Commun. Korean Math. Soc. **21** (2006), No. 3, pp. 451–464

SHARP WEIGHTED ESTIMATE FOR MULTILINEAR SINGULAR INTEGRAL OPERATOR

LIU LANZHE

ABSTRACT. In this paper, a sharp inequality for some multilinear singular integral operators are obtained. As the applications, we get the weighted $L^p(p > 1)$ norm inequalities and $L \log L$ type estimate for the multilinear operators.

1. Preliminaries and results

As the development of Calderón-Zygmund operators and their commutators, multilinear singular integral operators have been well studied. In this paper, we will study some multilinear singular integral operators as following.

Fix $\varepsilon > 0$. Let $T : S \to S'$ be a linear operator. Suppose that T is bounded on $L^p(R^n)$ for $1 < p < \infty$ and weak (L^1, L^1)-bounded and there exists a locally integrable function $K(x, y)$ on $R^n \times R^n \setminus \{(x, y) \in R^n \times R^n : x = y\}$ such that

$$Tf(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x, y)| \leq C|x - y|^{-n}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y - z|^{\varepsilon}|x - z|^{-n-\varepsilon}$$

when $2|y - z| \leq |x - z|$. Let m_j be the positive integers $(j = 1, \ldots, l)$, $m_1 + \cdots + m_l = m$ and A_j be the functions on R^n $(j = 1, \ldots, l)$. The

Received February 28, 2005.

2000 Mathematics Subject Classification: 42B20, 42B25.

Key words and phrases: multilinear operator, singular integral operator, sharp estimate, BMO, A_p-weight.

multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} K(x,y)f(y)dy,$$

where

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Note that when $m=0$, T^A is just the multilinear commutator of T and A (see [8]). While when $m>0$, T^A is non-trivial generalizations of the commutator. It is well known that multilinear operators are of great interest in harmonic analysis and have been studied by many authors (see [1-5]). In [7], Hu and Yang proved a variant sharp estimate for the multilinear singular integral operators. In [11], Perez and Trujillo-Gonzalez prove a sharp estimate for some multilinear commutator when $A_j \in Osc_{expL^{r_j}}$. The main purpose of this paper is to prove a sharp inequality for the multilinear singular integral operators. As the applications, we obtain the weighted $L^p(p>1)$ norm inequalities and $L\log L$ type estimate for the multilinear operators.

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^\#(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)-f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that(see[6])

$$f^\#(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y)-c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^\#$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^\#||_{L^\infty}$. For $0<r<\infty$, we denote $f_r^\#$ by

$$f_r^\#(x) = [(|f|^r)^\#(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator, that is, that $M(f)(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y)|dy$. For $k\in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k\ge 2$.

Let Φ be a Young function and $\tilde{\Phi}$ be the complementary associated to Φ, we define the Φ-average of a function of f over a cube Q by

$$||f||_{\Phi,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \leq 1\right\}$$

and the maximal function associated to Φ by

$$M_\Phi(f)(x) = \sup_{x\in Q} ||f||_{\Phi,Q};$$

The Young functions to be using in this paper are $\Phi(t) = exp(t^r)-1$ and $\Psi(t) = tlog^r(t+e)$, the corresponding Φ-average and maximal functions denoted by $||\cdot||_{expL^r,Q}$, M_{expL^r} and $||\cdot||_{L(logL)^r,Q}$, $M_{L(logL)^r}$. We have the following inequality, for any $r > 0$ and $m \in N$(see[11])

$$M(f) \leq M_{L(logL)^r}(f), \quad M_{L(logL)^m}(f) \approx M^{m+1}(f);$$

For $r \geq 1$, we denote that

$$||b||_{osc_{expL^r}} = \sup_Q ||b - b_Q||_{expL^r,Q},$$

the space Osc_{expL^r} is defined by

$$Osc_{expL^r} = \{b \in L^1_{log}(R^n) : ||b||_{osc_{expL^r}} < \infty\}.$$

It has been known that(see[11])

$$||b - b_{2Q}||_{expL^r,2^kQ} \leq Ck||b||_{Osc_{expL^r}}.$$

It is obvious that Osc_{expL^r} coincides with the BMO space if $r = 1$. And $Osc_{expL^r} \subset BMO$ if $r > 1$. We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$(see[6]).

Now we state our main results as following.

THEOREM 1. *Let $r_j \geq 1$ and $D^\alpha A_j \in Osc_{expL^{r_j}}$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$. Denote that $1/r = 1/r_1 + \cdots + 1/r_l$. Then for any $0 < p < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and $x \in R^n$,*

$$(T^A(f))_p^\#(x) \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right) M_{L(logL)^{1/r}}(f)(x).$$

THEOREM 2. *Let $r_j \geq 1$ and $D^\alpha A_j \in Osc_{expL^{r_j}}$ for all α with $|\alpha| = m_j$ and $j = 1, \ldots, l$.*

(1) *If* $1 < p < \infty$ *and* $w \in A_p$, *then*

$$||T^A(f)||_{L^p(w)} \le C \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) ||f||_{L^p(w)};$$

(2) *If* $w \in A_1$. *Denote that* $1/r = 1/r_1 + \cdots + 1/r_l$ *and* $\Phi(t) = tlog^{1/r}(t+e)$. *Then there exists a constant* $C > 0$ *such that for all* $\lambda > 0$,

$$\begin{aligned} & w(\{x \in R^n : |T^A(f)(x)| > \lambda\}) \\ & \le C \int_{R^n} \Phi\left(\lambda^{-1} \prod_{j=1}^{l} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}} \right) |f(x)| \right) w(x) dx. \end{aligned}$$

2. Some lemmas

We give some preliminaries lemmas.

LEMMA 1. ([3]) *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for all* α *with* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

LEMMA 2. ([6, p.485]) *Let* $0 < p < q < \infty$ *and for any function* $f \ge 0$. *We define that, for* $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda > 0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q},$$

$$N_{p,q}(f) = \sup_E ||f \chi_E||_{L^p} / ||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E *with* $0 < |E| < \infty$. *Then*

$$||f||_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p} ||f||_{WL^q}.$$

LEMMA 3. ([11]) *Let $r_j \geq 1$ for $j = 1, \dots, m$, we denote that $1/r = 1/r_1 + \dots + 1/r_m$. Then*

$$\frac{1}{|Q|}\int_Q |f_1(x)\cdots f_m(x)g(x)|dx$$
$$\leq ||f||_{expL^{r_1},Q}\cdots||f||_{expL^{r_m},Q}||g||_{L(logL)^{1/r},Q}.$$

3. Proof of Theorem

It is only to prove Theorem 1.

PROOF OF THEOREM 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |T^A(f)(x) - C_0|^p dx\right)^{1/p}$$
$$\leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(x).$$

Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let

$$\tilde{Q} = 5\sqrt{n}Q$$

and

$$\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j}\frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha.$$

Then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha\tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}
&T^A(f)(x)\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m}K(x,y)f(y)dy\\
&= \int_{R^n}\frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m}K(x,y)f_2(y)dy\\
&\quad+\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m}K(x,y)f_1(y)dy\\
&\quad-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)f_1(y)dy
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)f_1(y)dy\\
&+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}\\
&\times K(x,y)f_1(y)dy,
\end{aligned}
$$

then

$$
\begin{aligned}
&\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(x_0)\right|\\
\leq\ &\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)f_1(y)dy\right|\\
&+\left|\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)f_1(y)dy\right|\\
&+\left|\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)f_1(y)dy\right|\\
&+\Big|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}\\
&\times K(x,y)f_1(y)dy\Big|\\
&+\left|T^{\tilde{A}}(f_2)(x)-T^{\tilde{A}}(f_2)(x_0)\right|\\
:=\ &I_1(x)+I_2(x)+I_3(x)+I_4(x)+I_5(x),
\end{aligned}
$$

thus,

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\left|T^A(f)(x)-T^{\tilde{A}}(f_2)(x_0)\right|^p dx\right)^{1/p}\\
\leq\ &\left(\frac{C}{|Q|}\int_Q I_1(x)^p dx\right)^{1/p}+\left(\frac{C}{|Q|}\int_Q I_2(x)^p dx\right)^{1/p}\\
&+\left(\frac{C}{|Q|}\int_Q I_3(x)^p dx\right)+\left(\frac{C}{|Q|}\int_Q I_4(x)^p dx\right)\\
&+\left(\frac{C}{|Q|}\int_Q I_5(x)^p dx\right)^{1/p}\\
:=\ &I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j; x, y) \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}},$$

thus, by Lemma 2 and the weak type (1,1) of T, we obtain

$$\begin{aligned}
I_1 &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right)\left(\frac{1}{|Q|}\int_Q |T(f_1)(x)|^p dx\right)^{1/p} \\
&= C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right)|Q|^{-1}\frac{||T(f_1)\chi_Q||_{L^p}}{|Q|^{1/p-1}} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right)|Q|^{-1}||T(f_1)||_{WL^1} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right)|Q|^{-1}||f_1||_{L^1} \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right)M(f)(\tilde{x}) \\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(\tilde{x});
\end{aligned}$$

For I_2, note that $||\chi_Q||_{expL^{r_2},Q} \leq C$, similar to the proof of I_1 and by using Lemma 3, we get

$$\begin{aligned}
I_2 &\leq C\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{Osc_{expL^{r_2}}} \\
&\quad \times \sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 f_1)(x)|^p dx\right)^{1/p} \\
&\leq C\sum_{|\alpha_2|=m_2} ||D^{\alpha_2} A_2||_{Osc_{expL^{r_2}}}
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{|\alpha_1|=m_1} |Q|^{-1} ||T(D^{\alpha_1}\tilde{A}_1 f_1)(x)\chi_Q||_{WL^1} \\
\leq\ & C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}} \\
&\times \sum_{|\alpha_1|=m_1} \frac{1}{|Q|}\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) f_1(x)|dx \\
\leq\ & C \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}} ||\chi_Q||_{expL^{r_2},Q} \\
&\times \sum_{|\alpha_1|=m_1} ||D^{\alpha_1}A_1 - (D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL^{r_1},\tilde{Q}} ||f||_{L(logL)^{1/r},\tilde{Q}} \\
\leq\ & C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(f)(\tilde{x});
\end{aligned}
$$

For I_3, similar to the proof of I_2, we get

$$
I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(f)(\tilde{x});
$$

Similarly, for I_4, by using Lemma 3, we get

$$
\begin{aligned}
I_4 \leq\ & C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)(x)|^p dx \right)^{1/p} \\
\leq\ & C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} |Q|^{-1} ||T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 f_1)\chi_Q||_{WL^1} \\
\leq\ & C \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{|Q|}\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) f_1(x)|dx \\
\leq\ & C \sum_{|\alpha_1|=m_1} ||D^{\alpha_1}A_1 - (D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL^{r_1},\tilde{Q}} \\
&\times \sum_{|\alpha_2|=m_2} ||D^{\alpha_2}A_2 - (D^{\alpha_2}A_2)_{\tilde{Q}}||_{expL^{r_2},\tilde{Q}} ||f||_{L(logL)^{1/r},\tilde{Q}} \\
\leq\ & C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}} \right) M_{L(logL)^{1/r}}(f)(\tilde{x});
\end{aligned}
$$

For I_5, we write

$$\begin{aligned}
&T^{\tilde{A}}(f_2)(x) - T^{\tilde{A}}(f_2)(x_0)\\
= &\int_{R^n} \left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m}\right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) f_2(y) dy\\
&+ \int_{R^n} \Big(R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y)\Big)\\
&\times \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} K(x_0,y) f_2(y) dy\\
&+ \int_{R^n} \Big(R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)\Big)\\
&\times \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} K(x_0,y) f_2(y) dy\\
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \Big[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y)\\
&\qquad\qquad - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0,y)\Big]\\
&\times D^{\alpha_1}\tilde{A}_1(y) f_2(y) dy\\
&- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \Big[\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y)\\
&\qquad\qquad - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y)\Big]\\
&\times D^{\alpha_2}\tilde{A}_2(y) f_2(y) dy\\
&+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \Big[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y)\\
&\qquad\qquad - \frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m} K(x_0,y)\Big]\\
&\times D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y) f_2(y) dy\\
= &I_5^{(1)} + I_5^{(2)} + I_5^{(3)} + I_5^{(4)} + I_5^{(5)} + I_5^{(6)};
\end{aligned}$$

By Lemma 1, we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}$,

$$|R_{m_j}(\tilde{A}_j; x, y)| \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} (||D^{\alpha_j}A||_{Osc_{expL^{r_j}}}$$

$$
\begin{aligned}
&+|(D^{\alpha_j}A)_{\tilde{Q}(x,y)} - (D^{\alpha_j}A)_{\tilde{Q}}|) \\
\le\ & Ck|x-y|^{m_j} \sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A||_{Osc_{expL^{r_j}}}.
\end{aligned}
$$

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the conditions on K,

$$
\begin{aligned}
|I_5^{(1)}| \ \le\ & C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon}}\right) \\
& \times \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j;x,y)||f_2(y)|dy \\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right) \\
& \times \sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon}}\right)|f(y)|dy \\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right) \\
& \times \sum_{k=1}^{\infty} k^2(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|f(y)|dy \\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right) M(f)(\tilde{x});
\end{aligned}
$$

For $I_5^{(2)}$, by the formula (see [3]):

$$
R_{m_j}(\tilde{A};x,y) - R_{m_j}(\tilde{A};x_0,y) = \sum_{|\beta|<m_j}\frac{1}{\beta!}R_{m_j-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta
$$

and Lemma 1, we have

$$
\begin{aligned}
&|R_{m_j}(\tilde{A};x,y) - R_{m_j}(\tilde{A};x_0,y)| \\
&\le C\sum_{|\beta|<m_j}\sum_{|\alpha|=m_j}|x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{Osc_{expL^{r_j}}},
\end{aligned}
$$

thus

$$\begin{aligned}
|I_5^{(2)}| &\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)\\
&\quad\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x-x_0|}{|x_0-y|^{n+1}}|f(y)|dy\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M(f)(\tilde{x});
\end{aligned}$$

Similarly,

$$|I_5^{(3)}|\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M(f)(\tilde{x});$$

For $I_5^{(4)}$, similar to the proof of $I_5^{(1)}$, $I_5^{(2)}$ and I_2, we get

$$\begin{aligned}
|I_5^{(4)}| &\leq C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(x_0-y)^{\alpha_1}K(x_0,y)}{|x_0-y|^m}\right|\\
&\quad\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||f_2(y)|dy\\
&\quad+C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;x_0,y)|\\
&\quad\times\frac{|(x_0-y)^{\alpha_1}K(x_0,y)|}{|x_0-y|^m}|D^{\alpha_1}\tilde{A}_1(y)||f_2(y)|dy\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\\
&\quad\times\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|dy\\
&\leq C\sum_{|\alpha_2|=m_2}||D^{\alpha_2}A_2||_{Osc_{expL^{r_2}}}\sum_{|\alpha_1|=m_1}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\\
&\quad\times||D^{\alpha_1}A_1-(D^{\alpha_1}A_1)_{\tilde{Q}}||_{expL^{r_1},2^k\tilde{Q}}||f||_{L(logL)^{1/r},2^k\tilde{Q}}\\
&\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(\tilde{x});
\end{aligned}$$

Similarly,

$$|I_5^{(5)}| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(\tilde{x});$$

For $I_5^{(6)}$, by using Lemma 3, we obtain

$$\begin{aligned}|I_5^{(6)}| &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\\ &\quad\times\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(\tilde{x});\end{aligned}$$

Thus

$$|I_5| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

By Theorem 1 and the L^p-boundedness of $M_{L(logL)^{1/r}}$, we may obtain the conclusions (1)(2) of Theorem 2.

4. Example

In this section we shall apply Theorem 1 and 2 of the paper to the Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator (see [4], [6], [12]), the multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n}\frac{\prod_{j=1}^{l}R_{m_j+1}(A_j;x,y)}{|x-y|^m}K(x,y)f(y)dy.$$

In particular, the multilinear commutator related to T is(see[11])

$$T^A(f)(x) = \int_{R^n}\left[\prod_{j=1}^{l}(A_j(x)-A_j(y))\right]K(x,y)f(y)dy.$$

Then

$$(T^A(f))^{\#}_p(x)$$

$$(1) \qquad \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)M_{L(logL)^{1/r}}(f)(x)$$

for any $f \in C_0^\infty(R^n)$ and $0 < p < 1$;

$$(2) \qquad ||T^A(f)||_{L^p(w)} \leq C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)||f||_{L^p(w)}$$

for any $w \in A_p$ and $1 < p < \infty$;

$$w(\{x \in R^n : |T^A(f)(x)| > \lambda\})$$

$$(3) \qquad \leq C\int_{R^n}\Phi\left(\lambda^{-1}\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Osc_{expL^{r_j}}}\right)|f(x)|\right)w(x)dx$$

for any $w \in A_1$ and all $\lambda > 0$.

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College of Mathematics
Changsha University of Science and Technology
Changsha 410077, P. R. China
E-mail: lanzheliu@163.com