Commun. Korean Math. Soc. **21** (2006), No. 3, pp. 429–432

A NEW NON-MEASURABLE SET AS A VECTOR SPACE

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ABSTRACT. We use Cauchy's functional equation to construct a new non-measurable set which is a (vector) subspace of $\mathbb{R}$ and is of a codimension 1, considering $\mathbb{R}$, the set of real numbers, as a vector space over a field $\mathbb{Q}$ of rational numbers. Moreover, we show that $\mathbb{R}$ can be partitioned into a countable family of disjoint non-measurable subsets.

In every book on classical measure theory it is not hard to find a statement given by G. Vitali that there exists a subset of $\mathbb{R}$ which is not Lebesgue measurable. We call the subset a non-measurable set for simplicity.

In general the existence of non-measurable set is guaranteed by *the Axiom of Choice*, or equivalent theorems such as Hausdorff maximality principle, Zorn's lemma, and so on. Conversely, it is well known that the axiom of choice is essential for the existence.

Here we construct a new non-measurable set which is a (vector) subspace of $\mathbb{R}$, the set all real numbers, and has a codimension 1 when we consider $\mathbb{R}$ as a vector space over a field $\mathbb{Q}$ of rational numbers. We construct it via the Cauchy functional equation using a method which seems to be simpler, more heuristic and less logical, in author's opinion, than we have done with the classical one.

Moreover, it will be shown that $\mathbb{R}$ can be expressed as a countable union of disjoint family of the non-measurable subspace and its translations.

First, consider the following famous Lemma whose proof is seen, for example, in [1]:

Received May 20, 2005.
2000 Mathematics Subject Classification: 28A05.
Key words and phrases: non-measurable set.
This work was supported by grant No.1999-2-101-001-5 from KOSEF.

LEMMA 1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a (Lebesgue) measurable function satisfying*

$$(1) \qquad \textit{Cauchy's Equation}: \; f(x+y) = f(x) + f(y), \; x, y \in \mathbb{R}.$$

Then $f(x) = ax$, $x \in \mathbb{R}$, for a constant $a = f(1)$.

Throughout this $\mathbb{R}$ is considered as a vector space over a field $\mathbb{Q}$. First of all, with the help of Hausdorff maximality principle we take a maximal linearly independent subset Λ containing the number 1 as an element. We write the set as

$$\Lambda = \{1\} \cup \{v_\alpha | \alpha \in I\}$$

for a some index set I. Note here that I is uncountable and each v_α is an *irrational* number. The subset Λ is usually called a *Hamel basis* for $\mathbb{R}$, in a sense that every real number x can be *uniquely* expressed as

$$(2) \qquad x = r + \sum_{\alpha \in I} a_\alpha v_\alpha,$$

where r and a_α are rational numbers which are zero except only a finite number of them.

We define a linear map φ on the vector space $\mathbb{R}$ into itself by its values on the basis Λ as follows:

$$(3) \qquad \varphi(1) = 1, \quad \varphi(v_\alpha) = 0, \; \alpha \in I.$$

Then using the expression as (2), for every $x = r + \sum_{\alpha \in I} a_\alpha v_\alpha$ and $y = s + \sum_{\alpha \in I} b_\alpha v_\alpha$ we have

$$\varphi(x+y) = \varphi((r+s) + \sum_{\alpha \in I}(a_\alpha + b_\alpha)v_\alpha) = r + s = \varphi(x) + \varphi(y).$$

For each v_α in Λ and a sequence (r_j) in $\mathbb{Q}$ converging to v_α we see that $\varphi(r_j) = r_j$ converges to v_α as j goes to ∞, but $\varphi(v_\alpha) = 0$. This implies the discontinuity of φ at each point $x = v_\alpha$. In view of the above Lemma 1 we conclude that φ is eventually not measurable.

REMARK. The function defined above is a variant of an example seen in the book [2].

Now we denote by $\mathbb{Q}_\Lambda$ the set of all real numbers whose expressions with respect to the basis Λ are of the form $\sum_{\alpha \in I} a_\alpha v_\alpha$. In fact, $\mathbb{Q}_\Lambda$ is a subspace generated by the set $\Lambda \setminus \{1\}$. In other words, a real number x belongs to $\mathbb{Q}_\Lambda$ if and only if there exist indices $\alpha_1, \alpha_2, \dots, \alpha_k \in I$ such that

$$x = a_{\alpha_1} v_{\alpha_1} + a_{\alpha_2} v_{\alpha_2} + \cdots + a_{\alpha_k} v_{\alpha_k},$$

for some rational numbers $a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_k}$.

THEOREM 2. *The set* $\mathbb{Q}_\Lambda$ *is a subspace of codimension* 1 *and non-measurable as a subset of* $\mathbb{R}$.

PROOF. It is easy to see that $\mathbb{Q}_\Lambda$ is a subspace and has a codimension 1, since $\mathbb{R} \approx \mathbb{Q} \oplus \mathbb{Q}_\Lambda$.

To prove the second statement let (r_j) be an enumeration of $\mathbb{Q}$ and $r + \mathbb{Q}_\Lambda$ be a translation of $\mathbb{Q}_\Lambda$ by a number r. Then we have

$$\mathbb{R} = \bigcup_{j=1}^{\infty} (r_j + \mathbb{Q}_\Lambda). \tag{4}$$

Here, it is not hard to see that $(r_i + \mathbb{Q}_\Lambda) \cap (r_j + \mathbb{Q}_\Lambda) = \phi$ for $i \neq j$, using the unique expression of real numbers with respect to the Hamel basis and $\varphi(x) = r_j$ for all $x \in r_j + \mathbb{Q}_\Lambda$ from its values defined by (3).

Now we recall that the function φ defined above is a non-measurable function. Therefore, we find a real number γ for which the set $A = \{x | \varphi(x) > \gamma\}$ is non-measurable. Hence, we can write

$$A = \bigcup_{r_j > \gamma}^{\infty} (r_j + \mathbb{Q}_\Lambda).$$

Therefore, there exists a j_0 with $r_{j_0} > \gamma$ such that $r_{j_0} + \mathbb{Q}_\Lambda$ is non-measurable. Otherwise, the set A would be measurable, which leads a contradiction. But, since the Lebesgue measure is translation-invariant, the set $\mathbb{Q}_\Lambda$ is also non-measurable. □

In fact, since the subspace $\mathbb{Q}_\Lambda$ is non-measurable, so is its every translation $r + \mathbb{Q}_\Lambda$. Therefore, in view of (4) we can say as follows:

COROLLARY 3. $\mathbb{R}$ *can be partitioned into a countable family of disjoint non-measurable subset, i.e.,*

$$\mathbb{R} = \bigcup_{j=1}^{\infty} (r_j + \mathbb{Q}_\Lambda),$$

where (r_j) *be an enumeration of* $\mathbb{Q}$.

References

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