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HYPONORMAL WEIGHTED SHIFT OPERATORS AND TRUNCATED COMPLEX MOMENT PROBLEMS

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ABSTRACT. In this paper, we present some recent developments on hyponormal operator theory and truncated Curto-Fialkow and Embry complex moment problems.

1. Hyponormal operators

1.1. Definitions

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, and *subnormal* if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $\lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_k T^k$ is hyponormal for every $\lambda_i \in \mathbb{C}, i = 1, \ldots, k$, then T is said to be *weakly* k-hyponormal. In particular, the weakly 2-hyponormal is often said to be *quadratically* hyponormal. It is well known that normal $\Rightarrow$ subnormal $\Rightarrow$ hyponormal, with converses false.

For $A, B \in \mathcal{L}(\mathcal{H})$, let $[A, B] = AB - BA$. We say that an n-tuple $T = (T_1, \ldots, T_n)$ of operators on $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k \geq 1$ and $T \in \mathcal{L}(\mathcal{H})$, T is *k-hyponormal* if $(I, T, \ldots, T^k)$ is hyponormal. The Bram-Halmos characterization of subnormality may be described as follows: $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if T is k-hyponormal for every $k \geq 1$.

Let $\{e_n\}_{n=0}^\infty$ be the canonical orthonormal basis for $l^2(\mathbb{Z}_+)$, and $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a bounded sequence of positive numbers. Let W_α be the

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unilateral weighted shift defined on $l^2(\mathbb{Z}_+)$ by $W_\alpha e_n := \alpha_n e_{n+1}$, $\forall n \geq 0$. The numbers

$$\gamma_0 := 1, \quad \gamma_1 := \alpha_0^2, \quad \gamma_2 := \alpha_0^2\alpha_1^2, \ldots, \quad \gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2, \ldots$$

are called the *moments* of W_α. It is well known that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1} (\forall n \geq 0)$.

A weighted shift W_α is said to be *recursively generated* if there exist the smallest number $r \geq 1$ and $\psi_i (i = 0, \ldots, r-1) \in \mathbb{R}$ such that $\gamma_n = \psi_{r-1}\gamma_{n-1} + \cdots + \psi_0\gamma_{n-r}$ $(n \geq r)$, equivalently,

$$\alpha_n^2 = \psi_{r-1} + \frac{\psi_{r-2}}{\alpha_{n-1}^2} + \cdots + \frac{\psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-r+1}^2} \qquad (n \geq r-1).$$

We call r the *rank* of γ. In addition, a weighted shift W_α is *non-recursively generated* if it is not recursively generated. Note that a subnormal weighted shift is recursively generated if and only if the corresponding probability measure has finite support ([20, P.6]).

1.2. Backward extension problem

Let $x > 0$ and $\alpha(x) : x, \alpha_0, \alpha_1, \ldots$ be an augmented weight sequence. In [4], Curto-Fialkow introduced a backward extension $W_{\alpha(x)}$ of the weighted shift W_α and described

$$HE(\alpha(x), q) = \{x \in \mathbb{R}_+ : W_{\alpha(x)} \text{ is } q\text{-hyponormal}\}$$

when W_α is k-hyponormal and $1 \leq q \leq k$. This description was called *backward extension problem* of W_α with a variable x as initial weight. In particular, it follows from [2] that if $\alpha(x) : x, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{5}}, \ldots$, then there exists a sequence $\{\lambda_k\}_{k=1}^\infty$ of positive numbers with $\lim \lambda_k = \sqrt{\frac{1}{2}}$ such that $\lambda_k > \lambda_{k+1}$ $(k \geq 1)$ and $HE(\alpha(x), k) = (0, \lambda_k]$, where $\lambda_1 = \sqrt{\frac{2}{3}}, \lambda_2 = \frac{3}{4}, \ldots$, and $W_{\alpha(x)}$ is subnormal if and only if $0 < x \leq \sqrt{\frac{1}{2}}$. In [11], Jung and Li obtained a formula to describe $HE(\alpha(x), q)$, when W_α is subnormal.

For the moment sequence $\{\gamma_n\}_{n=0}^\infty$ of W_α, we let

$$A(i,j) := \begin{pmatrix} \gamma_i & \gamma_{i+1} & \cdots & \gamma_{i+j} \\ \gamma_{i+1} & \gamma_{i+2} & \cdots & \gamma_{i+j+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{i+j} & \gamma_{i+j+1} & \cdots & \gamma_{i+2j} \end{pmatrix}$$

If a subnormal weighted shift W_α is recursively generated and rank $\gamma = r$, then $\det A(i, r-1) \neq 0$ and $\det A(i,j) = 0$ for any $i \geq 0$, $j \geq r$. Note that if a subnormal weighted shift W_α is non-recursively generated, then $\det A(i,j) \neq 0$ for any positive integers i and j([5]).

Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a bounded sequence of positive numbers. Let $x > 0$ and let $\alpha(x) : x, \alpha_0, \alpha_1, \ldots$ be an augmented weight sequence. Assume that W_α is non-recursively generated subnormal weighted shift. For brevity, let us put $t := \frac{1}{x^2}$. Then it follows from [2, Theorem 4] that $W_{\alpha(x)}$ is k-hyponormal if and only if

$$D_k(t) := \begin{pmatrix} t & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{pmatrix}$$

is nonnegative. Note that $d_k(t) := \det D_k(t)$ is the polynomial of t with degree 1. Since W_α is non-recursively generated, the coefficient $\det A(1, k-1)$ of t in the polynomial $d_k(t)$ is nonzero. Hence $d_k(t)$ has the unique zero. In fact, $\det A(1, k-1) > 0$, since W_α is subnormal.

THEOREM 1.1. ([11, Theorem 2.1]) *Let $\alpha : \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ be a bounded sequence of positive numbers. Assume that W_α is non-recursively generated subnormal weighted shift. Let $t_k := t_k(\alpha)$ be the unique zero of* $\det D_k(t)$. *Then*

$$t_{k+1}(\alpha) = t_k(\alpha) + \frac{[\det A(0,k)]^2}{\det A(1,k-1) \cdot \det A(1,k)}, \quad \textit{for all } k = 1, 2, \ldots.$$

Given any non-recursively generated subnormal weighted shift W_α, by Theorem 1.1, the backward one step extension of W_α provides several examples being distinct the classes of k-hyponormal operators. Indeed, we may recapture Curto's example [2, Proposition 7] as following.

EXAMPLE 1.2. For $x > 0$, let $W_{\alpha(x)}$ be the weighted shift (with Bergman tail) whose weight sequence is given by $\alpha_0 := x, \alpha_n := \sqrt{\frac{n+1}{n+2}}$ $(n \geq 1)$. It follows from [2] that $W_{\alpha(x)}$ is subnormal if and only if $0 < x \leq \sqrt{\frac{1}{2}}$. Since the Berger measure corresponded by $W_{\alpha(x)}$ is not finite, $W_{\alpha(x)}$ is not recursively generated. Applying Theorem 1.1, we have $t_1 = \frac{3}{2}$, $t_2 = \frac{16}{9}$, $t_3 = \frac{15}{8}$, $t_4 = \frac{48}{25}$, $t_5 = \frac{35}{18}, \ldots$. Hence $\lambda_1 = \sqrt{\frac{2}{3}}$, $\lambda_2 = \frac{3}{4}$, $\lambda_3 = \sqrt{\frac{8}{15}}$, $\lambda_4 = \sqrt{\frac{25}{48}}$, $\lambda_5 = \sqrt{\frac{18}{35}}, \ldots$, and $HE(\alpha, \infty) = (0, \sqrt{\frac{1}{2}}]$.

In [10], the authors considered backward two and three step extension problems.

1.3. On subnormal completion problem

C. Berger's characterization of subnormality for unilateral weighted shifts states that W_α is subnormal if and only if the sequences $\{\gamma_n\}_{n=0}^{\infty}$ can be interpolated by a probability measure μ supported on $[0, \|W_\alpha\|^2]$, i.e., $\gamma_n = \int t^n d\mu(t)$, for all $n \geq 0$ ([1, III. 8.16]). It establishes a connection between probability measure theory and the classical theory of moments.

J. Stampfli posed the following

SUBNORMAL COMPLETION PROBLEM: *Given $m \geq 0$ and an initial segment of positive weights $\alpha : \alpha_0, \ldots, \alpha_m$, seeks necessary and sufficient conditions for the existence of a subnormal weighted shift $W_{\hat{\alpha}}$ whose first $(m+1)$ weights are $\alpha_0, \ldots, \alpha_m$.*

J. Stampfli solve the problem for $m = 2$. That is, $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ is subnormal if and only if $0 < \alpha_0 < \alpha_1 < \alpha_2$ ([22]).

Curto-Fialkow gives the following result which solves the subnormal completion problem.

THEOREM 1.3. ([3, Theorem 3.5]) *Let $\alpha : \alpha_0, \ldots, \alpha_m (m \geq 0)$ be an initial segment of positive weights, and let*

$$k := \left[\frac{m+1}{2}\right], \quad l := \left[\frac{m}{2}\right] + 1.$$

The following statements are equivalent:

(1) *$W_{\hat{\alpha}}$ is a subnormal completion of α;*

(2) *α has a recursive subnormal completion;*

(3) *α has a subnormal completion;*

(4) *α has an l-hyponormal completion;*

(5) *$A(k) \geq 0, B(l-1) \geq 0$, and $v(k+1, k) \in \operatorname{Ran} A(k)$ if m is even, $v(k+1, k-1) \in \operatorname{Ran} B(l-1)$ if m is odd, where*

$$A(j) := \begin{pmatrix} \gamma_0 & \cdots & \gamma_j \\ \vdots & \ddots & \vdots \\ \gamma_j & \cdots & \gamma_{2j} \end{pmatrix}, \quad B(j) := \begin{pmatrix} \gamma_1 & \cdots & \gamma_{j+1} \\ \vdots & \ddots & \vdots \\ \gamma_{j+1} & \cdots & \gamma_{2j+1} \end{pmatrix},$$

$$v(i,j) := \begin{pmatrix} \gamma_i \\ \vdots \\ \gamma_{i+j} \end{pmatrix}.$$

Assume that W_α is subnormal. Then

$$W_\alpha \text{ is RG} \iff \operatorname{supp} \mu \text{ is finite.}$$

To find μ, consider $g(t) := t^r - (\psi_0 + \cdots + \psi_{r-1}t^{r-1})$. g has exactly r zeros, $0 \le s_0 < s_1 < \cdots < s_{r-1} = \|W_\alpha\|^2$. Let

$$\begin{pmatrix} \rho_0 \\ \rho_1 \\ \vdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ s_0 & s_1 & \cdots & s_{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_0^{r-1} & s_1^{r-1} & \cdots & s_{r-1}^{r-1} \end{pmatrix}^{-1} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{r-1} \end{pmatrix}.$$

Then $\mu := \rho_0 \delta_{s_0} + \cdots + \rho_{r-1}\delta_{s_{r-1}}$.

2. Curto and Fialkow's moment problem

2.1. Classical moment problem

Given an infinite sequence of complex numbers $\gamma = \{\gamma_0, \gamma_1, \ldots,\}$ and a subset $K \subseteq \mathbb{C}$, the *K-power moment problem* with data γ entails finding a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (i \ge 0) \ \text{ and } \ \operatorname{supp} \mu \subseteq K.$$

For $K = [0, +\infty), K = \mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}, K = \mathbb{R}$ and $K = [a, b]\ (a, b \in \mathbb{R})$, we also said Stieljes, Toeplitz, Hamburger and Hausdorff moment problem, respectively.

For $0 \le m < \infty$, let $\gamma = (\gamma_0, \cdots, \gamma_m) \in \mathbb{C}^{m+1}$, and consider the *truncated K-power moment problem*

$$\int t^i d\mu(t) = \gamma_i \ (0 \le i \le m) \ \text{ and } \ \operatorname{supp} \mu \subseteq K.$$

2.1.1. *Solutions of the truncated Stieljes moment problem.* Let $\gamma = (\gamma_0, \ldots, \gamma_m) \in \mathbb{C}^{m+1}$. The truncated Stieljes moment problem is finding a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (0 \le i \le m) \ \text{ and } \ \operatorname{supp} \mu \subseteq [0, +\infty). \tag{2.1}$$

THEOREM 2.1. (Odd Case, [5, Theorem 5.1]) *Let* $\gamma = (\gamma_0, \ldots, \gamma_{2k+1})$, $\gamma_0 > 0$, *and let* $r := \operatorname{rank} \gamma$. *The following are equivalent:*

(i) *The truncated Stieljes moment problem* (2.1) *is soluble;*

(ii) *There exists an r-atomic representing measure of* γ*;*

(iii) $A(k) \ge 0, B(k) \ge 0$ *and* $v(k+1, k) \in \operatorname{Ran} A(k)$.

THEOREM 2.2. (Even Case, [5, Theorem 5.3]) *Let* $\gamma = (\gamma_0, \ldots, \gamma_{2k})$, $\gamma_0 > 0$, *and let* $r := \operatorname{rank} \gamma$. *The following are equivalent:*

(i) *The truncated Stieljes moment problem* (2.1) *is soluble;*

(ii) *There exists an r-atomic representing measure of γ;*
(iii) $A(k) \geq 0, B(k-1) \geq 0$ *and* $v(k+1, k-1) \in \operatorname{Ran} B(k-1)$.

2.1.2. *Solutions of the truncated Toeplitz moment problem.* Let

$$\gamma = (\gamma_{-k}, \dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots, \gamma_k) \in \mathbb{C}^{2k+1} \text{ with } \gamma_0 > 0 \text{ and } \gamma_{-j} = \overline{\gamma}_j.$$

The truncated Toeplitz moment problem is finding a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (-k \leq i \leq k) \quad \text{and} \quad \operatorname{supp} \mu \subseteq \mathbb{T}. \tag{2.2}$$

We obtain the Toeplitz matrix

$$T_\gamma(k) = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{r-1} & \gamma_r & \cdots & \gamma_k \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{r-2} & \gamma_{r-1} & \cdots & \gamma_{k-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{1-r} & \gamma_{2-r} & \cdots & \gamma_0 & \gamma_1 & \cdots & \gamma_{k+1-r} \\ \gamma_{-r} & \gamma_{1-r} & \cdots & \gamma_{-1} & \gamma_0 & \cdots & \gamma_{k-r} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{-k} & \gamma_{1-k} & \cdots & \gamma_{-k+r-1} & \gamma_{-k+r} & \cdots & \gamma_0 \end{pmatrix}.$$

THEOREM 2.3. ([5, Theorem 6.12]) *Let* $\gamma = (\gamma_{-k}, \dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots, \gamma_k) \in \mathbb{C}^{2k+1}$ with $\gamma_0 > 0$ and $\gamma_{-j} = \overline{\gamma}_j$, *and let* $r := \operatorname{rank} \gamma$. *Then the truncated Toeplitz moment problem* (2.2) *is soluble if and only if* $T_\gamma(k) \geq 0$. *In this case,* μ *can be chosen to have* r *atoms.*

2.1.3. *Solutions of the truncated Hamburger moment problem.* Let $\gamma = (\gamma_0, \dots, \gamma_m) \in \mathbb{C}^{m+1}$. The truncated Hamburger moment problem is finding a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (0 \leq i \leq m) \quad \text{and} \quad \operatorname{supp} \mu \subseteq \mathbb{R}. \tag{2.3}$$

THEOREM 2.4. (Odd Case, [5, Theorem 3.1]) *Let* $\gamma = (\gamma_0, \dots, \gamma_{2k+1}) \in \mathbb{R}^{2k+2}, \gamma_0 > 0$, *and let* $r := \operatorname{rank} \gamma$. *The following are equivalent:*
(i) *The truncated Stieljes moment problem* (2.3) *is soluble;*
(ii) *There exists an r-atomic representing measure of γ;*
(iii) $A(k) \geq 0$ *and* $v(k+1, k) \in \operatorname{Ran} A(k)$.

THEOREM 2.5. (Even Case, [5, Theorem 3.9]) *Let* $\gamma = (\gamma_0, \dots, \gamma_{2k})$, $\gamma_0 > 0$, *and let* $r := \operatorname{rank} \gamma$. *The following are equivalent:*
(i) *The truncated Stieljes moment problem* (2.3) *is soluble;*
(ii) *There exists an r-atomic representing measure of γ;*
(iii) $A(k) \geq 0$ *and* $\operatorname{Rank} A(k) = r$.

2.1.4. *Solutions of the truncated Hausdorff moment problem.* Let $a < b$ and $\gamma = (\gamma_0, \ldots, \gamma_m) \in \mathbb{R}^{m+1}$. The truncated Hausdorff moment problem is finding a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^i d\mu(t) = \gamma_i \ (0 \le i \le m) \ \text{ and } \ \operatorname{supp} \mu \subseteq [a, b]. \tag{2.4}$$

THEOREM 2.6. (Odd Case, [5, Theorem 4.1]) *Let* $\gamma = (\gamma_0, \ldots, \gamma_{2k+1})$, $\gamma_0 > 0$, *and let* $r :=$ rank γ. *The following are equivalent:*

(i) *The truncated Stieljes moment problem* (2.4) *is soluble;*

(ii) *There exists an r-atomic representing measure of* γ;

(iii) $A(k) \ge 0$ *and* $v(k+1, k) \in \operatorname{Ran} A(k)$ *and* $bA(k) \ge B(k) \ge aA(k)$.

THEOREM 2.7. (Even Case, [5, Theorem 4.3]) *Let* $\gamma = (\gamma_0, \ldots, \gamma_{2k})$, $\gamma_0 > 0$, *and let* $r :=$ rank γ. *The following are equivalent:*

(i) *The truncated Stieljes moment problem* (2.4) *is soluble;*

(ii) *There exists an r-atomic representing measure of* γ;

(iii) $A(k) \ge 0$ *and there exists* $\gamma_{2k+1} \in \mathbb{R}$ *such that* $v(k+1, k) \in \operatorname{Ran} A(k)$ *and* $bA(k) \ge B(k) \ge aA(k)$.

2.2. Curto and Fialkow's truncated complex moment problem

Given a closed subset $K \subseteq \mathbb{C}$ and a doubly indexed finite sequence of complex numbers

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \ldots, \gamma_{0,2n}, \gamma_{1,2n-1}, \ldots, \gamma_{2n-1,1}, \gamma_{2n,0}, \tag{2.5}$$

with $\gamma_{00} > 0$, $\gamma_{ji} = \overline{\gamma_{ij}}$. The *truncated K complex moment problem* entails finding a positive Borel measure μ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \le i + j \le 2n) \quad \text{and} \quad \operatorname{supp} \mu \subseteq K. \tag{2.6}$$

Any sequence γ as in (2.5) is a *truncated moment sequence* and any measure μ as in (2.6) is a *representing measure* for γ.

For $n \ge 1$, let $m \equiv m(n) = (n+1)(n+2)/2$. For $A \in M_m(\mathbb{C})$ (the $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering : $1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots, Z^n, \ldots, \bar{Z}^n$; rows and columns indexed by $1, Z, Z^2, \ldots, Z^n$ are said to be *analytic*. For $0 \le i + j \le n, 0 \le l + k \le n$, we denote the entry in row $\bar{Z}^l Z^k$, column $\bar{Z}^i Z^j$ by $A_{(l,k)(i,j)}$. For the truncated moment sequence (2.5), we define $M(n)(\gamma) \in M_m(\mathbb{C})$ as follows : for $0 \le i + j \le n, 0 \le l + k \le n$,

$$M(n)_{(l,k)(i,j)} = \gamma_{i+k,j+l}.$$

For example, if $n = 1$, the *quadratic moment problem* for $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix}. \tag{2.7}$$

The quadratic moment problem was solved completely.

THEOREM 2.8. ([6, Theorem 6.1]) *$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ admits a representing measure if and only if the associated moment matrix $M(1)$ in (2.7) is positive.*

As an application of Theorem 2.8, the author in [14] solved two variable subnormal completion problem.

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, and $r =$ rank $M(1)$.

THEOREM 2.9. ([7, Theorem 3.1 and Theorem 1.8]) *Suppose $M(1)$ is positive. The following statements are equivalent.*

(i) *γ has a representing measure supported in $\mathbb{T}$ (or $\mathbb{D}$);*

(ii) *γ has a r-atomic representing measure supported in $\mathbb{T}$ (or $\mathbb{D}$);*

(iii) $\gamma_{11} = \gamma_{00}$ (*or* $\gamma_{11} \leq \gamma_{00}$).

Let $P_n \subseteq \mathbb{C}[z, \bar{z}]$ denote the complex polynomials in $z, \bar{z}$ of total degree $\leq n$. Clearly, $\dim P_n = m$. For $p \in P_n$,

$$p(z,\bar{z}) = \sum_{0\leq i+j\leq n} a_{ij}\bar{z}^i z^j, \qquad \bar{p}(z,\bar{z}) = \sum_{0\leq i+j\leq n} \bar{a}_{ij} z^i \bar{z}^j,$$
$$\hat{p} \equiv (a_{00}, a_{01}, a_{10}, \dots, a_{0n}, \dots, a_{n0}) \in \mathbb{C}^m.$$

The basic connection between $M(n)(\gamma)$ and any representing measure μ is provided by the identity

$$\int f\bar{g}d\mu = (M(n)\hat{f}, \hat{g}) \qquad (f, g \in P_n);$$

in particular, $(M(n)\hat{f}, \hat{f}) = \int |f|^2 d\mu \geq 0$, so $M(n) \geq 0$.

For $k, l \in \mathbb{Z}_+$, let $A \in M_k(\mathbb{C}), A = A^*, B \in M_{k,l}(\mathbb{C}), C \in M_l(\mathbb{C})$; we refer to any matrix of the form

$$\tilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

as an *extension* of A.

PROPOSITION 2.10. ([21]) *For $A \geq 0$, the following are equivalent:*

(1) $\tilde{A} \geq 0$;

(2) *There exists $W_{k,l}(\mathbb{C})$ such that $AW = B$ and $C \geq W^*AW$.*

The following is the main theorem.

THEOREM 2.11. ([6, Theorem 5.13]) *γ has a rank $M(n)$-atomic representing measure if and only if $M(n)$ admits a flat extension $M(n+1)$, i.e., rank $M(n+1)$ = rank $M(n)$.*

Let $r :=$ rank $M(n)$ and let $\mathcal{C}_{M(n)}$ denote the column space of $M(n)$, so that in $\mathcal{C}_{M(r)}$ there is a dependence relation of the form $Z^r = c_0 1 + c_1 Z + \cdots + c_{r-1} Z^{r-1}$. The polynomial $z^r - (c_0 + \cdots + c_{r-1} z^{r-1})$ has r distinct roots, $z_0, \ldots, z_{r-1}$, which provide the support for the unique representing measure for $\gamma^{(2n+2)}$ corresponding to the flat extension $M(n+1)$. The densities of this measure, $\rho_0, \ldots, \rho_{r-1}$, are determined by the Vandermonde equation

$$V(z_0, \ldots, z_{r-1})(\rho_0, \ldots, \rho_{r-1})^t = (\gamma_{00}, \ldots, \gamma_{0,r-1})^t.$$

Then the representing measure is $\mu := \sum_{i=0}^{r-1} \rho_i \delta_{z_i}$.

2.3. Solutions of the quartic moment problem

If $n = 2$, the moment sequence

$$\gamma \equiv \gamma^{(4)} : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40}$$

corresponds to

$$M(2)(\gamma) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

THE QUARTIC MOMENT PROBLEM. *If $M(2)(\gamma)$ is positive, does $M(2)(\gamma)$ have a representing measure ?*

First, for the singular (i.e., $\det M(2) = 0$) quartic moment problem, Curto-Fialkow obtained the following partial solutions.

THEOREM 2.12. ([8, Theorem 1.10]) *Suppose $M(2)(\gamma)$ is positive and recursively generated. Then γ has a rank $M(2)$-atomic representing measure in each of the following cases:*

(i) *$\{1, Z, \bar{Z}, Z^2\}$ is linearly dependent in $\mathcal{C}_{M(2)}$;*

(ii) *$\{1, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, $\bar{Z}Z \in \langle 1, Z, \bar{Z} \rangle$, and the moments γ_{ij} are all real, with the possible exception of γ_{04};*

(iii) *$\{1, Z, \bar{Z}, Z^2\}$ is a basis for $\mathcal{C}_{M(2)}$, $\bar{Z}Z \in \langle 1, Z, \bar{Z} \rangle$, and the reduced C-block test $c_{11} = c_{22}$ passes;*

(iv) $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ *is a basis for* $\mathcal{C}_{M(2)}, \bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, *and the reduced C-block test* $c_{11} = c_{22}$ *passes for some choice of* γ_{05}.

The followings are an extension of Theorem 2.12.

THEOREM 2.13. ([13, Theorem 3.2]) *If*
(i) $M(2)$ *is positive,*
(ii) $\{1, Z, \bar{Z}, Z^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$,
(iii) $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$,
then $M(2)$ *has the unique flat extension* $M(3)$.

THEOREM 2.14. ([13, Theorem 3.4]) *If*
(i) $M(2)$ *is positive,*
(ii) $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$,
(iii) $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2\rangle$,
then $M(2)$ *has the unique flat extension* $M(3)$.
The following example is very important.

EXAMPLE 2.15. ([13, Example 2.4]) If the truncated moment sequence γ of order 4 is given by

$$\gamma : 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 1,\ 0,\ 0,\ 1,\ 1,\ 1,\ 2,\ 1,\ 1,$$

then we have

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

(i) $M(2) \geq 0$;
(ii) rank $M(2) = 4$ and thus $M(2)$ satisfies the property (RG) ;
(iii) γ has no representing measure.
The authors [9] and [15] also obtained some results for the singular quartic moment problem.

THEOREM 2.16. ([9, Theorem 1.3]) *Suppose* $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2\}$ *is independent in* $\mathcal{C}_{M(2)}$, *and* $\bar{Z}Z = A1 + BZ + C\bar{Z} + DZ^2, D \neq 0$. *The following are equivalent:*
(i) $\gamma^{(4)}$ *admits a 4-atomic (minimal) representing measure;*
(ii) $M(2)$ *admits a flat extension* $M(3)$;
(iii) *there exists* $\gamma_{23} \in \mathbb{C}$ *such that* $\bar{\gamma}_{23} = A\gamma_{21} + B\gamma_{22} + C\gamma_{31} + D\gamma_{23}$.

THEOREM 2.17. ([9, Theorem 4.1]) *Suppose* $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. $\gamma^{(4)}$ *admits a 5-atomic (minimal) representing* *measure if and only if there exists* $\gamma_{23} \in \mathbb{C}$ *such that the* C*-block of* $[M(2); B(3)[\gamma_{23}]]$ *satisfies* $C_{21} = C_{32}$.

THEOREM 2.18. ([9, Theorem 1.5]) *Suppose* $M(2) \geq 0$, $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. $\gamma^{(4)}$ *admits a representing measure.*

For nonsingular quartic moment matrices, if $M(2) > 0$ then $[M(2); B(2)]$ is a flat extension of $M(2)$ if and only if the C-block satisfies $c_{11} = c_{22}$ and $c_{21} = c_{32}$.

In [13] the authors obtained the partial solutions.

We consider the case that $M(1) = I$ and $\gamma_{12} = \gamma_{03} = 0$. Thus $M(2)$ is of the form

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ 1 & 0 & 0 & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ 0 & 0 & 0 & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}. \tag{2.8}$$

THEOREM 2.19. ([18, Theorem 4.4]) *Let* $M(2)$ *be a positive and nonsingular moment matrix as in* (2.8) *and* c_{ij} *be the cofactor of the* i*-th row* j*-th column entry of* $M(2)$, $i, j = 1, 2, 3, 4, 5, 6$. *Put*

$$a := \frac{c_{22}c_{64}c_{65}}{|c_{64}|^2 - |c_{65}|^2 + c_{66}(c_{55} - c_{66})},$$

$$b := \frac{c_{22}{}^2|c_{65}|^2}{|c_{64}|^2 - |c_{65}|^2 + c_{66}(c_{55} - c_{66})},$$

$$c := \frac{c_{22}{}^2|c_{65}|(|c_{64}|^2 - 1)}{|c_{64}|^2 - |c_{65}|^2 + c_{66}(c_{55} - c_{66})}.$$

If

$$S_t := \{t \in \mathbb{R}^+ \mid c \leq t^2,\ c - |a|b \leq t^2 - bt \leq |a|b + c,\ |a|b + c \leq t^2 + bt\},$$

is not empty, then $M(2)$ *admits a flat extension* $M(3)$.

EXAMPLE 2.20. ([18, Example 4.6]) Assume that $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = 0, \gamma_{12} = 0, \gamma_{04} = 0, \gamma_{13} = 0$, and $\gamma_{22} = 2$. Then

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

which is positive and invertible. In fact, by Mathematica, the eigenvalues of $M(2)$ are $1, 1, 2, 2, \frac{1}{2}(3-\sqrt{5}), \frac{1}{2}(3+\sqrt{5})$. Also a straightforward calculation gives that $a = 0, b = 8$ and $c = 0$, so $S_t = \{0, 8\}$. Thus by Theorem 2.19, $M(2)$ admits a flat extension $M(3)$. In fact, by a straightforward calculation, we can obtain a representing measure $\mu = \sum_{0\le i\le 5} \rho_i \delta_{z_i}$, where

$$\begin{aligned}
z_0 &\approx 0.618034 - 0.618034i, & \rho_0 &\approx 0.315738,\\
z_1 &\approx -1.61803 - 1.61803i, & \rho_1 &\approx 0.0175955,\\
z_2 &\approx -0.84425 - 0.226216i, & \rho_2 &\approx 0.315738,\\
z_3 &\approx 2.21028 + 0.592242i, & \rho_3 &\approx 0.0175955,\\
z_4 &\approx -0.592242 - 2.21028i, & \rho_4 &\approx 0.0175955,\\
z_5 &\approx 0.226216 + 0.84425i, & \rho_5 &\approx 0.315738.
\end{aligned}$$

OPEN PROBLEM. *Does arbitrary nonsingular positive quartic moment matrix $M(2)$ admit a representing measure?*

3. Embry's moment problem

3.1. The truncated Embry complex moment problem

As a subcollection of the collection in (2.5), we consider

$$(3.1) \qquad \gamma \equiv \{\gamma_{ij}\} \ \ (0 \le i+j \le 2n, \ \ |i-j| \le n)$$

with $\gamma_{00} > 0$, $\gamma_{ji} = \overline{\gamma_{ij}}$. The *truncated Embry K complex moment problem* entails finding a positive Borel measure μ such that

$$(3.2) \ \gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \le i+j \le 2n, |i-j| \le n) \quad \text{and} \quad \operatorname{supp}\mu \subseteq K.$$

Any measure μ as in (3.2) is a *representing measure* for γ.

For $n \in \mathbb{N}$, let $m = m(n) := ([\frac{n}{2}]+1)([\frac{n+1}{2}]+1)$. We define the moment matrix $E(n) \equiv E(n)(\gamma)$ in $M_m(\mathbb{C})$ as follows: $E(n)_{(k,l)(i,j)} := \gamma_{l+i,j+k}$. Note that $E(n)$ is a submatrix of $M(n)$, i.e., $E(n)_{(k,l),(i,j)} = M(n)_{(k,l),(i,j)}$ when $k \le l$ and $i \le j$.

Basically, the positivity of $E(n)$ is necessary.

THEOREM 3.1. ([12, Proposition 3.10]) *Let $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i+j \le 2n, \ |i-j| \le n)$ be given.*

(i) *If n is even number, then γ has a* rank *$E(n)$-atomic representing measure if and only if $E(n) \ge 0$ and $E(n)$ admits a flat extension $E(n+2)$.*

(ii) *If n is odd number, then γ has a* rank *$E(n)$-atomic representing measure if and only if $E(n) \geq 0$ and $E(n)$ admits a flat extension $E(n+1)$.*

3.2. Quadratic Embry moment problem

Let $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$ with $\gamma_{00} > 0, \gamma_{10} = \overline{\gamma_{01}}$ and $\gamma_{11} \in \mathbb{R}$. The *quadratic moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu(z), \qquad (0 \leq i+j \leq 2,\ |i-j| \leq 1).$$

As in section 3.1, we can obtain the moment matrix

$$E(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix}.$$

Let $r = \text{rank } E(1)$. We can obtain the following

THEOREM 3.2. ([17, Theorem 2.1]) *The following statements are equivalent.*

(i) *γ has a representing measure;*

(ii) *γ has an r-atomic representing measure;*

(iii) *$E(1) \geq 0$.*

In this case, if $r = 1$, there exists a unique representing measure $\mu = \gamma_{00}\delta_{\frac{\gamma_{01}}{\gamma_{00}}}$; if $r = 2$, the 2-atomic representing measures contain a sub-parameter by a circle.

Furthermore, we have

THEOREM 3.3. ([17, Theorem 3.1, Theorem 4.1]) *Suppose $E(1)$ is positive. The following statements are equivalent.*

(i) *γ has a representing measure supported in $\mathbb{T}$ (or $\mathbb{D}$);*

(ii) *γ has an r-atomic representing measure supported in $\mathbb{T}$ (or $\mathbb{D}$);*

(iii) *$\gamma_{11} = \gamma_{00}$ (or $\gamma_{11} \leq \gamma_{00}$).*

3.3. Quartic Embry moment problem

Let $\gamma \equiv \{\gamma_{ij}\}$ $(0 \leq i+j \leq 4,\ |i-j| \leq 2)$ with $\gamma_{00} > 0,\ \gamma_{ji} = \overline{\gamma_{ij}}$. The *quartic Embry moment problem* entails finding a positive Borel measure μ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \leq i+j \leq 4, |i-j| \leq 2).$$

We obtain the moment matrix

$$E(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

Assume that $E(2)$ is positive and let $r := \text{rank}\, E(2)$. Then obviously $1 \leq r \leq 4$. The singular case is of $\det E(2) = 0$, i.e., $r = 1, 2$ and 3.

3.3.1. *The case of $r = 1$.* By a direct computation, we have the following proposition.

PROPOSITION 3.4. ([16, Proposition 2.1]) *Assume that $E(2) \geq 0$ and $r = 1$. Then there exists the unique flat extension $E(3)$ of $E(2)$. Therefore γ admits the unique 1-atomic representing measure $\mu = \gamma_{00}\delta_{\frac{\gamma_{01}}{\gamma_{00}}}$.*

3.3.2. *The case of $r = 2$.* Assume that rank $E(2) = 2$. Then

$$Z^2 = \alpha 1 + \beta Z \quad \text{and} \quad \bar{Z}Z = \alpha' 1 + \beta' Z,$$

for some complex numbers $\alpha, \beta, \alpha', \beta'$. By a direct computation, we have

$$\begin{aligned} \alpha &= -\frac{\gamma_{01}\gamma_{12} - \gamma_{02}\gamma_{11}}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}, \quad \beta = \frac{\gamma_{00}\gamma_{12} - \gamma_{10}\gamma_{02}}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}, \\ \alpha' &= -\frac{\gamma_{01}\gamma_{21} - \gamma_{11}^2}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}, \quad \beta' = \frac{\gamma_{00}\gamma_{21} - \gamma_{10}\gamma_{11}}{\gamma_{00}\gamma_{11} - \gamma_{10}\gamma_{01}}. \end{aligned}$$

PROPOSITION 3.5. ([16, Proposition 2.3]) *Assume that $E(2) \geq 0$ and $r = 2$. If*

$$\bar{\alpha}\gamma_{12} + \bar{\beta}\gamma_{22} = \alpha'\gamma_{21} + \beta'\gamma_{22},$$

then there exists a unique flat extension $E(3)$ of $E(2)$. Therefore, γ admits unique 2-atomic representing measure $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1}$, the two atoms z_0, z_1 are the roots of

$$z^2 - (\alpha + \beta z) = 0,$$

and the densities are

$$\rho_0 = \frac{\gamma_{01} - \gamma_{00}z_1}{z_0 - z_1} \quad \text{and} \quad \rho_1 = \frac{z_0\gamma_{00} - \gamma_{01}}{z_0 - z_1}.$$

3.3.3. *The case of $r = 3$.* For a positive $n \times n$ matrix A, let us denote by $[A]_k$ the compression of A to the first k rows and k columns. We denote by M_{ij} the determinant of the cofactor of $E(2)$ with respect to (i, j) and $\Delta_d = \det([E(2)]_d)$, for $d = 1, 2, 3$, and 4.

We now assume that rank $E(2) = 3$. Then there exist a_0, a_1, a_2 in $\mathbb{C}$ such that

$$\bar{Z}Z = a_0 \mathit{1} + a_1 Z + a_2 Z^2. \tag{3.3}$$

In fact,

$$a_0 = \frac{M_{41}}{\Delta_3}, \quad a_1 = -\frac{M_{42}}{\Delta_3}, \quad a_2 = \frac{M_{43}}{\Delta_3}.$$

To establish a flat extension $E(3)$, we should choose suitable γ_{23}. By (3.3) we have

$$\bar{Z}Z^2 = a_0 Z + a_1 Z^2 + a_2 Z^3.$$

Let us take

$$\gamma_{23} := a_0\gamma_{12} + a_1\gamma_{13} + a_2\gamma_{14}.$$

Since $\{\mathit{1}, Z, Z^2, Z^3\}$ is linearly dependent, we have $Z^3 = b_0 \mathit{1} + b_1 Z + b_2 Z^2$, for some $b_i \in \mathbb{C}$. Then

$$b_0 = \frac{1}{\Delta_3}\begin{vmatrix} \gamma_{03} & \gamma_{01} & \gamma_{02} \\ \gamma_{13} & \gamma_{11} & \gamma_{12} \\ \gamma_{23} & \gamma_{21} & \gamma_{22} \end{vmatrix}, \quad b_1 = \frac{1}{\Delta_3}\begin{vmatrix} \gamma_{00} & \gamma_{03} & \gamma_{02} \\ \gamma_{10} & \gamma_{13} & \gamma_{12} \\ \gamma_{20} & \gamma_{23} & \gamma_{22} \end{vmatrix},$$

$$b_2 = \frac{1}{\Delta_3}\begin{vmatrix} \gamma_{00} & \gamma_{01} & \gamma_{03} \\ \gamma_{10} & \gamma_{11} & \gamma_{13} \\ \gamma_{20} & \gamma_{21} & \gamma_{23} \end{vmatrix}.$$

Define $\gamma_{14} := b_0\gamma_{11} + b_1\gamma_{12} + b_2\gamma_{13}$.

THEOREM 3.6. ([16, Theorem 2.11]) *Assume that $E(2)$ is positive and $r = 3$. Then γ admits a 3-atomic representing measure if and only if we may take γ_{03} satisfying*

$$\begin{aligned} b_0\gamma_{30} + b_1\gamma_{31} + b_2\gamma_{32} &= a_0\gamma_{22} + a_1\gamma_{23} + a_2\gamma_{24}, \\ b_0\gamma_{41} + b_1\gamma_{42} + b_2\gamma_{43} &= a_0\gamma_{33} + a_1\gamma_{34} + a_2\gamma_{35}. \end{aligned}$$

By Theorem 3.6, we have the following result.

PROPOSITION 3.7. ([16, Proposition 2.14]) *Assume that $E(2) \geq 0$ and $r = 3$. If $\bar{Z}Z = \mathit{1}$, then there exists a 3-atomic representing measure for γ.*

OPEN PROBLEM. *Assume that $E(2)$ is positive and invertible. Does there exist any representing measure?*

The following result is on the unit circle $\mathbb{T}$.

THEOREM 3.8. ([19]) *If $\bar{Z}Z = 1$, then the following statements are equivalent.*

(i) *γ admits a representing measure on $\mathbb{T}$;*

(ii) *γ admits an r-atomic representing measure on $\mathbb{T}$;*

(iii) *$E(2) \geq 0$ and $\gamma_{00} = \gamma_{11} = \gamma_{22}$.*

OPEN PROBLEM. *Discuss the quartic Embry moment problem on the unit disc $\mathbb{D}$.*

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