RELATIONS IN THE TAUTOLOGICAL RING BY LOCALIZATION

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ABSTRACT. We give a way to obtain formulas for $\pi_* \psi_{n+1}^k$ in terms of ψ and λ-classes where $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}(g = 0, 1, 2)$ by the localization theorem. By using the formulas, we obtain Kontsevich–Manin type reconstruction theorems for $\overline{M}_{0,n}(\mathbb{P}^m), \overline{M}_{1,n}$, and $\overline{M}_{2,n}$. We also (re)produce a lot of well-known relations in tautological rings, such as WDVV equation, the Mumford relations, the string and dilaton equations $(g = 0, 1, 2)$ etc. and new formulas for $\pi_*(\lambda_g \psi_{n+1}^k + \cdots + \psi_{n+1}^{g+k})$.

1. Introduction

Let $M_{g,n}$ be the moduli of genus g smooth curves with n distinct marked points defined over the complex numbers. There is a compactification of $M_{g,n}$ denoted by $\overline{M}_{g,n}$ which is the moduli of genus g stable curves with n marked points. A genus g stable curve with n marked points is an arithmetic genus g complete connected nodal curve with distinct smooth n marked points and finite automorphisms. $\overline{M}_{g,n}$ has a stratification according to topological types.

Let $A^*(\overline{M}_{g,n})$ be the Chow ring with $\mathbb{Q}$-coefficients. The system of tautological rings is defined to be the set of smallest $\mathbb{Q}$-subalgebras containing $1(\neq 0)$ of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n})$$

satisfying the following two properties:

1. The system is closed under pushforward and pullback via all the maps forgetting the last marking (Figure 1):

$$\pi : \overline{M}_{g,n} \to \overline{M}_{g,n-1}.$$

FIGURE 1. Forgetting the last marked point $*$

2. The system is closed under pushforward and pullback via all the gluing maps (Figures 2–3):

$$\iota : \overline{M}_{g_1,n_1\cup*} \times \overline{M}_{g_2,n_2\cup\#} \to \overline{M}_{g_1+g_2,n_1+n_2}$$

$$\iota : \overline{M}_{g,n\cup\{*,\#\}} \to \overline{M}_{g+1,n}$$
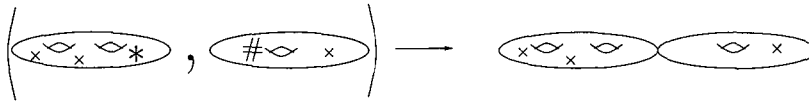
with attachments along the marking $*$ and $\#$.

FIGURE 2. Gluing $*$ and $\#$

FIGURE 3. Gluing $*$ and $\#$

While the definition appears restrictive, the standard ψ, κ and λ-classes all lie in the tautological ring [17]. For example,

$$-\pi_*(\iota_*([\overline{M}_{g,n}] \times [\overline{M}_{0,3}])^2) = \psi_i$$

where $\iota : \overline{M}_{g,\{1,2,\dots,i-1,*,i+1,\dots,n\}} \times \overline{M}_{0,\{\#,i,n+1\}} \to \overline{M}_{g,n+1}$.

The tautological rings possess a rich conjectural structure [7].

PROBLEM 1. *Find out relations in tautological rings and ways to compute intersection numbers recursively from the moduli stacks with smaller g or n.*

We can ask the same problem for other moduli stacks.

In this paper we will answer the above problem in lower genus cases $(g = 0, 1, 2)$ for $\overline{M}_{g,n}$ and genus 0 case for $\overline{M}_{0,n}(\mathbb{P}^m)$. The technique we will use is the localization theorem for equivariant Chow group [1], [3], [6], [12]. More precisely, we give a way to obtain formulas for $\pi_*\psi_{n+1}^k$ in terms of ψ and λ where $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}(g = 0, 1, 2)$ by the localization theorem. By using the formulas, we obtain Kontsevich–Manin type reconstruction theorems. We also (re)produce a lot of well-known relations in tautological rings, such as WDVV equation, the Mumford relations, the string and dilaton equations $(g = 0, 1, 2)$ etc. and new formulas for $\pi_*(\lambda_g\psi_{n+1}^k + \cdots + \psi_{n+1}^{g+k})$.

2. Background

2.1. Dimension formula

In this section, we list virtual dimensions for the stacks which we will deal with in the preceding sections.

Let X be a smooth projective variety and $\beta \in A_1(X)$. Let $\overline{M}_{g,n}(X,\beta)$ be the moduli of genus g, n-pointed stable maps to a projective variety X with β class. A genus g, n-pointed stable map to X with β class is a map from a genus g nodal curve with n smooth distinct points to X whose image is class $\beta \in A_1(X)$ [8]. By deformation theory and the Riemann-Roch formula, we can calculate the virtual dimension of $\overline{M}_{g,n}(X,\beta)$;

$$\text{vir. dim. } \overline{M}_{g,n}(X,\beta) = (1-g)(dimX - 3) - \int_\beta \omega_X + n.$$

The special cases which we will use frequently in this paper are:

$$\text{vir. dim. } \overline{M}_{g,n} = \text{dim. } \overline{M}_{g,n} = 3g - 3 + n$$

$$\text{vir. dim. } \overline{M}_{g,n}(\mathbb{P}^1, d) = 2g - 2 + 2d + n.$$

2.2. Virtual localization

The higher genus Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbb{P}^m, d)$ are in general non-reduced, reducible, singular, so we can not apply the usual localization formula [1]. The answer to overcome this difficulty is the virtual localization theorem by Graber and Pandharipande [12].

THEOREM 2 (The virtual localization theorem). ([12] §1) *Suppose* $f : X \to X'$ *is a* $T = (C^*)^{m+1}$-*equivariant map of proper Deligne–Mumford quotient stacks with a* T-*equivariant perfect obstruction theory. If* $i' : F' \hookrightarrow X'$*is a fixed substack and* $c \in A_T^*(X)$*, let* $f_{F_i} : F_i \to F'$ *be the restriction of* f *to each of the fixed substacks* $F_i \subset f^{-1}(F')$*. Then*

$$\sum_{F_i} f_{F_i *} \frac{i_{F_i}^* c}{\epsilon_T(F_i^{vir})} = \frac{i'^* f_* c}{\epsilon_T(F'^{vir})}$$

where $i_{F_i} : F_i \hookrightarrow X$ and $\epsilon_T(F^{\text{vir}})$ is the virtual equivariant Euler class of "virtual" normal bundle F^{vir}.

REMARK 3.   1. If X and X' are nonsingular with the trivial perfect obstruction theories ([2] §4), then the virtual localization formula reduces to the standard localization formula.

2. The conditions in the theorem are satisfied for the Kontsevich–Manin spaces $\overline{M}_{g,n}(\mathbb{P}^m, d)$ with the induced action by the diagonal action of T on $\mathbb{P}^m$, and $\epsilon_T(F^{\text{vir}})$ can be explicitly computed in terms of ψ and λ-classes ([12] §4).

2.3. $\mathbb{C}^*$-action on $\mathbb{P}^1$

We define a $T = \mathbb{C}^*$-action on $\mathbb{P}^1$ for $a \in T$ and $(x_0 : x_1) \in \mathbb{P}^1$ by $a \cdot (x_0 : x_1) = (x_0 : ax_1)$. There are two fixed points $0 = (0 : 1)$ and $\infty = (1 : 0)$.

This T-action induces T-actions on $\overline{M}_{g,n}(\mathbb{P}^1, d)$.

3. Pushforward of ψ_{n+1}^k

In this section, we will obtain the formulas $\pi_* \psi_{n+1}^k$, where $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ for $g < 3$. We will explain the detail for $g = 0$ case.

Consider the following map

$$f : \overline{M}_{0,n}(\mathbb{P}^1, 1) \to \overline{M}_{0,n} \times \mathbb{P}^1;$$

$$\left(g : (C; p_1, \ldots, p_n) \to X \right) \mapsto \left((C; p_1, \ldots, p_n)^{\text{st}}, g(p_1) \right)$$

where $(C; p_1, \ldots, p_n)^{\text{st}}$ is a marked curve obtained by contracting all the rational components which have at most two special points.

We know that $f_*(1) = 0$ by dimension counting $(dim(\overline{M}_{0,n} \times \mathbb{P}^1) < dim(\overline{M}_{0,n}(\mathbb{P}^1, 1))$. If we choose one component of fixed loci $F' = \overline{M}_{0,n} \times 0 (\cong \overline{M}_{0,n})$ in the target space, by the localization theorem we have

$$(1) \qquad\qquad \sum_{F_i} f_{F_i *} \frac{i_{F_i}^* 1}{\epsilon_T(F_i)} = 0.$$

Now, we have to write up all the components of fixed loci in the domain which maps to F'. The general fixed loci are as in Figure 4. There are three types of degeneration (Figure 5). The first one is the case $m = n$, which is isomorphic to $\overline{M}_{0,n+1}$. The second and third ones have only one point on a rational tail. These are isomorphic to $\overline{M}_{0,n}$.
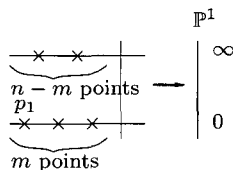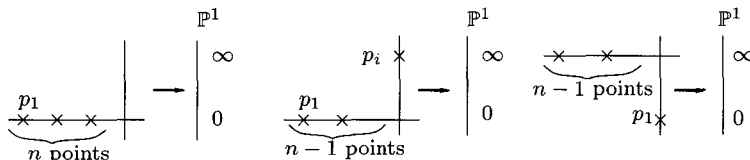
FIGURE 4. General Fixed Locus

FIGURE 5. Three Types of Degeneration

Calculating the equivariant Euler classes for each fixed loci as in [12], we obtain

$$(2) \quad \pi_*\Big(\frac{1}{t(t-\psi_{n+1})}\Big) + \sum_{i=2}^{n} id_*\Big(\frac{1}{t(t-\psi_i)(-t)}\Big) + id_*\Big(\frac{1}{(-t)(-t-\psi_1)t}\Big)$$

$$+ \sum_{1 \in I |I|,|I^c| \geq 2} \iota_*\Big(\frac{1}{t(t-\psi_{|I|+1})}\frac{1}{(-t)(-t-\psi_{|I^c|+1})}\Big) = 0$$

where $I \subset \{1, 2, \ldots, n\}$.

To compute $\pi_*\psi_{n+1}^k$, calculate the coefficient of t^{-k-2}.

EXAMPLE 4.   1. If we take the coefficient of t^{-2}, only the first term will contribute, so $\pi_*1 = 0$.
2. If we take the coefficient of t^{-3}, the last summation will not contribute. We will obtain $\pi_*\psi_{n+1} = n - 2$.
3. If we take the coefficient of t^{-4}, We will obtain $\pi_*\psi_{n+1}^2 = \sum_{i=1}^{n} \psi_i - \Delta$ where Δ is all the boundary divisors.

For genus 1 case, we consider

$$f : \overline{M}_{1,n}(\mathbb{P}^1, 1) \to \overline{M}_{1,n} \times \mathbb{P}^1.$$

and do the similar computation. In genus 1 case, instead of $\pi_*\psi_{n+1}^k$, we will have $\pi_*(\lambda_1\psi_{n+1}^{k-1} + \psi_{n+1}^k)$. But we can compute ψ_{n+1}^k inductively because λ-classes will be pushforwarded to the same λ-classes or we know what λ-class is from [17].

For genus 2 case, we consider

$$f : \overline{M}_{2,n}(\mathbb{P}^1, 1) \to \overline{M}_{2,n} \times \mathbb{P}^1.$$

But in this case, we have $\dim(\overline{M}_{2,n}(\mathbb{P}^1,1)) = \dim(\overline{M}_{2,n} \times \mathbb{P}^1)$, so that $f_*(1) = c$. In fact we know that $c = 4$ [15] or by a further similar localization calculation in §8.1. In genus 2 case, instead of $\pi_* \psi_{n+1}^k$, we will have $\pi_*(\lambda_2 \psi_{n+1}^{k-2} + \lambda_1 \psi_{n+1}^{k-1} + \psi_{n+1}^k)$. But we can compute ψ_{n+1}^k just as genus 1 case. For λ-class, see [9].

4. The case of $\overline{M}_{0,n+1}(\mathbb{P}^m, d)$

In this section, we will generalize the method in §3 to $\overline{M}_{0,n+1}(\mathbb{P}^m, d)$. This will give a simplified version of the result in [3]. In this case, we have pullbacks of powers of the hyperplane class besides pushforward of powers of ψ-class. Even to compute the pushforward of power of ψ-class, we need the knowledge of mixed classes because the general fixed locus will be a fiber product over $\mathbb{P}^m$.

As before, we have analogous maps to π and ι, and we will call these maps π and ι. Besides these, we have one more family of maps $ev_i : \overline{M}_{0,n+1}(\mathbb{P}^m, d) \to \mathbb{P}^m$, that is the evaluation map at i-th point.

$$
\begin{array}{ccc}
\overline{M}_{0,n+1}(\mathbb{P}^m, d) & \xrightarrow{ev_i} & \mathbb{P}^m \\
\Big\downarrow \pi & & \\
\overline{M}_{0,n}(\mathbb{P}^m, d) & &
\end{array}
$$

we want to find formulas for $\pi_*(\psi_{n+1}^k \cap ev_i^* H^l)$.

This time we will consider the following map:

$$f : \overline{M}_{0,n}(\mathbb{P}^m \times \mathbb{P}^1, (d,1)) \to \overline{M}_{0,n}(\mathbb{P}^m, d) \times \mathbb{P}^1.$$

To include a power of the hyperplane class, we consider the following map to the linear sigma model [4], [10]:

$$
\begin{aligned}
g : \quad & \overline{M}_{0,n}(\mathbb{P}^m \times \mathbb{P}^1, (d,1)) \longrightarrow \overline{M}_{0,0}(\mathbb{P}^m \times \mathbb{P}^1, (d,1)) \\
& \longrightarrow \mathbb{P}_d^m = \mathbb{P}(Sym^d(V) \otimes \mathbb{C}^{m+1})
\end{aligned}
$$

where V is $\mathbb{P}^1 = \mathbb{P}(V^*)$. Denote $H_T = c_1^T(\mathcal{O}_{\mathbb{P}_d^m}(1))$. Consider one component of fixed loci $F' = \overline{M}_{0,n}(\mathbb{P}^m, d) \times 0 (\cong \overline{M}_{0,n}(\mathbb{P}^m, d))$. Then by the localization theorem, we have

$$\sum_{F_i} f_{F_i *} \frac{i_{F_i}^*(g^* H_T)^l}{e_T(F_i)} = \frac{i'^* f_*(g^* H_T)^l}{e_T(F')}.$$

As in §3, we need to classify all the fixed loci which map to F', then write down the Euler classes of each fixed loci and restrictions of $(g^*H_T)^l$ to each fixed loci. The general fixed loci F_i's are as in Figure 6.

As §3, we have three types of degeneration. In each type the rational tail growing out from 0 or ∞ on $\mathbb{P}^1$ maps to $\mathbb{P}^m$ with total degree d. The Euler classes are exactly same as §3.

Now we need to know the restriction of $(g^*H_T)^l$ to each fixed loci. Following calculations in [4], we will obtain $f_*((g^*H_T)^l)|_{F'}$ is a polynomial in t, $p(t)$ and $(g^*H_T)^l|_{F_i} = ev^*\{(H - et)^l\}$.

So we have

$$(3)\quad \pi_*\left(\frac{ev_{n+1}^*H^l}{t(t-\psi_{n+1})}\right) + \sum_{i=2}^{n} id_*\left(\frac{ev_i^*H^l}{t(t-\psi_i)(-t)}\right) + id_*\left(\frac{ev_1^*(H-dt)^l}{(-t)(-t-\psi_1)t}\right)$$

$$+ \sum_{\substack{\text{non-degenerate fixed loci}}} \iota_*\left(\frac{1}{t(t-\psi_{|I|+1})}\frac{ev_{|I|+1}^*(H-et)^l}{(-t)(-t-\psi_{|I^c|+1})}\right) = \frac{p(t)}{t}$$
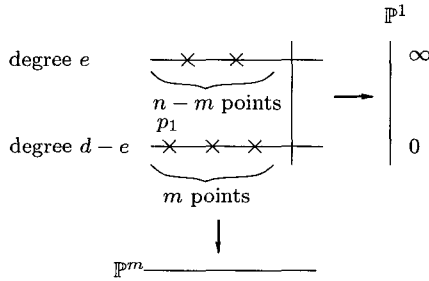

FIGURE 6. General Fixed Locus

By calculating the coefficient of t^{-k} ($k \geq 2$), we can obtain the formula for $\pi_*(\psi_{n+1}^{k-2} \cap ev_{n+1}^*H^l)$.

5. Reconstruction theorem

In this section we explain why it is enough to know $\pi_*(\psi_{n+1}^{-k+1} \cap ev_{n+1}^*H^l)$ to compute all the genus-zero Gromov–Witten invariants.

By knowing the projection formula and $\psi_i = \pi^*\psi_i + D_{i,n+1}$ [15], we have

$$\int_{\overline{M}_{0,n}(\mathbb{P}^m,d)} \prod_{i=1}^{n}\{\psi_i^{k_i} \cap ev_i^*H^{l_i}\} \cap \pi_*(\psi_{n+1}^k \cap ev_{n+1}^*H^l) =$$

$$\int_{\overline{M}_{0,n+1}(\mathbb{P}^m,d)} \prod_{i=1}^{n} \{ (\psi_i - D_{i,n+1})^{k_i} \cap ev_i^* H^{l_i} \} \cap \psi_{n+1}^k \cap ev_{n+1}^* H^l$$

where $D_{i,n+1}$ is a divisor such that i-th marked point and $(n+1)$-th marked point are only marked points on a rational tail.

If we expand the right hand side, you have $\int_{\overline{M}_{0,n+1}(\mathbb{P}^m,d)} \prod_{i=1}^{n} \{ (\psi_i)^{k_i} \cap ev_i^* H^{l_i} \} \cap \psi_{n+1}^k \cap ev_{n+1}^* H^l$ and other terms. But other integral terms have $D_{i,n+1}$ in their integrand, so they are integral on smaller moduli spaces. Thus inductively we can calculate Gromov–Witten invariants starting from one-point invariants.

By this method, we can obtain the string, dilaton and divisor equations.

REMARK 5.   1. If $m = 0$, that is $\mathbb{P}^0$, then $\overline{M}_{0,3}$ is the smallest moduli in this family. So in this case we can calculate Gromov–Witten invariants starting from three-point invariants instead of one-point invariants.
   2. We have similar theorems for $\overline{M}_{g,n}(g = 1,2)$.
   3. By pulling back the formula of $\pi_* \psi_{n+1}^k$ on $\overline{M}_{g,n}$ to $\overline{M}_{g,n}(X,\beta)$, we can obtain the sting and dilaton equation for genus $1,2$.

6. Genus 0 relations

6.1. The WDVV equation

The WDVV equation is one of the crucial equations in Gromov-Witten Invariants (it implies associativity of quantum cohomology, flatness of Dubrovin connection). But the well-known proof depends on the knowledge of $\overline{M}_{0,4} \cong \mathbb{P}^1$. In this section we will prove the WDVV equation without knowing the above isomorphism.

Consider the following map

$$f : \overline{M}_{0,4}(\mathbb{P}^1, 1) \to \overline{M}_{0,4} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

$$\left(g : (C; p_1, p_2, p_3) \to \mathbb{P}^1 \right) \mapsto \left((C; p_1, p_2, p_3)^{\text{st}}, g(p_1), g(p_2), g(p_3) \right).$$

From now on, for maps we will consider in this paper, the first component is forgetting a map to $\mathbb{P}^1$, the i-th $\mathbb{P}^1$ is the image of the i-th point.

We know that $f_*(1) = c$ for some constant c by dimension counting. So if we choose one component of fixed loci $F' = \overline{M}_{0,4} \times 0 \times 0 \times \infty$ in the

target space, then there are two fixed loci which map to F' (Figure 7). By the localization theorem, we will have

$$\iota_*\left(\frac{1}{t(t-\psi_3')}\frac{1}{(-t)(-t-\psi_3'')}\right) + id_*\left(\frac{1}{t(t-\psi_3)(-t)}\right) = \frac{c}{-t^3}$$

here "$'$" and "$''$" indicate ψ-classes live on different components. We will use this notation in the rest of this paper.
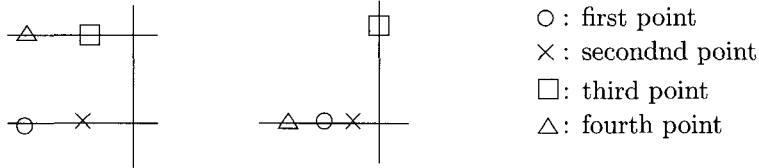

$\bigcirc$: first point
$\times$: secondnd point
$\square$: third point
$\triangle$: fourth point

FIGURE 7. Two Fixed Loci

By taking the coefficient of t^{-4}, we obtain

$$\psi_3 =$$

By changing the role of points, we can prove all three boundary divisors are rationally equivalent to ψ_3.

6.2. Relation $\psi_i = \pi^*\psi_i + D_{i,n+1}$

It is enough to prove for $\overline{M}_{0,5}$. The cases of $\overline{M}_{0,n}(X,\beta)$ will follow by distributing appropriate points and degrees. We will prove $\psi_3 = \pi^*\psi_3 + D_{3,5}$.

Consider the following map as before

$$f : \overline{M}_{0,5}(\mathbb{P}^1, 1) \to \overline{M}_{0,5} \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

We know that $f_*(1) = c$ for some constant c by dimension counting. So if we choose one component of fixed loci $F' = \overline{M}_{0,4} \times 0 \times 0 \times \infty$ in the target space, then there are four fixed loci which map to F'. By the localization theorem, we have

$$(4) \quad id_*\left(\frac{1}{t(t-\psi_3)(-t)}\right) + \iota_*\left(\frac{1}{t(t-\psi_4')}\frac{1}{(-t)(-t-\psi_3'')}\right)$$

$$+ \iota_*\left(\frac{1}{t(t-\psi_4')}\frac{1}{(-t)(-t-\psi_3'')}\right) + \iota_*\left(\frac{1}{t(t-\psi_3')}\frac{1}{(-t)(-t-\psi_4'')}\right) = \frac{c}{-t^3}.$$
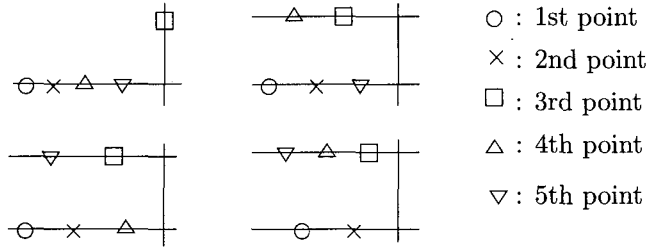
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FIGURE 8. Four Fixed Loci

By taking the coefficient of t^{-4}, we will obtain

But we know $\pi^*\psi_3$ from §6.1,

So this implies $\psi_3 = \pi^*\psi_3 + D_{3,5}$.

7. Genus 1 relations

7.1. Relation $\psi_1 = \psi_2$ on $\overline{M}_{1,2}$

Consider the following map $f : \overline{M}_{1,2}(\mathbb{P}^1, 1) \to \overline{M}_{1,2} \times \mathbb{P}^1$. We have four fixed loci which map to $\overline{M}_{1,2} \times 0$ (Figure 9). By the virtual localization theorem, we obtain

$$\pi_*\left(\frac{-\lambda_1+t}{t(t-\psi_3)}\right) + id_*\left(\frac{-\lambda_1+t}{t(t-\psi_2)(-t)}\right)$$
$$+id_*\left(\frac{-\lambda_1-t}{(-t)(-t-\psi_1)t}\right) + \iota_*\left(\frac{-\lambda_1-t}{(-t)(-t-\psi_1'')t^2}\right) = 0.$$
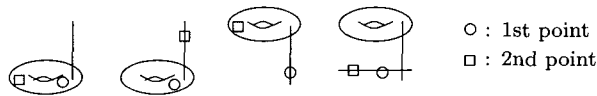

FIGURE 9. Four Fixed Loci

By taking the coefficient of t^{-4}, we have

(5) $\pi_*(-\lambda_1\psi_3 + \psi_3^2) = (-\lambda_1 + \psi_2) - (-\lambda_1 + \psi_1) + \iota_*(1).$

By switching the role of the first point and the second point, we will get

(6) $\pi_*(-\lambda_1\psi_3 + \psi_3^2) = (-\lambda_1 + \psi_1) - (-\lambda_1 + \psi_2) + \iota_*(1).$

By equating (5) and (6), we have

$$(7) \qquad\qquad -\lambda_1 + \psi_2 = -\lambda_1 + \psi_1.$$

So we have $\psi_1 = \psi_2$ on $\overline{M}_{1,2}$.

7.2. Relation $\lambda_1 = \frac{\delta_0}{24}$

In stead of considering a degree 1 map to $\mathbb{P}^1$, we consider a degree 2 map and then obtain $\lambda_1 = \frac{\delta_0}{24}$. Consider the map $\overline{M}_{1,1}(\mathbb{P}^1, 2) \rightarrow \overline{M}_{1,1} \times \mathbb{P}^1$. This time there are 8 fixed loci which map to $\overline{M}_{1,1} \times 0$ (Figure 10). By calculating the Euler classes for each fixed loci, we obtain

$$
\begin{aligned}
(8) \quad & \pi_*^2 \left(\frac{-\lambda_1 + t}{2t(t - \psi_2)(t - \psi_3)} \right) + \pi_* \left(\frac{-\lambda_1 + t}{2\frac{t}{2}(\frac{t}{2} - \psi_2)t(-t)} \right) \\
& + \pi_* \left(\frac{-\lambda_1 - t}{(-t)(-t - \psi_1)(-t - \psi_2)t} \right) + id_* \left(\frac{-\lambda_1 - t}{2(\frac{-t}{2})(-\frac{t}{2} - \psi_1)t\frac{t}{2}(-t)} \right) \\
& + \iota_* \left(\frac{1}{2} \frac{1}{(-t)(-2t)t^3} \right) + \pi_* \left(\frac{-\lambda_1 + t}{(-t)(-2t)t(t - \psi_2)} \right) \\
& + \pi_* \left(\frac{-\lambda_1 + t}{(-t)(-2t)t(t - \psi_1)t} \right) + id_* \left(\frac{-\lambda_1 - t}{t^3(-t)(-t - \psi_1)} \right) = 0.
\end{aligned}
$$

| indicates a rational component which maps to $\mathbb{P}^1$ by degree 1
| indicates a rational component which maps to $\mathbb{P}^1$ by degree 2
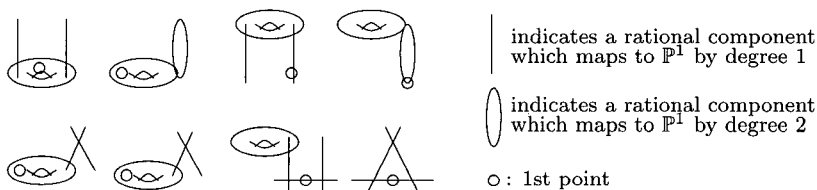o : 1st point

FIGURE 10. Eight Fixed Loci

Taking the coefficient of t^{-5} and simplifying, we can get the formula. Knowing the string equation and $\lambda_1 = \psi_1$ on $\overline{M}_{1,1}$, you can see that the last three terms won't contribute.

8. Genus 2 relations

8.1. Relations $c_0 = 4$ and $10\lambda_1 = \delta_0 + 2\delta_1$ on $\overline{M}_{2,0}$

In this section, we will obtain a well-known relation $10\lambda_1 = \delta_0 + 2\delta_1$ on $\overline{M}_{2,0}$ where δ_0 is a divisor which is the closure of the locus of irreducible

singular curves, and δ_1 is the locus of singular curves $C_1 \bigcup C_2, C_1 \bigcap C_2 = \{\text{one point}\}$, genus of C_i is 1. Denote $\Delta = \delta_0 + \delta_1$.

Consider the following map $f : \overline{M}_{2,1}(\mathbb{P}^1, 2) \to \overline{M}_{2,1} \times \mathbb{P}^1$. By dimension counting, we know that $f_*(1) = 0$. We have fifteen fixed loci which map to $\overline{M}_{2,1} \times 0$ (Figure 11). Computing the Euler classes of each fixed loci, we get

$$\pi_*^2\left(\frac{\lambda_2 - \lambda_1 t + t^2}{2t(t - \psi_2)(t - \psi_3)}\right) + \pi_*\left(\frac{\lambda_2 - \lambda_1 t + t^2}{t(\frac{t}{2})(\frac{t}{2} - \psi_2)(-t)}\right)$$

$$+ \pi_*\left(\frac{\lambda_2 + \lambda_1 t + t^2}{t(-t)(-t - \psi_1)(-t - \psi_2)}\right)$$

$$+ \pi_*\left(\frac{\lambda_2 - \lambda_1 t + t^2}{t(t - \psi_2)(-t)(-2t)}\right)$$

$$+ id_*\left(\frac{\lambda_2 - \lambda_1 t + t^2}{t(t - \psi_1)(-t)(-2t)}\right) + id_*\left(\frac{\lambda_2 + \lambda_1 t + t^2}{2t(\frac{t}{2})(-t)(-\frac{t}{2})(\frac{t}{2} - \psi_1)}\right)$$

$$+ id_*\left(\frac{\lambda_2 + \lambda_1 t + t^2}{t^3(-t)(-t - \psi_1)}\right)$$

$$+ \iota_*\left(\frac{1}{2}\frac{-\lambda_1 + t}{t(t - \psi_2)(t - \psi_3)(-t)(-2t)}\right)$$

$$+ \pi_*\iota_*\left(\frac{-\lambda_1 + t}{t(t - \psi_2')(t - \psi_3')}\frac{-\lambda_1 - t}{(-t)(-t - \psi_1'')}\right)$$

$$+ \iota_*\left(\frac{1}{2}\frac{-\lambda_1 + t}{t(\frac{t}{2})(\frac{t}{2} - \psi_2')}\frac{-\lambda_1 - t}{-t(\frac{t}{2})(\frac{t}{2} - \psi_1'')}\right)$$

$$+ \iota_*\left(\frac{-\lambda_1 + t}{t(t - \psi_2')}\frac{-\lambda_1 + t}{t(t - \psi_1'')}\frac{1}{(-t)(-2t)}\right)$$

$$+ \iota_*\left(\frac{-\lambda_1 + t}{t^2(t - \psi_1')}\frac{-\lambda_1 - t}{(-t)(-t - \psi_1'')(-t - \psi_2'')}\right)$$

$$+ \pi_*\iota_*\left(\frac{-\lambda_1 + t}{t(t - \psi_2')}\frac{-\lambda_1 - t}{(-t)(-t - \psi_1'')(-t - \psi_2'')}\right)$$

$$+ \iota_*\left(\frac{1}{2}\frac{-\lambda_1 - t}{(-t)(-t - \psi_1')}\frac{-\lambda_1 - t}{(-t)(-t - \psi_1'')}\frac{1}{t^3}\right)$$

$$+ \iota_*\left(\frac{1}{2}\frac{1}{t^3}\frac{-\lambda_1 - t}{(-t)(-t - \psi_1'')(-t - \psi_2'')}\right)$$
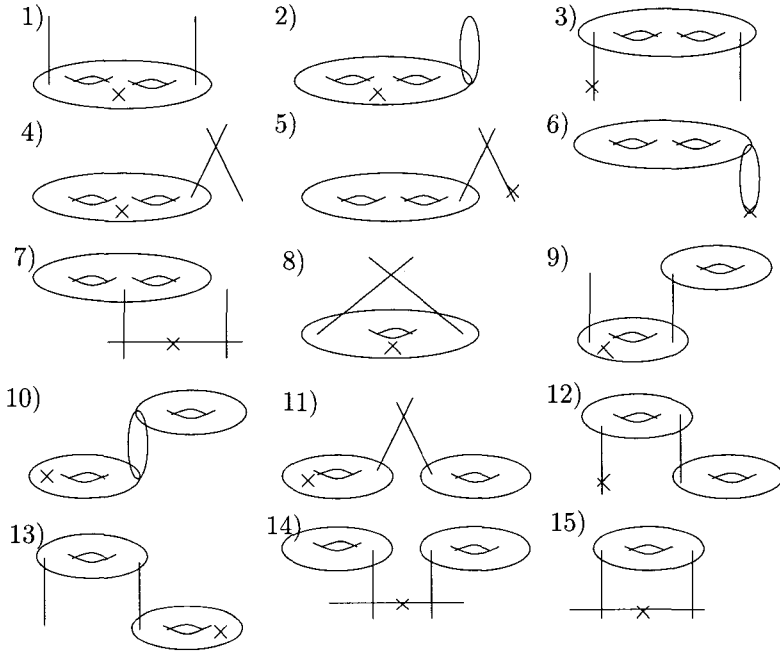
$$= 0.$$

FIGURE 11. Fifteen Fixed Loci

To compute c_0 in §3, we will take the coefficient of t^{-3}. Then only the first seven terms will contribute to the calculation.

$$(9) \qquad \frac{1}{2}\pi_*^2(\psi_2^2 + \psi_2\psi_3 + \psi_3^2) - 2\pi_*(-\lambda_1 + 2\psi_2) + \pi_*(-\lambda_1 + \psi_1 + \psi_2)$$
$$+ \frac{1}{2}\pi_*(-\lambda_1 + \psi_2) - 4 + 1 = 0.$$

For the calculation of the first term, you can assume $\psi_2^2 = \psi_3^2$ because you will pushforward twice by π. So the first term will be computed by the following way.

$$\frac{1}{2}\pi_*^2(\psi_2^2 + \psi_2\psi_3 + \psi_3^2) = \frac{1}{2}\pi_*^2(2\psi_2^2 + \psi_2\psi_3)$$

$$(10) \qquad\qquad\qquad\qquad = \frac{1}{2}\pi_*(2\psi_2 + c_0\psi_2)$$

$$\qquad\qquad\qquad\qquad = \frac{1}{2}(c_0 + 2)(c_0 - 1).$$

The other terms can be calculated by a similar way. Then we obtain

$$\frac{1}{2}c_0(c_0 - 4) = 0.$$

We know that c_0 is nonzero, so $c_0 = 4$.

The coefficients of t^{-4} give $10\lambda_1 = \frac{\delta_0}{2} + 2\delta_1$ on $\overline{M}_{2,1}$. Multiplying the relation by ψ_1 and pushforward to $\overline{M}_{2,0}$, we have what we wanted.

8.2. Relation $\kappa_1 = 2\lambda_1 + \frac{\delta_1}{2}$ on $\overline{M}_{2,0}$

Consider the following map $f : \overline{M}_{2,1}(\mathbb{P}^1, 1) \to \overline{M}_{2,1} \times \mathbb{P}^1$. As before, we will have a codimemsion 2 relation in the tautological ring of $\overline{M}_{2,1}$. If we pushforward this relation to $\overline{M}_{2,0}$ by $\pi : \overline{M}_{2,1} \to \overline{M}_{2,0}$, we will have $\kappa_1 = 2\lambda_1 + \delta_1$ on $\overline{M}_{2,0}$. Combining with the relation in §8.1, we have

$$\lambda_1 = \frac{\kappa_1 + \Delta}{12}.$$

8.3. Mumford's relation

In this section, we will prove a Mumford's relation

$$\psi_1^2 - \lambda_1\psi_1 + \lambda_2 = \text{⬯⬯}.$$

Consider the following map $f : \overline{M}_{2,2}(\mathbb{P}^1, 1) \to \overline{M}_{2,2} \times \mathbb{P}^1$ just as before. There are six fixed loci which map to $\overline{M}_{2,2} \times 0$ (Figure 12).

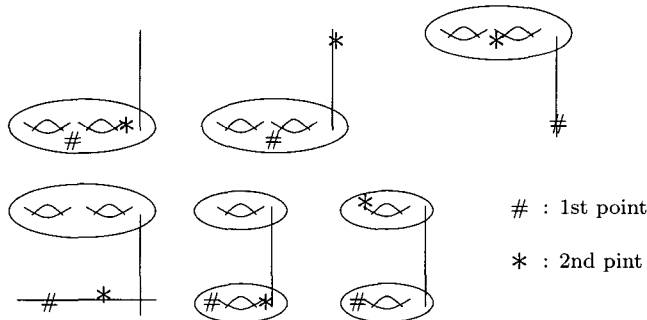# : 1st point

* : 2nd pint

FIGURE 12. Six Fixed Loci

We will have

$$\pi_*(\lambda_2\psi_3 - \lambda_1\psi_3^2 + \psi_3^3) = (-\lambda_1\psi_2 + \psi_2^2) - (-\lambda_1\psi_1 + \psi_1^2)$$

$$+2 \;\text{⬯⬯} + \text{⬯⬯} + \iota_*(-\lambda_1 + \psi_1).$$

By changing the role of the first and second points, we have

$$\pi_*(\lambda_2\psi_3 - \lambda_1\psi_3^2 + \psi_3^3) = (-\lambda_1\psi_1 + \psi_1^2) - (-\lambda_1\psi_2 + \psi_2^2)$$

$$+2 \;\text{⬯⬯} + \text{⬯⬯} + \iota_*(-\lambda_1 + \psi_1).$$

By equating, we obtain the following relation

$$(-\lambda_1\psi_2 + \psi_2^2) + \text{⟨◇⟩#⟨◇⟩} = (-\lambda_1\psi_1 + \psi_1^2) + \text{⟨◇⟩*⟨◇⟩}$$

on $\overline{M}_{2,2}$. Multiplying this relation by ψ_2 and then pushforwarding to $\overline{M}_{2,1}$, we have

$$\pi_*(\psi_2 * \text{L. H. S.}) = -4\lambda_2 + \lambda_1\psi_1 - \psi_1^2 + 4 \quad \text{⟨◇⟩#⟨◇⟩}$$

$$\pi_*(\psi_2 * \text{R.H.S.}) = -3\lambda_1\psi_1 + 3\psi_1^2.$$

Thus we will have what we wanted.

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