-|z|)}{\omega(1-|z|)}|\nabla h(z)|\right],$$

$$M_1(h) = \sup_{z\in\mathbf{D}} \frac{1}{\omega(1-|z|)}\mathbb{B}(|h-h(z)|)(z),$$

$$M_2(h) = \sup_{z\in\mathbf{D}} \frac{1}{\omega(1-|z|)}\left[\mathbb{B}(|h|^2)(z) - |(\mathbb{B}h)(z)|^2\right]^{1/2}.$$

(i) *Let $h \in L^1(\mathbf{D})$. If h is harmonic in $\mathbf{D}$, then we have*

$$M_0(h) \sim M_1(h).$$

(ii) *Let $h \in L^2(\mathbf{D})$. If h is harmonic in $\mathbf{D}$, then we have*

$$M_0(h) \sim M_2(h).$$

Let ω be a regular modulus of continuity. By the same argument as the proof of Hardy-Littlewood lemma for harmonic Lipschitz functions, we can prove that

$$\|h\|_{\Lambda_\omega(\mathbf{D})} \sim \sup_{\mathbf{D}}|h| + M_0(f) \tag{1}$$

for functions h harmonic in $\mathbf{D}$. By (1) and Theorem 2, we have the following.

COROLLARY 3. *Let ω be a regular modulus of continuity. Let h be harmonic in $\mathbf{D}$.*

(i) *If $h \in L^1(\mathbf{D})$, then we have*

$$\|h\|_{\Lambda_\omega(\mathbf{D})} \sim \sup_{z\in\mathbf{D}} |h| + M_1(h).$$

(ii) *If $h \in L^2(\mathbf{D})$, then we have*

$$\|h\|_{\Lambda_\omega(\mathbf{D})} \sim \sup_{z\in\mathbf{D}} |h| + M_2(h).$$

The Berezin transform was introduced by Berezin in [3] and [4] and most applications of the Berezin transform so far have been in study of Hankel and Toeplitz operators (see [16]). The characterization of the BMO function in the Bergman metric by the Berezin transform was begun by Zhu in his thesis [15] and then developed by Békollé, Berger, Coburn, and Zhu in [2]. We state the results in Section 2. Their results motivated the authors to consider the characterization of the harmonic Lipschitz function in terms of the Berezin transform.

2. Preliminaries

LEMMA 4. *Let Ω be an open subset in $\mathbb{R}^N$. Let ℓ be a line segment lying entirely in Ω. Suppose that $f : \Omega \to \mathbb{R}$ is continuous and*

$$\liminf_{x\to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} = 0, \quad \text{for each} \quad x_0 \in \ell.$$

Then f is constant on ℓ.

PROOF. Let $a, b \in \Omega$ and $\ell = \{ta + (1-t)b : 0 \le t \le 1\}$. Suppose that f is not constant on ℓ. Then there are $t_1, t_2 \in [0, 1]$ such that

$$f(a_0) = f(t_1 a + (1-t_1)b) < f(t_2 a + (1-t_2)b) = f(b_0).$$

We may assume that $t_1 < t_2$. We choose a line L with slope $\alpha > 0$ separating these two points $(a_0, f(a_0)), (b_0, f(b_0))$ in the section $\{(x, s) : x \in \ell, s \in \mathbb{R}\}$. Let

$$t_0 = \sup\{t : (ta + (1-t)b, f(ta + (1-t)b)) \in L\}.$$

Then for $t_0 < t < 1$ the point $(ta + (1-t)b, f(ta + (1-t)b)$ lies above L. Thus we have

$$\liminf_{t \to t_0^+} \frac{|f(ta + (1-t)b)) - f(t_0 a + (1-t_0)b))|}{|(t-t_0)a + (t-t_0)b|} \geq \alpha > 0.$$

This is a contradiction. □

COROLLARY 5. *Let Ω be a connected open subset in $\mathbb{R}^N$. If*

$$\liminf_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} = 0, \quad \text{for each} \quad x_0 \in \Omega,$$

then f is constant on Ω.

PROOF. Let $a, b \in \Omega$. Since Ω is connected, a and b can be joined by a polygonal curve lying entirely in Ω. By Lemma 4, $f(a) = f(b)$. □

PROPOSITION 6. *Let ω be a modulus of continuity. Let Ω be a connected open subset in $\mathbb{R}^N$. If $\liminf_{t \to 0^+} (\omega(t)/t) = 0$, then $\Lambda_\omega(\Omega) = \{constants\}$.*

PROOF. We have

$$\liminf_{y \to 0} \frac{|f(x+y) - f(x)|}{|y|} \lesssim \liminf_{y \to 0} \frac{\omega(|y|)}{|y|} = 0, \quad \text{for each} \quad x \in \Omega.$$

By Corollary 5, we get the result. □

LEMMA 7. *Let ω be a regular modulus of continuity. There is $C > 0$ such that*

$$\frac{\omega(\tau)}{\tau} \leq C \frac{\omega(t)}{t} \quad \text{for} \quad t \leq \tau.$$

PROOF. Since ω is non-decreasing, for $t \leq \tau$, we have

$$\frac{\omega(\tau)}{\tau} = 2^2 \tau \frac{\omega(\tau)}{(2\tau)^2} \leq 2^2 \int_\tau^{2\tau} \frac{\omega(s)}{s^2} ds \leq 2^2 \int_t^\infty \frac{\omega(s)}{s^2} ds \lesssim \frac{\omega(t)}{t}.$$

□

REMARK 1. By Lemma 7, if ω is regular, then $\omega(t)/t$ is bounded below and so it excludes $\omega(t) = t^\alpha$ for $\alpha > 1$.

COROLLARY 8. *Let ω be a regular modulus of continuity. Let $M \geq 1$. Let C be the constant in Lemma 7. Then we get*

$$\omega(Mt) \leq CM\omega(t) \quad \text{for all} \quad t > 0.$$

3. BMO in the Bergman metric

The Bergman metric on $\mathbf{D}$, also called the hyperbolic metric or the Poincaré metric, is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad z,w \in \mathbf{D}.$$

For any $z \in \mathbf{D}$ and $r > 0$, let

$$D(z,r) = \{w \in \mathbf{D} : \beta(z,w) < r\}$$

be the Bergman metric disk with center z and radius r.

Given a function $h \in L^1(\mathbf{D})$, we define an averaging function $\hat{h}_r(z)$ on $\mathbf{D}$ as follows:

$$\hat{h}_r(z) = \frac{1}{|D(z,r)|}\int_{D(z,r)} h(w)dA(w), \quad z \in \mathbf{D}.$$

For $h \in L^2(\mathbf{D})$, we define the mean oscillation of h at z in the Bergman metric to be

$$MO_r(h)(z) = \left(\frac{1}{|D(z,r)|}\int_{D(z,r)} |h(w) - \hat{h}_r(z)|^2 dA(w)\right)^{1/2}.$$

Let $BMO_r = BMO_r(\mathbf{D})$ denote the space of all integrable functions h such that

$$\|h\|_r = \sup\{MO_r(h)(z) : z \in \mathbf{D}\} < +\infty.$$

In fact, BMO_r is independent of r and can be described in terms of the Berezin transform.

THEOREM 9 ([2], [11], [16]). *Suppose $0 < r < +\infty$ and that the function h is locally square-integrable in $\mathbf{D}$. Then $h \in BMO_r$ if and only if $h \in L^2(\mathbf{D})$,*

$$\sup_{z\in\mathbf{D}}[\mathbb{B}(|h|^2)(z) - |\mathbb{B}h(z)|^2]^{1/2} < +\infty.$$

4. Proof of Theorem 2

LEMMA 10 ([7]). *Let $1 \le p < \infty$. Then*

$$\int_{\mathbf{D}} |h|^p dA \lesssim \int_{\mathbf{D}} (1-|z|)^p |\nabla h(z)|^p dA + |h(0)|^p$$

for functions h harmonic on $\mathbf{D}$.

LEMMA 11. *Let $1 \le p < \infty$. Let $h \in L^p(\mathbf{D})$ be harmonic in $\mathbf{D}$. Then*

$$|\nabla h(0)|^p \lesssim \int_{\mathbf{D}} |h(\zeta) - h(0)|^p dA(\zeta).$$

PROOF. Let $\chi : \mathbb{C} \to \mathbb{R}$ be a cut-off function satisfying:

$$\chi \in C_0^\infty, \quad \chi \ge 0 \text{ on } \mathbf{D}, \quad \chi(z) = \chi(|z|), \quad \text{supp}\chi \subset \mathbf{D}, \quad \int_{\mathbf{D}} \chi dA = 1.$$

For $\epsilon > 0$, define

$$\chi_\epsilon(z) = \frac{1}{\epsilon^2}\chi\left(\frac{z}{\epsilon}\right) \quad \text{and} \quad \mathbf{D}_\epsilon(z) = \{\zeta : |z - \zeta| < \epsilon\}.$$

Since h is harmonic, we have

$$h(z) - h(0) = \int_{\zeta \in \mathbf{D}_\epsilon(z)} (h(\zeta) - h(0))\chi_\epsilon(z - \zeta) dA(\zeta).$$

It follows that

$$\begin{aligned} |\nabla h(z)| &\le \sup_{\zeta \in \mathbf{D}_\epsilon(z)} |\nabla \chi_\epsilon(z - \zeta)| \int_{\zeta \in \mathbf{D}_\epsilon(z)} |h(\zeta) - h(0)| dA(\zeta) \\ &\lesssim \frac{1}{\epsilon^3} \int_{\mathbf{D}} |h(\zeta) - h(0)| dA(\zeta). \end{aligned}$$

Thus we have

$$|\nabla h(0)|^p \lesssim \frac{1}{\epsilon^{3p}} \int_{\zeta \in \mathbf{D}} |h(\zeta) - h(0)|^p dA(\zeta).$$

□

LEMMA 12. *Let ω be a regular modulus of continuity. Then we have*

$$\int_{\mathbf{D}} \frac{\omega(1 - |w|)}{|1 - \bar{z}w|^3} dA(w) \lesssim \frac{\omega(1 - |z|)}{1 - |z|}.$$

PROOF. By the polar coordinate and the inequality in [14, Proposition 1.4.10], we have

$$\begin{aligned} \int_{\mathbf{D}} \frac{\omega(1 - |w|)}{|1 - \bar{z}w|^3} dA(w) &= \int_0^1 \int_{\mathbf{T}} \frac{\omega(1 - r)}{|1 - r\bar{z}\zeta|^3} d\sigma(\zeta) r dr \\ &\lesssim \int_0^1 \frac{\omega(1 - r)}{(1 - r|z|)^2} dr. \end{aligned}$$

Note that $1 - r|z| = (1 - r) + (1 - |z|) - (1 - |z|)(1 - r)$. Thus we have

$$\int_0^1 \frac{\omega(1 - r)}{(1 - r|z|)^2} dr = \int_0^1 \frac{\omega(s)}{[s + (1 - |z|) - (1 - |z|)s]^2} ds,$$

by putting $1 - r = s$. We decompose the integral by two parts as following

$$\begin{aligned}&\int_0^1 \frac{\omega(s)}{[s+(1-|z|)-(1-|z|)s]^2}ds \\ &= \int_0^{1-|z|} + \int_{1-|z|}^1 \frac{\omega(s)}{[s+(1-|z|)-(1-|z|)s]^2}ds.\end{aligned}$$

For the first part we have

$$\begin{aligned}\int_0^{1-|z|} \frac{\omega(s)}{[s+(1-|z|)-(1-|z|)s]^2}ds &\lesssim \int_0^{1-|z|} \frac{\omega(s)}{s[s|z|+(1-|z|)]}ds \\ &\lesssim \frac{1}{1-|z|}\int_0^{1-|z|} \frac{\omega(s)}{s}ds \\ &\lesssim \frac{\omega(1-|z|)}{1-|z|}.\end{aligned}$$

Now for the second part we have

$$\int_{1-|z|}^1 \frac{\omega(s)}{[s+(1-|z|)-(1-|z|)s]^2}ds \lesssim \int_{1-|z|}^1 \frac{\omega(s)}{s^2}ds \lesssim \frac{\omega(1-|z|)}{1-|z|}.$$

We use the regular condition of ω at the last step of estimates for two parts of the integral. Thus we complete the proof. □

For any $z \in \mathbf{D}$, let φ_z be the Möbius transformation on $\mathbf{D}$ defined by

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w}, \quad z, w \in \mathbf{D}.$$

Then the Möbius map φ_z has the following properties:

$$\begin{aligned}\varphi_z'(w) &= \frac{-(1-|z|^2)}{(1-\bar{z}w)^2} = -k_z(w), \\ (1-|w|^2)k_z(w) &= 1 - |\varphi_z(w)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\bar{z}w|^2}.\end{aligned}$$

The real Jacobian determinant of φ_z at w is

$$\det J_{\mathbb{R}}\varphi_z(w) = |\varphi_z'(w)|^2 = |k_z(w)|^2 = \frac{(1-|z|^2)^2}{|1-\bar{z}w|^4}.$$

PROOF OF THEOREM 2. (i) By the change of variable, we have that

$$\begin{aligned}\mathbb{B}(|h-h(z)|)(z) &= \int_{\mathbf{D}} |h(w)-h(z)|\mathbb{B}(z,w)dA(w)\\ &= \int_{\mathbf{D}} |(h\circ\varphi_z)(w)-h(z)|dA(w).\end{aligned}$$

By Lemma 10, we have

$$\int_{\mathbf{D}} |h\circ\varphi_z(w)-h(z)|dA \lesssim \int_{\mathbf{D}} (1-|w|^2)|\nabla(h\circ\varphi_z)(w)|dA.$$

We note that

$$|\nabla(h\circ\varphi_z)(w) \lesssim |\nabla h(\varphi_z(w)|k_z(w) \quad \text{and} \quad (1-|w|^2)k_z(w) = 1-|\varphi_z(w)|^2.$$

Thus it follows that

$$\int_{\mathbf{D}} (1-|w|^2)|\nabla(h\circ\varphi_z)(w)|dA \lesssim \int_{\mathbf{D}} (1-|\zeta|^2)|\nabla h(\zeta)||k_z(\zeta)|^2dA.$$

By Lemma 12, we have

$$\begin{aligned}&\int_{\mathbf{D}} (1-|\zeta|^2)|\nabla h(\zeta)||k_z(\zeta)|^2dA\\ &\lesssim \sup_{\zeta\in\mathbf{D}}\left[\frac{(1-|\zeta|}{\omega(1-|\zeta|)}|\nabla h(\zeta)|\right]\int_{\mathbf{D}}\frac{\omega(1-|\zeta|)(1-|z|)^2}{|1-\bar{z}\zeta|^4}dA(\zeta)\\ &\lesssim \sup_{\zeta\in\mathbf{D}}\left[\frac{(1-|\zeta|}{\omega(1-|\zeta|)}|\nabla h(\zeta)|\right](1-|z|)\int_{\mathbf{D}}\frac{\omega(1-|\zeta|)}{|1-\bar{z}\zeta|^3}dA(\zeta)\\ &\lesssim \sup_{\zeta\in\mathbf{D}}\left[\frac{(1-|\zeta|}{\omega(1-|\zeta|)}|\nabla h(\zeta)|\right]\omega(1-|z|).\end{aligned}\tag{2}$$

Then, from (2), it follows that

$$\frac{1}{\omega(1-|z|)}\int_{\mathbf{D}} |h(\zeta)-h(z)|\mathbb{B}(z,\zeta)dA(\zeta) \lesssim \sup_{\zeta\in\mathbf{D}}\left[\frac{(1-|\zeta|}{\omega(1-|\zeta|)}|\nabla h(\zeta)|\right].$$

Thus we have

$$M_1(h) \lesssim M_0(h).$$

For $p=1$ we replace h by $h\circ\varphi_z$ in Lemma 11. We have

$$\begin{aligned}(1-|z|^2)|\nabla h(z)| &\lesssim \int_{\mathbf{D}} |(h\circ\varphi_z)(w)-h(z)|dA(w)\\ &= \int_{\mathbf{D}} |h(\zeta)-h(z)|\mathbb{B}(z,\zeta)dA(\zeta).\end{aligned}$$

Thus we have

$$\frac{(1-|z|)}{\omega(1-|z|)}|\nabla h(z)| \lesssim \frac{1}{\omega(1-|z|)}\mathbb{B}(|h-h(z)|)(z)$$

and hence $M_0(f) \lesssim M_1(f)$.

(ii) For $h \in L^2(\mathbf{D})$, as in the proof of (i), we have

$$\begin{aligned}\mathbb{B}(|h|^2)(z) - |(\mathbb{B}h)(z)|^2 &= \int_{\mathbf{D}} |(h\circ\varphi_z)(w) - h(z)|^2 dA(w)\\ &\lesssim \int_{\mathbf{D}} |\nabla h(\zeta)|^2(1-|\zeta|^2)^2\frac{(1-|z|^2)^2}{|1-\bar{z}\zeta|^4}dA(w).\end{aligned}$$

Thus it follows that

$$\frac{1}{\omega(1-|z|)}\left[\mathbb{B}(|h|^2)(z) - |(\mathbb{B}h)(z)|^2\right]^{1/2} \lesssim \sup_{\zeta\in\mathbf{D}}\left[\frac{(1-|\zeta|)}{\omega(1-|z|)}|\nabla h(\zeta)|\right]$$

and hence $M_2(h) \lesssim M_0(h)$.

For $p = 2$ we replace h by $h\circ\varphi_z$ in Lemma 11. Then we have

$$\begin{aligned}(1-|z|)^2|\nabla h(z)|^2 &\lesssim \int_{\mathbf{D}} |h(\zeta) - h(z)|^2\mathbb{B}(z,\zeta)dA(\zeta)\\ &= \mathbb{B}(|h|^2)(z) - |(\mathbb{B}h)(z)|^2\end{aligned}$$

and so that

$$\frac{(1-|z|)}{\omega(1-|z|)}|\nabla h(z)| \lesssim \frac{1}{\omega(1-|z|)}\left[\mathbb{B}(|h|^2)(z) - |(\mathbb{B}h)(z)|^2\right]^{1/2}.$$

Thus it follows that $M_0(h) \lesssim M_2(h)$. □

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