Commun. Korean Math. Soc. 21 (2006), No. 3, pp. 491–496

ON THE COMPACT RIEMANNIAN MANIFOLDS WITH SOME GEODESICAL PROPERTIES

Kwang-Soon Park

ABSTRACT. In the paper, we study an n-dimensional compact Riemannian manifold (M, g) with the property that the lengthes of the images $c(\mathbb{R})$ in M of any geodesic curves $c : \mathbb{R} \mapsto M$ are finite.

1. Introduction

In the differential geometry, we know

THEOREM 1.1. (H. Whitney) *Any n-dimensional differentiable manifold can be embedded into* $\mathbb{R}^{2n+1}$ [5].

So, in order to study any differentiable manifolds, it is sufficient to consider only the submanifolds of $\mathbb{R}^m$ for each integer $m \geq 1$.

We also see

THEOREM 1.2. (J. Nash) *Every Riemannian manifold (M, g) can be isometrically embedded into some Euclidean space* $\mathbb{R}^n$ ([1], [3], [4]).

Thus, by Theorem 1.2, in the paper we consider a Riemannian manifold (M, g) as a submanifold of some Euclidean space $\mathbb{R}^n$. Now, we can consider the following Question:

QUESTION 1.1. Let $M = S^1 \times S^2$. Is there a geodesic curve $c : \mathbb{R} \mapsto M$ such that the length of its image $c(\mathbb{R})$ in M is infinite?

More generally, we can also give the following Question:

QUESTION 1.2. Let (M, g) be an n-dimensional compact Riemannian manifold. Is there a geodesic curve $c : \mathbb{R} \mapsto M$ such that the length of its image $c(\mathbb{R})$ in M is infinite?

Received December 28, 2005.
2000 Mathematics Subject Classification: 53C22.
Key words and phrases: Riemannian manifold, geodesic curve.

For the answers of the above Questions, we have the following Main Theorems.

MAIN THEOREM 1. *Let (M, g) be an n-dimensional compact Riemannian manifold with the property that for each geodesic curve $c : \mathbb{R} \mapsto M$, the length of its image $c(\mathbb{R})$ in M is finite. Then we have*

$$\textit{either } \pi_1(M) = \mathbb{Z} \textit{ or } \pi_1(M) = \mathbb{Z}_p \textit{ for some } p \in \mathbb{N}.$$

MAIN THEOREM 2. *Let (M, g) be an n-dimensional compact Riemannian manifold with the property that for each geodesic curve $c : \mathbb{R} \mapsto M$, the length of its image $c(\mathbb{R})$ in M is finite. If $n \geq 2$, then we obtain*

$$\pi_1(M) = \mathbb{Z}_p \textit{ for some } p \in \mathbb{N}.$$

But we know

THEOREM 1.3. (Synge) *Any compact oriented even-dimensional Riemannian manifold with positive sectional curvature is simply connected* [2].

So, we can compare Main Theorem 2 with Theorem 1.3.
In the paper, we will use the following notations:
$\mathbb{R}$ = the set of real numbers
$\mathbb{R}^+$ = the set of positive real numbers
$\mathbb{Q}$ = the set of rational numbers
$\mathbb{N}$ = the set of natural numbers
$\mathbb{Z}$ = the set of integers
$\mathbb{Z}_p$ = the quotient group $\mathbb{Z}/p\mathbb{Z}$
I = the closed interval $[0, 1]$
$L(c)$ = the length of a curve c
T_pM = the tangent space of M at $p \in M$
∇ = the Levi-Civita connection of a metric g in M
$|v| = \sqrt{g(v, v)}$ for $v \in T_pM$
$U_pM = \{v \in T_pM \mid |v| = 1\}$
$\gamma_t(p, v)$ = the (closed) geodesic curve in (M, g) with the initial conditions: $\gamma_0(p, v) = p$, $\gamma_0'(p, v) = v$
S^n = the n-dimensional unit sphere
$\pi_1(M)$ = the fundamental group of M

2. Proofs of the main theorems

Let (M, g) be an n-dimensional compact Riemannian manifold with the property that for each geodesic curve $c : \mathbb{R} \mapsto M$, the length of its image $c(\mathbb{R})$ in M is finite. Then we have

PROPOSITION 2.1. *Every complete geodesic curve in (M, g) is closed.*

PROOF. Suppose that there exists a nontrivial complete geodesic curve $c : \mathbb{R} \mapsto M$ such that c is not closed. Define a curve $\widetilde{c} : \mathbb{R} \mapsto M$ by

$$\widetilde{c}(s) := c(\frac{s}{|c'(0)|}) \quad \text{for } s \in \mathbb{R}.$$

Then we know that $\widetilde{c}$ is also geodesic in (M, g) such that

$$|\widetilde{c}'(s)| = 1 \quad \text{for } s \in \mathbb{R}.$$

By the definition of a geodesic curve, i.e., since every geodesic curve $\widetilde{c}(t)$ with $\widetilde{c}(0) = p$ and $\widetilde{c}'(0) = v$ in (M, g) is the unique solution of the initial value problem:

$$\nabla_{\widetilde{c}'}\widetilde{c}' \equiv 0, \quad \widetilde{c}(0) = p, \quad \widetilde{c}'(0) = v,$$

we obtain that the length of the image of the curve $\widetilde{c}(s)$ from 0 to s_0 is equal to s_0.

$$\text{i.e.,} \quad L(\widetilde{c}([0, s_0])) = s_0.$$

That is, if the curve $\widetilde{c}(s)$ has self-intersection points in the image $\widetilde{c}(\mathbb{R}) \subset M$, then they must intersect transversely.

Thus,

$$L(\widetilde{c}([0, \infty])) = \lim_{s_0 \to \infty} L(\widetilde{c}([0, s_0])) = \lim_{s_0 \to \infty} s_0 = \infty.$$

This contradicts the hypothesis. Therefore, the result follows. □

PROPOSITION 2.2. *For each non-zero tangent vector $v, v' \in T_pM$, $p \in M$, there is a homotopy between the closed geodesic curve $\gamma_t(p, v)$ and the closed geodesic curve $\gamma_t(p, v')$.*

PROOF. Define a map $\widetilde{t} : U_pM \mapsto \mathbb{R}$ by

$$\widetilde{t}(w) := \min\{t_0 \in \mathbb{R}^+ \mid \gamma_{t_0}(p, w) = p \text{ and } \gamma'_{t_0}(p, w) = w\} \quad \text{for } w \in U_pM.$$

Then by Proposition 2.1, $\widetilde{t}$ is well-defined. For each non-zero tangent vector $v, v' \in T_pM$, define a map $F : I \times I \mapsto M$ by

$$F(t, s) := \gamma_{\widehat{t}}(p, \widehat{v}),$$

where $\widehat{t} = \dfrac{\widetilde{t}(\frac{\widehat{v}}{|\widehat{v}|})}{|\widehat{v}|} \cdot t$ and $\widehat{v} = (1-s)v + sv'$ for $s, t \in I$. Then clearly, F is a homotopy between the closed curve $\gamma_t(p, v)$ and the closed curve $\gamma_t(p, v')$. □

PROOF OF THE MAIN THEOREM 1. Since each class in $\pi_1(M)$ can be represented by a closed geodesic curve [2], by Proposition 2, the result follows. □

EXAMPLE 2.1. a) For the unit circle S^1, we know that the lengthes of the images $c(\mathbb{R})$ of any non-trivial geodesic curves $c : \mathbb{R} \mapsto S^1$ are equal to 2π, and so finite. But $\pi_1(S^1) = \mathbb{Z}$.

b) For the unit sphere S^2, let $c : \mathbb{R} \mapsto S^2$ be a non-trivial geodesic curve in S^2. Then the length of the image $c(\mathbb{R}) \subset S^2$ is equal to 2π, and so finite. But $\pi_1(S^2) = 0$.

c) For the generalized flat torus $T^2 := \mathbb{R}^2/s\mathbb{Z} \oplus t\mathbb{Z}$ with $s, t \in \mathbb{R}^+$, we know that $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. But let $\pi : \mathbb{R}^2 \mapsto T^2$ be the natural projection. Then for any straight lines $l : y = ax + b$ with $a, b \in \mathbb{R}$ in $\mathbb{R}^2$, we have

$$L(c) = \begin{cases} \sqrt{(sn)^2 + (tm)^2} & \text{if } a = \frac{m}{n} \in \mathbb{Q} - \{0\} \text{ and } m, n \text{ integers}, \\ \infty & \text{if } a \in \mathbb{R} - \mathbb{Q}, \\ s & \text{if } a = 0, \\ t & \text{if } l : x = p, \ p \in \mathbb{R}, \end{cases}$$

where c is the image $\pi(l)$ in T^2. So, the torus T^2 has a geodesic curve whose image is of infinite length in T^2.

d) Let (M, g) be an n-dimensional compact connected Riemannian manifold such that neither $\pi_1(M) = \mathbb{Z}$ nor $\pi_1(M) = \mathbb{Z}_p$ for some $p \in \mathbb{N}$. Then there exists a geodesic curve in (M, g) such that the length of its image in M is infinite.

PROOF OF THE MAIN THEOREM 2. Consider the map $\widetilde{t} : U_pM \mapsto \mathbb{R}$, defined by

$$\widetilde{t}(v) := \min\{t_0 \in \mathbb{R}^+ \mid \gamma_{t_0}(p, v) = p \text{ and } \gamma'_{t_0}(p, v) = v\} \quad \text{for } v \in U_pM.$$

Let $\widetilde{o}$ be the north pole of the unit sphere S^n and $T : T_{\widetilde{o}}S^n \mapsto T_pM$ an isometry. For each closed geodesic curve $\gamma_t(p, v) : [0, \widetilde{t}(v)] \mapsto M$ with $v \in U_pM$, let $w := T^{-1}(v) \in U_{\widetilde{o}}S^n$. Then define a map $f_v : S^n \mapsto M$ by

$$f_v(\widetilde{o}) := p$$

and for any

$$\overline{w} \in U_{\widetilde{o}}S^n, \quad f_v(c_t(\widetilde{o}, \overline{w})) := \gamma_{\widehat{t}}(p, T(\overline{w})) \text{ for } \begin{cases} t \in [0,\pi) & \text{if } \overline{w} \neq w \\ t \in [0,\pi] & \text{if } \overline{w} = w, \end{cases}$$

where $\widehat{t} = \dfrac{\widetilde{t}(T(\overline{w}))}{2\pi} \cdot t$ and $c_t(\widetilde{o}, \overline{w})$ is the geodesic curve in S^n such that $c_0(\widetilde{o}, \overline{w}) = \widetilde{o}$ and $c_0'(\widetilde{o}, \overline{w}) = \overline{w}$. Then by the definition of f_v, obviously, f_v is a well-defined map. Let $\widehat{o}$ be the south pole of S^n. Then we can also show that

$$f_v\mid_{S^n - \{\widehat{o}\}} : S^n - \{\widehat{o}\} \mapsto M \quad \text{is continuous,}$$

where $f_v\mid_{S^n-\{\widehat{o}\}}$ is the restriction map of f_v to the subset $S^n - \{\widehat{o}\}$.

For each $v \in U_pM$, we get $w = T^{-1}(v) \in U_{\widetilde{o}}S^n$. Then conveniently, we may assume $w = (-1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$. Consider the map $F : I \times I \mapsto S^n$, given by

$$F(t,s) := (-s\sin 2\pi t, \sqrt{2s(1-s)(1-\sin 2\pi t)}, 0, \cdots, 0, 1-s+s\cos 2\pi t)$$

for $(t,s) \in I \times I$ and the inclusion map $i : I \times [0,1) \hookrightarrow I \times I$.

Let $\widetilde{F} := f_v \circ F \circ i : I \times [0,1) \mapsto M$. Then define the map $\widehat{F} : I \times I \mapsto M$ by

$$\begin{cases} \widehat{F}(t,s) := \widetilde{F}(t,s), & (t,s) \in I \times [0,1) \\ \widehat{F}(t,1) := \lim_{s \to 1} \widetilde{F}(t,s), & (t,1) \in I \times \{1\}. \end{cases}$$

It is easy to show that $\widehat{F}$ is continuous. By Main Theorem 1, there is an element α in $\pi_1(M)$ such that α generates the group $\pi_1(M)$. Then we have

$$\alpha = [\gamma_t(p, v_0)] \quad \text{for some } v_0 \in U_pM,$$

where $[\gamma_t(p, v_0)]$ denotes the class represented by the closed geodesic curve $\gamma_t(p, v_0)$. Consider the map f_{v_0} and the map $\widehat{F}$ obtained by using f_{v_0} instead of f_v as above. Then

$$0 = [p] = [\widehat{F}(t,0)] = [\widehat{F}(t,1)] = [\gamma_t(p,v_0) + \widetilde{C}] = [\gamma_t(p,v_0)] + [\widetilde{C}],$$

where $\widetilde{C} = (\widehat{F}(t,1) - \gamma_t(p,v_0)) \cup \{\gamma_{\frac{\widetilde{t}(v_0)}{2}}(p,v_0)\}$ and to obtain the last equality, with the abuse of the notations we handle those classes in the last equation with the new base point $\gamma_{\frac{\widetilde{t}(v_0)}{2}}(p,v_0)$ instead of the old original base point p. Since the closed curves $\gamma_t(p,v_0)$ and $\widetilde{C}$ have the same orientation in some sense, by Main Theorem 1, there exists a non-negative integer n such that $[\widetilde{C}] = n\alpha$. Thus we have

$$(n+1)\alpha = 0.$$

Therefore, we conclude

$$\pi_1(M) = \mathbb{Z}_p \quad \text{for some } p \in \mathbb{N}.$$

□

REMARK 2.1. a) Let $\mathbb{P}^n$ be the n-dimensional real projective space with $n \geq 2$. i.e., let $A : S^n \mapsto S^n$ be the antipodal map, defined by $A(q) = -q$ for $q \in S^n$. Then we have $\mathbb{P}^n := S^n/\{id, A\}$. We know $\pi_1(\mathbb{P}^n) = \mathbb{Z}_2$. But with some computations, we have

$$0 = [\widehat{F}(t,1)] = 2\alpha.$$

Let $c : \mathbb{R} \mapsto \mathbb{P}^n$ be a geodesic curve in $\mathbb{P}^n$. Then the length of the image $c(\mathbb{R}) \subset \mathbb{P}^n$ is equal to or less than π, and so finite.

b) Let $M = S^1 \times S^2$. Since $\pi_1(M) = \mathbb{Z}$ and $\dim M \geq 2$, we know that there exists a geodesic curve in M such that the length of its image in M is infinite.

c) If $\pi_1(M) \supset \mathbb{Z}$ with $\dim M \geq 2$, then for each point $p \in M$ there is a tangent vector $v \in U_pM$ such that the complete geodesic curve $\gamma_t(p, v)$ with $\gamma_0(p, v) = p$ and $\gamma_0'(p, v) = v$ has the length of its image in M to be infinite.

d) Let N be a k-dimensional compact Riemannian manifold with $k \geq 1$. Let $M = S^1 \times N$. Since $\pi_1(M) \supset \mathbb{Z}$, we must have a geodesic curve in M whose image is of infinite length.

References

[1] M. Günther, *Zum Einbettungssatz von J. Nash*, Math. Nachr. **144** (1989), 165–187.

[2] J. Jost *Riemannian geometry and geometric analysis*, Springer-Verlag, Berlin Heidelberg, 1995.

[3] J. Nash, *C^1 isometric imbeddings*, Ann. of Math.(2) **60** (1954), 383–396.

[4] ———, *The imbedding problem for Riemannian manifolds*, Ann. of Math.(2) **63** (1956), 20–63.

[5] H. Whitney, *Differentiable manifolds*, Ann. of Math. **37** (1936), 645–680.

School of Mathematical Sciences
Seoul National University
Seoul 151-747, Korea
E-mail: parkksn@math.snu.ac.kr